THE PRIME NUMBER THEOREM

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ABSTRACT. The Prime Number Theorem is an intriguing result describing, for large enough x, the close approximation of the number of primes less than or equal to x with $\frac{x}{log x}$. I wish to prove this result, and by doing so describe a variety of other interesting relationships in close accordance with the Prime Number Theorem. This process will illustrate the nature of an important function, $\varphi(x)$ which also has a close connection with the Riemann Zeta Function.

CONTENTS

1. A FEW PRELIMINARY DEFINITIONS

Definition 1.1. We define $\varphi(x) = \sum_{p \leq x} \log p$ where p is a prime number.

Definition 1.2. Given two functions f and g of common variable, x, where g is positive, and both f and g are defined for all sufficiently large x, we denote

$$f = O(g)$$

to mean that there exists some constant $C > 0$ such that for all x sufficiently large, $|f(x)| \leq Cg(x)$

i.e. $f = O(x)$ means that $|f(x)| \leq Cx$ for some $C > 0$ and large enough x

2. THE MAIN LEMMA

This section is devoted to an important Theorem, which describes, in general, the association between a bounded, piecewise continuous function and its Laplace Transform which will have great importance in some later results needed to prove the Prime Number Theorem.

Theorem 2.1. *Let f be a bounded, piecewise continuous function defined on $\mathbb{R}_{\geq 0}$. Now define*

$$g(z) = \int_0^\infty f(t)e^{-zt} \quad for\ \Re(z) > 0$$

Suppose g extends to an analytic function on $\Re(z) \geq 0$, then

$$\int_0^\infty f(t)dt \ exists\ and\ equals\ g(0)$$

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Proof. We begin by defining an entire function, g_T where

$$g_T(z) = \int_0^T f(t)e^{-zt}dt$$

for $T > 0$

We then have to show that $\lim_{T\to\infty} g_T(0) = g(0)$.

To do this, we define a path, C, around 0 which is composed of, for $\delta > 0$, the union of C^+ and C^- where

C^+ is the semicircle $|z| = R$ for $\Re(z) \geq 0$ and

C^- is the line $\Re(z) = -\delta$ and the portion of the circle $|z| = R$ for $-\delta \leq \Re(z) < 0$
Now, due to the assumption we made that g extends to an analytic function on $\Re(z) \geq 0$, we can choose δ to be arbitrarily small so that g will be analytic on the region bounded by C. We now recall the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z}d\zeta$$

where C_R is the circle of radius R, and $z \in C_R$

Thus, because $0 \in C$ we see that

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_C \frac{g(z) - g_T(z)}{z}dz = \frac{1}{2\pi i}\int_C (g(z) - g_T(z))\, e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}$$

which is seen because 0 is a simple pole for $\frac{1}{2\pi i}\int_C \frac{g(z) - g_T(z)}{z}dz$, and the term $e^{Tz}\left(1 + \frac{z^2}{R^2}\right)$ evaluated at 0 is just 1. We now split the proof up into 3 claims, which intend to show that $|g(0) - g_T(0)| \to 0$ as $T \to \infty$.

Claim 1:

$$\left|\frac{1}{2\pi i}\int_{C^+} (g(z) - g_T(z))\, e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq \frac{2B}{R}$$

where B is a bound for f.

Proof. First, for $|z| = R$ we see that

$$\left|e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| = e^{\Re(z)T}\left|\frac{R^2 + z^2}{R^2}\right|\left|\frac{1}{z}\right| = e^{\Re(z)T}\left|\frac{R}{z} + \frac{z}{R}\right|\frac{1}{R} = e^{\Re(z)T}\frac{2|\Re(z)|}{R^2}$$

Second, for $\Re(z) > 0$ we get

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}dt\right| \leq B\int_T^\infty |e^{-zt}|dt \leq e^{-\Re(z)T}\frac{B}{\Re(z)}$$

Taking the product of these two results gives us

$$\left|\frac{1}{2\pi i}\int_{C^+} (g(z) - g_T(z))\, e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq e^{-\Re(z)T}e^{\Re(z)T}\frac{B}{\Re(z)}\frac{2|\Re(z)|}{R^2} = \frac{2B}{R^2}$$

Multiplying this by R, the radius of C^+, gives us $\frac{2B}{R}$, which is our bound for the integral over C^+. □

Now that we have this result, we need to look at the expression under the integral sign for g_T and g separately.

