EXPLORATIONS OF SPERNER'S LEMMA AND ITS CONNECTIONS TO BROUWER'S FIXED POINT THEOREM

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Abstract. We discuss Sperner's Lemma in the form of two different proofs. Connections can be made to graph theory and cochains in simplicial complexes. This result is then used to prove Brouwer's Fixed Point Theorem in a nontraditional manner. Our method provides a more constructive approach to the theorem in contrast to the usual proof. We also mention the connection to the No-Retraction Theorem which is used in the usual proof.

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1. Introduction

Sperner's Lemma is a combinatorial result about a triangulation of an n-simplex. More specifically, it deals with colorings of vertices of a triangulation and counting full color simplexes. There are various proofs of Sperner's Lemma, many of which use vastly different techniques. In this paper we discuss two different methods of proving the result and the connections between them.

On the other hand, Brouwer's Fixed Point Theorem is a well known result dealing with continuous functions on the closed unit ball in $\mathbb{R}^n$. While at first these theorems seem unrelated, it turns out that we can use Sperner's Lemma to directly prove Brouwer's Theorem.

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2. Sperner's Lemma

We now present two proofs of Sperner's Lemma. The first is a combinatorial proof involving graphs, while the second relies on cochains and simplicial complexes. First we establish some notation which will be used in both proofs. Let T be the standard n-simplex in $\mathbb{R}^{n+1}$. Let T^k be the face of T which doesn't contain the k^{th} vertex of T. That is, T^k is the face opposite the k^{th} vertex. Let $\mathcal{T}$ be a triangulation of T. We can color the vertices of $\mathcal{T}$ in a particular way. Namely, color the $n+1$ vertices of T with colors $c_1, \dots c_{n+1}$ respectively. Color each vertex of $\mathcal{T}$ in such a way that if a vertex of $\mathcal{T}$ lies on T^k then it is not colored c_k. We call such a coloring of $\mathcal{T}$ a *Sperner labeling* or *Sperner coloring.* The first proof of Sperner's Lemma follows.

Theorem 2.1 (Sperner's Lemma)**.** *Suppose $\mathcal{T}$ is given a Sperner Labeling. Then there are an odd number of simplexes in $\mathcal{T}$ which contain a vertex of every color.*

Proof. First consider the case $n = 1$. Then T is the line segment with ends colored c_1 and c_2, and the triangulation $\mathcal{T}$ consists of finitely many points on T which are colored c_1 or c_2. Since T starts with c_1 and ends with c_2, there must be an odd number of segments with both colors.

Induct on n and assume that any such triangulation of an $(n-1)$-simplex has an odd number of $(n-1)$-simplexes with full color. Create a graph where each vertex corresponds to an n-simplex in $\mathcal{T}$ and there is one additional vertex. Two vertices are connected by an edge if the two simplexes they represent share a face which is colored with every color except c_{n+1}. The external vertex is connected by an edge to any n-simplex whose intersection with ∂T is an $(n-1)$-simplex with all colors but c_{n+1}. Note that every such boundary $(n-1)$-simplex must be contained in T^{n+1} and that T^{n+1} is an $(n-1)$-simplex whose induced colored triangulation is a Sperner labeling with colors $c_1, c_2, \dots, c_n$. It follows that T^{n+1} has an odd number of $(n-1)$-simplexes with full color. Note that here "full color" means colors c_1 through c_n, but not c_{n+1}. Thus the degree of the external vertex is odd. Since in any simple graph there are an even number of vertices with odd degree, there must be an odd number of simplexes in $\mathcal{T}$ with odd degree.

Let $\sigma = \{v_1, v_2, \dots, v_{n+1}\}$ be a simplex of $\mathcal{T}$ with degree at least 2. Without loss of generality, let $j = \{v_1, v_2, \dots, v_n\}$ and $k = \{v_2, v_3, \dots, v_{n+1}\}$ be the vertex sets of two full color faces. It follows that v_1 and v_{n+1} must be the same color, but since any other face will contain both these vertices, σ can have degree at most 2. Therefore each simplex with odd degree must have degree 1 and, consequently, exactly once face with colors $c_1, \ldots, c_n$. Since these are exactly the full-colored simplexes, we see that there are an odd number of simplexes in $\mathcal{T}$ with full color. $\square$

We now consider a proof of Sperner's Lemma using cochains. For the purposes of this proof, we take cochains with values in $\mathbb{F}_2$. For a simplicial complex X, a *cochain* in $C^n(X)$ is a map from the set of n-simplexes in X to $\mathbb{F}_2$. If α is an n-simplex in X, then there exists a cochain in $C^n(X)$ which has value 1 on α and 0 on all other simplexes. and we can identify α with this cochain. In this way we will identify cochains in $C^n(X)$ with formal sums of n-simplexes.

For a simplicial complex X, the coboundary map $\partial^* : C^{n-1}(X) \to C^n(X)$ is defined as follows: If $\alpha \in C^{n-1}(X)$ is an $(n-1)$-simplex, then $\partial^*(\alpha)$ is the sum of all simplexes in $C^n(X)$ which have α as a face. We can extend ∂^* to all of C^{n-1} using linearity. If $\alpha \in \ker(\partial^*)$ then α is a *cocyle*.

Given another simplicial complex Y and a simplicial map $\gamma : X \to Y$, we can also define the map $\gamma^* : C^n(Y) \to C^n(X)$. If β is an n-simplex in Y, then $\gamma^*(\beta)$ is the sum of all n-simplexes in X which are mapped on to β by γ. We extend γ^* to all of $C^n(X)$ by linearity. It can be checked that $\gamma^* \circ \partial^* = \partial^* \circ \gamma^*$.

For a simplicial complex X and a subcomplex $A \subseteq X$, let α be a cocycle in $C^{n-1}(A)$. Then we can extend α arbitrarily to obtain $\overline{\alpha} \in C^{n-1}(X)$. If we take $\partial^*(\overline{\alpha})$, then this coboundary is a cocycle in $C^n(X, A)$. Although this assignment is not unique for cochains, it can be checked that it gives a well-defined map in cohomology $\delta : H^{n-1}(A) \to H^n(X, A)$. Here $\delta(\alpha) = \partial^*(\overline{\alpha})$ is the connecting homomorphism in the long exact sequence of cohomology coming from the short exact sequence

$$0 \leftarrow C^n(A) \leftarrow C^n(X) \leftarrow C^n(X, A) \leftarrow 0.$$

It follows that given a simplicial map of pairs $\gamma : (X', A') \to (X, A)$, we have the following commutative diagram.

$$\begin{array}{ccc} H^{n-1}(A') & \xrightarrow{\delta} & H^n(X', A') \\ \uparrow{\scriptstyle (\gamma|_A)^*} & & \uparrow{\scriptstyle \gamma^*} \\ H^{n-1}(A) & \xrightarrow{\delta} & H^n(X, A) \end{array}$$

We will now use the equality

$$\gamma^*(\partial^*(\overline{\alpha})) = \partial^*(\gamma^*(\overline{\alpha})), \tag{1}$$

to prove Sperner's Lemma using cochains.

Cochain Level Proof of Sperner's Lemma. As in the combinatorial proof, we proceed by induction on n, where $n = 1$ is clear. Suppose that for $n > 1$ we have an odd number of full colored n-simplexes in any such triangulation $\mathcal{T}$.

Let $\phi : (\mathcal{T}, \partial\mathcal{T}), \rightarrow (T, \partial T)$ be the simplicial map which sends a vertex of $\mathcal{T}$ to the vertex of T with the same color. Note that a simplex of $\mathcal{T}$ has full color if and only if its image under ϕ is T. Let ∂T be the boundary of T, defined as the union of all its faces. Recall that T^{n+1} is the face of T which does not contain the color c_{n+1}. We proceed by using equation (1) with ϕ in place of γ and T^{n+1} in place of α.