Claim 2:

$$\left|\frac{1}{2\pi i}\int_{C^-} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq \frac{2B}{R}$$

Proof. Notice that we have

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}dt\right| \leq B\int_0^T e^{-\Re(z)T}dt \leq \frac{Be^{-\Re(z)T}}{-\Re(z)}$$

And just as in the first claim, we have that, for $|z| = R$ on C^-

$$\left|e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\right| = -e^{\Re(z)T}\frac{2|\Re(z)|}{R^2}$$

Thus, multiplying these two results together yields us,

$$-e^{\Re(z)T}e^{-\Re(z)T}\frac{2|\Re(z)|}{R^2}\frac{B}{-\Re(z)} = \frac{2B}{R^2}$$

And again, multiplying this value by, R, the radius of the semicirlce, gives us $\frac{2B}{R}$ as a bound for the expression under the integral sign for $g_T(z)$ □

So now we need to look at the expression under the integral for $g(z)$

Claim 3:

$$\int_{C^-} g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \to 0 \quad as\ T \to \infty.$$

Proof. We see that for any z in the region bounded by C^-,

$$e^{Tz} \to 0$$

as $T \to \infty$. This result follows from the region being a compact subset of $\mathbb{C}$. Thus, because $g(z)\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}$ does not depend on T, and is a bounded expression for all z in the region bounded by C^-, we see that for all $\Re(z) \geq -\delta$ and $|z| \leq R$ that

$$g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z} \to 0\ as\ T \to \infty$$

Hence it follows that

$$\int_{C^-} g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \to 0 \quad as\ T \to \infty.$$

□

Knowing this, we see that for any $\epsilon > 0$, choose R large enough so that $\frac{2B}{R} < \frac{\epsilon}{3}$ and T large enough so that

$$\left|\int_{C^-} g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| < \frac{\epsilon}{3}$$

Then we see that $|g(0) - g_T(0)| < \epsilon$, and thus the limit;

$$\lim_{T\to\infty} g_T(0) = \int_0^\infty f(t)dt$$

exists. □

3. The Behavior of $\varphi(x)$

The purpose of this section is to describe some of the unique properties of $\varphi(x)$, leading to the result that $\varphi(x) \to x$ as $x \to \infty$, which is essential to prove the Prime Number Theorem.

Theorem 3.1. $\varphi(x) = O(x)$

Proof. We need to show that there exists a constant $C > 0$ such that for x large enough, $\varphi(x) = \sum_{p \le x} \log p \le Cx$

To do this, let $n \in \mathbb{N}$ and let p be prime, then we see that

$$2^{2n} = (1{+}1)^{2n} = \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} \ge \frac{(2n)!}{n!(2n-n)!} \ge \prod_{n<p\le 2n} p = e^{\left(\prod_{n<p\le 2n} \log p\right)} = e^{\varphi(2n)-\varphi(n)}$$

Taking the natural log of the left and right ends of the above relation, we find that

$$2n \log 2 \ge \varphi(2n) - \varphi(n)$$

Thus we see that for any constant $C > \log 2$,

$$\varphi(x) - \varphi(\frac{x}{2}) \le Cx$$
$$\varphi(\frac{x}{2}) - \varphi(\frac{x}{2^2}) \le C(\frac{x}{2})$$
$$\vdots$$
$$\varphi(\frac{x}{2^m}) - \varphi(\frac{x}{2^{m+1}}) \le C(\frac{x}{2^m})$$

Summing these up as m goes to ∞, we get

$$\varphi(x) \le Cx(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) = 2Cx$$

which proves the theorem for $C^{'} > 2C > \log 2$ □

Proposition 3.2.

$$\sum_p \frac{\log p}{p^s} = s \int_1^\infty \frac{\sum_{p \le x} \log p}{x^{s+1}} dx$$

for all $\Re(s) > 1$

Proof. First, we look at the integral between any two consecutive primes, p_1, p_2 where $\sum_{p \le x} \log p$ is constant:

$$s \int_{p_1}^{p_2} \frac{\sum_{p \le x} \log p}{x^{s+1}} dx = \left(\sum_{p \le p_1} \log p \right) s \int_{p_1}^{p_2} \frac{1}{x^{s+1}} dx = \left(\sum_{p \le p_1} \log p \right) s \left(\frac{x^{-s}}{-s} \right)_{p_1}^{p_2}$$

$$= \left(\sum_{p \le p_1} \log p \right) \left(\frac{1}{p_1^s} - \frac{1}{p_2^s} \right)$$

And then, using the summation by parts formula:

$$\sum_{k=0}^{n} a_k b_k = a_n B_n - \sum_{k=0}^{n-1} B_k (a_{k+1} - a_k)$$

where $B_n = \sum_{k=0}^{n} b_k$ we see that by letting $B_n = \sum_{p \le p_n} \log p$ and $a_n = \left(\frac{1}{p_n^s}\right)$ we get

$$s \int_{p_1}^{p_2} \frac{\sum_{p\le x} \log p}{x^{s+1}} dx + s \int_{p_2}^{p_3} \frac{\sum_{p\le x} \log p}{x^{s+1}} dx + \ldots + s \int_{p_{n-1}}^{p_n} \frac{\sum_{p\le x} \log p}{x^{s+1}} dx$$

$$= \left(\sum_{p\le p_1} \log p\right) \left(\frac{1}{p_1^s} - \frac{1}{p_2^s}\right) + \ldots + \left(\sum_{p\le p_{n-1}} \log p\right) \left(\frac{1}{p_{n-1}^s} - \frac{1}{p_n^s}\right)$$

$$= \sum_{i=1}^{n-1} \left[\left(\sum_{p\le p_i} \log p\right) \left(\frac{1}{p_i^s} - \frac{1}{p_{i+1}^s}\right)\right] = -\sum_{i=1}^{n-1} \left[\left(\sum_{p\le p_i} \log p\right) \left(\frac{1}{p_{i+1}^s} - \frac{1}{p_i^s}\right)\right]$$

$$= \sum_{i=1}^{n} \frac{\log p_i}{p_i^s} - \left(\sum_{p\le p_n} \log p\right) \frac{1}{p_n^s}$$

And using the fact that $\sum_{p\le x} \log p = \varphi(x) = O(x)$ we see that $\left(\sum_{p\le p_n} \log p\right) \frac{1}{p_n^s} \to 0$ as $n \to \infty$.

Thus, taking n to ∞ gives us $s \int_1^\infty \frac{\sum_{p\le x} \log p}{x^{s+1}} dx = \sum_{i=1}^{\infty} \frac{\log p_i}{p_i^s} = \sum_p \frac{\log p}{p^s}$

□

Lemma 3.3.

$$\int_1^\infty \frac{\varphi(x) - x}{x^2} dx$$

converges

Proof. We shall make the substitution $x = e^t$, $dx = e^t dt$, so

$$\int_1^\infty \frac{\varphi(x) - x}{x^2} dx = \int_0^\infty \frac{\varphi(e^t) - e^t}{e^{2t}} e^t dt = \int_0^\infty \frac{\varphi(e^t) - e^t}{e^t} dt = \int_0^\infty f(t) dt$$

where we define $f(t) = \frac{\varphi(e^t) - e^t}{e^t}$. We have seen that $\varphi(x) = O(x)$

And by Theorem 2.1, since $\varphi(x)$ is piecewise continuous and bounded by $O(x)$,

$$\int_0^\infty f(t) dt = g(0)$$

where $g(z) = \int_0^\infty f(t) e^{-zt} dt$. So since $\int_1^\infty \frac{\varphi(x) - x}{x^2} dt = \int_0^\infty f(t) dt$, it is enough to show that $\int_0^\infty f(t) dt$ converges, and by Theorem 2.1 we can just show that the Laplace transform of f is analytic for $\Re(z) \ge 0$

Claim: $g(z) = \frac{1}{z+1} \sum_p \frac{\log p}{p^{z+1}} - \frac{1}{z}$ where p is prime.

If this is true, then it follows that $g(z)$ is meromorphic, hence analytic, for $\Re(z) > 0$

We already have the equality $\displaystyle\sum_p \frac{\log p}{p^s} = s\int_1^\infty \frac{\sum_{p\le x}\log p}{x^{s+1}}dx = s\int_1^\infty \frac{\varphi(x)}{x^{s+1}}dx$,

So it follows that,

$$\begin{aligned}
g(z) &= \int_0^\infty f(t)e^{-zt}dt = \int_0^\infty \frac{\varphi(e^t)-e^t}{e^t}e^{-zt}dt = \int_0^\infty \frac{\varphi(e^t)-e^t}{e^{2t}}e^t e^{-zt}dt \\
&= \int_0^\infty \frac{\varphi(e^t)-e^t}{e^{2t}e^{zt}}e^t dt = \int_0^\infty \frac{\varphi(e^t)-e^t}{e^{t(z+2)}}e^t dt = \int_1^\infty \frac{\varphi(x)-x}{x^{z+2}}dx \\
&= \int_1^\infty \frac{\varphi(x)}{x^{z+2}}dx - \int_1^\infty \frac{x}{x^{z+2}}dx = \int_1^\infty \frac{\varphi(x)}{x^{z+2}}dx - \int_1^\infty \frac{1}{x^{z+1}}dx = \int_1^\infty \frac{\varphi(x)}{x^{z+2}}dx + \frac{1}{z}(\frac{1}{\infty} - \frac{1}{1^z}) \\
&= \int_1^\infty \frac{\varphi(x)}{x^{z+2}}dx - \frac{1}{z} = \frac{1}{z+1}\sum_p \frac{\log p}{p^{z+1}} - \frac{1}{z}
\end{aligned}$$