We can consider T^{n+1} as an $(n-1)$-cochain of ∂T. Since $C^n(\partial T) = 0$, we have T^{n+1} is cocycle. If we extend T^{n+1} to a cochain in $C^{n-1}(T)$, we find that the only choice is $\overline{T^{n+1}} = T^{n+1}$ since all $(n-1)$-simplexes are contained in ∂T. We then have

$$\phi^*(\partial^*(T^{n+1})) = \partial^*(\phi^*(T^{n+1})). \tag{2}$$

Label the n-simplexes in $\mathcal{T}$ which have full color as $\rho_1, \ldots \rho_e$. Now consider all $(n-1)$-simplexes in $\mathcal{T}$ which ϕ maps to T^{n+1}. These are precisely the $(n-1)$-simplexes which have every color but c_{n+1}. Let $\sigma_1, \ldots, \sigma_h$ be the $(n-1)$-simplexes in $\mathcal{T}$ contained in T^{n+1} which are mapped onto T^{n+1} by ϕ. Likewise, let $\tau_1, \ldots, \tau_g$ be the $(n-1)$-simplexes of $\mathcal{T}$ that are not in T^{n+1} which are mapped onto T^{n+1} by ϕ. Note that $\tau_1, \ldots, \tau_g$ are contained within the interior of $\mathcal{T}$. Moreover, $\sigma_1, \ldots, \sigma_h, \tau_1, \ldots, \tau_g$ constitute all $(n-1)$-simplexes in $\mathcal{T}$ whose image under ϕ is precisely T^{n+1}. We know this because $\mathcal{T}$ is given a Sperner coloring.

Now, note that by definition $\partial^*(T^{n+1})$ is the sum of all n-simplexes which have T^{n+1} as a face. Thus $\partial^*(T^{n+1}) = T$. Recall also that for a simplex β we have that $\phi^*(\beta)$ is the sum of all simplexes which ϕ maps onto β. It follows then that

$$\phi^*(\partial^*(T^{n+1})) = \phi^*(T) = \rho_1 + \cdots + \rho_e.$$

On the other hand $\phi^*(T^{n+1})$ is precisely $\sigma_1 + \cdots + \sigma_h + \tau_1 + \cdots + \tau_g$. Then equation (2) becomes

$$\rho_1 + \cdots + \rho_e = \partial^*(\sigma_1 + \cdots + \sigma_h + \tau_1 + \cdots + \tau_g).$$

Note that each σ_i is contained in T^{n+1} and so it is contained in precisely one n-simplex, $\widehat{\sigma_i}$, in $\mathcal{T}$. Also, since each of these simplexes has unique intersection with ∂T the h n-simplexes, $\widehat{\sigma_1}, \ldots, \widehat{\sigma_h}$ are distinct. Using this, and the linearity of ∂^*, we now have

$$\rho_1 + \cdots + \rho_e = \widehat{\sigma_1} + \cdots + \widehat{\sigma_h} + \partial^*(\tau_1) + \cdots + \partial^*(\tau_g).$$

Since each τ_i is contained in the interior of $\mathcal{T}$, it is the face of exactly two n-simplexes. Thus $\partial^*(\tau_i)$ is the sum of two n-simplexes. Since our cochains have values in $\mathbb{F}_2$, when we sum this equation mod 2 we obtain $h + 2g \equiv e \pmod 2$ which simplifies to $h \equiv e \pmod 2$. Therefore the parity of h is always the same as that of e. Since we've inductively assumed h is odd, it must also be the case that e is odd. But $\rho_1, \ldots, \rho_e$ are precisely the simplexes in $\mathcal{T}$ with full color. □

3. Brouwer's Fixed Point Theorem

The usual proof of Brouwer's Fixed Point Theorem is founded on the No-Retraction Theorem. Using this and the homology of the n-ball, one obtains the result through contradiction. The general idea is to construct a retraction from the closed n-ball to its boundary. Assuming there are no fixed points, this is done by considering the line defined by x and $f(x)$; this must intersect the boundary at some point and so x gets mapped to the boundary in this way. Since there are no fixed points, the map is valid for each x in the ball. But it can be shown that the n-ball has trivial homology while its boundary has infinite homology. Since the retraction induces a homomorphism on homology, we arrive at a contradiction.

Here we present a proof of Brouwer's Theorem using Sperner's Lemma. The proof is more straightforward and direct, and it also uses familiar and simple concepts. It should be noted that while the standard proof is non-constructive, this proof both allows for a more intuitive view of why the theorem is true and gives a semi-constructive approach to finding the fixed point.

Theorem 3.1 (Brouwer's Fixed Point Theorem)**.** *Every continuous function from the closed unit ball in $\mathbb{R}^n$ to itself has a fixed point.*

Proof. Let T be the n-simplex in $\mathbb{R}^{n+1}$ defined by the set of points

$$T = \left\{ (x_1, x_2, \ldots, x_{n+1}) \mid x_i \geq 0, \sum_{i=1}^{n+1} x_i = 1 \right\}.$$

Let $f : T \to T$ be continuous and define

$$f(x) = f((x^1, x^2, \ldots, x^{n+1})) = (f(x)^1, f(x)^2, \ldots, f(x)^{n+1}).$$

We first make the observation that for a point $x \in T$, if $f(x)^i - x^i \geq 0$ for all $1 \leq i \leq n+1$, then x must be a fixed point. This follows from the fact that both x and $f(x)$ are in T and so both points must have coordinates which sum to 1.

For a point $x \in T$, define k to be any index i which minimizes the quantity $f(x)^i - x^i$. Note that k is defined for each point in T. Assign the color c_k to each vertex of T based on its value of k. Note that the j^{th} vertex of T is the point with $x^j = 1$ and $x^i = 0$ for all $i \neq j$. Thus the j^{th} vertex of T has the property that $f(x)^j - x^j \leq 0$ and $f(x)^i - x^i \geq 0$ for all $i \neq j$. Moreover, in the case of equality, the j^{th} vertex is a fixed point and we're done. Assuming inequality, this vertex must be colored c_j. This shows that each vertex of T is colored a distinct color according to its index.

Take the Barycentric subdivision $\mathcal{T}_1$ of T. Color all vertices of $\mathcal{T}_1$ in the same way as the vertices of T were colored. If x is a vertex of $\mathcal{T}_1$ on T^j, then $x^j = 0$. Then $f(x)^j - x^j \geq 0$ and we see that x cannot be colored c_j unless it is a fixed point. Thus, we have a Sperner coloring of $\mathcal{T}_1$. One simplex of $\mathcal{T}_1$, by Sperner's Lemma, has full color, so call this simplex σ_1. Now take the Barycentric subdivision of $\mathcal{T}_1$ to obtain $\mathcal{T}_2$. By the same argument, labeling the vertices in the same fashion gives a Sperner coloring of $\mathcal{T}_2$ and so there exists some simplex σ_2 with full color. Continue in this way to obtain a sequence of simplexes, $T = \sigma_0, \sigma_1, \sigma_2, \ldots$.

Consider the sequence of vertices from $\sigma_0, \sigma_1, \sigma_2 \ldots$ which are colored c_k. Call this sequence (p_n^k). Since (p_n^1) is bounded, it has a convergent subsequence $(p_{n_j}^1)$. Now let p be the limit of $(p_{n_j}^1)$. Note that since the diameter of σ_i goes to 0 as i goes to ∞, any neighborhood of p will contain all but finitely many of the simplexes (σ_{n_j}). Thus the induced subsequences of each (p_n^k) all converge to p. Note that because of how each sequence is defined, we have, $f(p_{n_j}^k)^i \leq (p_{n_j}^k)^i$ for each $i = 1, 2, \ldots, n+1$. Since f is continuous, we can take the limit and obtain $f(p)^i \leq p^i$. But this is only possible if $f(p) = p$. □

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References

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