Thus, g is analytic for $\Re(z) \ge 0$, and since g is the Laplace transform of f, it follows that

$$\int_0^\infty f(t)dt = \int_1^\infty \frac{\varphi(x)-x}{x^2}dx$$

converges. □

Theorem 3.4. $\displaystyle\lim_{n\to\infty} \varphi(x) = x$

Proof.

Claim 1: $\{x \mid \varphi(x) \ge \lambda x\}$ is bounded for $\lambda > 1$
So let M_1 be an upper bound. i.e. for all $x > M_1$, $\varphi(x) < \lambda x$
Suppose not, then there exists some $\lambda > 1$ such that for all $x > M_1$,

$$\frac{\varphi(x)}{x} \ge \lambda$$

$\displaystyle\varphi(x) = \sum_{p\le x}\log p$ is increasing, so we have for $x > M_1$,

$$\int_x^{\lambda x} \frac{\varphi(t)-t}{t^2}dt \ge \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt = \int_1^\lambda \frac{\lambda - t}{t^2}dt > 0$$

The third integral does not depend on x, and we know for a convergent integral, given $\epsilon > 0$, there exists $N > 0$ such that

$$\int_N^\infty f(t)dt < \epsilon$$

so if we let $0 < \epsilon < \displaystyle\int_1^\lambda \frac{\lambda - t}{t^2}dt$, we see that for all x

$$\int_x^{\lambda x} \frac{\varphi(t)-t}{t^2}dt \ge \int_1^\lambda \frac{\lambda - t}{t^2}dt > \epsilon > 0$$

So it is clear that

$$\int_x^{\lambda x} \frac{\varphi(t)-t}{t^2}dt$$

does not converge, which we have seen from Lemma 3.3 is false. Hence there is a contradiction, and $\{x \mid \varphi(x) \geq \lambda x\}$ is bounded for $\lambda > 1$

Claim 2: $\{x \mid \varphi(x) \leq \lambda x\}$ is bounded for $0 < \lambda < 1$

Let M_2 be an upper bound. i.e. for all $x > M_2$, $\varphi(x) > \lambda x$

Suppose not, the there exists $0 < \lambda < 1$ such that for all $x > M_2$,

$$\frac{\varphi(x)}{x} \leq \lambda$$

and,

$$\int_{\lambda x}^{x} \frac{\varphi(t) - t}{t^2} dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2} dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2} dt < 0$$

So as in the previous claim,

$$\int_{\lambda x}^{x} \frac{\varphi(x) - t}{t^2} dt$$

does not converge, and hence, $\{x \mid \varphi(x) \leq \lambda x\}$ is bounded for $0 < \lambda < 1$

Thus, for $\lambda = 1$ and for x large enough, we have $\varphi(x) = x$ □

4. The Prime Number Theorem

Theorem 4.1. *If $\pi(x) =$ the number of prime numbers $\leq$ an integer x, then*

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1$$

Proof. Let $\varphi(x)$ be defined as earlier. $\varphi(x) = \sum_{p \leq x} \log p$ where p is prime.

and since $\sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x$ because $p \leq x$, we see that $\sum_{p \leq x} \log x = \pi(x) \log(x)$.

so,

$$\frac{\varphi(x)}{\log(x)} \leq \pi(x)$$

Now given $\epsilon > 0$,

$$\begin{aligned} \varphi(x) \geq \sum_{x^{1-\epsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\epsilon} \leq p \leq x} \log x^{1-\epsilon} &= \sum_{x^{1-\epsilon} \leq p \leq x} (1-\epsilon) \log x \\ &= (1-\epsilon) \log x [\pi(x) + O(x^{1-\epsilon})] \end{aligned}$$

Thus we have

$$(1-\epsilon)[\pi(x) + O(x^{1-\epsilon})] \leq \frac{\varphi(x)}{\log x} \leq \pi(x)$$

And knowing that $\varphi(x) \to x$ as $x \to \infty$ We see that as $x \to \infty$, $\pi(x) \to \dfrac{x}{\log x}$ □

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