DYNAMICAL SYSTEMS

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Contents

Abstract. This paper seeks to establish the foundation for examining dynamical systems. Dynamical systems are, very broadly, systems that can be modelled by systems of differential equations. In this paper, we will see how to examine the qualitative structure of a system of differential equations and how to model it geometrically, and what information can be gained from such an analysis. We will see what it means for focal points to be stable and unstable, and how we can apply this to examining population growth and evolution, bifurcations, and other applications.

1. Introduction

This paper is based on Arrowsmith and Place's book, *Dynamical Systems*. I have included corresponding references for propositions, theorems, and definitions. The images included in this paper are also from their book.

Definition 1.1. (Arrowsmith and Place 1.1.1) Let $X(t, x)$ be a real-valued function of the real variables t and x, with domain $D \subseteq \mathbb{R}^2$. A function $x(t)$, with t in some open interval $I \subseteq \mathbb{R}$, which satisfies

$$x'(t) = \frac{dx}{dt} = X(t, x(t)) \tag{1.2}$$

is said to be a solution satisfying x'.

In other words, $x(t)$ is only a solution if $(t, x(t)) \subseteq D$ for each $t \in I$. We take I to be the largest interval for which $x(t)$ satisfies (1.2).

Proposition 1.3. ***(AP 1.1.3)*** *If X is continuous in an open domain, $D' \subseteq D$, then given any pair $(t_0, x(t_0))$ when $x(t) \in D'$, there exists a solution $x(t), t \in I$, of $x' = X(t, x)$ such that $t_0 \in I$ and $x(t_0) = x_0$.*

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This proposition gives us the conditions sufficient to show the existence of a solution. This proposition does not exclude the possibility that $x(t_0) = x_0$ for more than one solution $x(t)$. The following proposition gives a sufficient condition for each pair in D' to occur in one and only one solution of (1.3).

Proposition 1.4. ***(AP 1.1.2)*** *If X and $\frac{\partial X}{\partial x}$ are continuous in an open domain $D' \subseteq D$, then given any $(t_0, x_0) \in D'$ there exists a unique solution $x(t)$ of $x' = X(t, x)$ such that $x(t_0) = x_0$.*

We can represent this solution by the graph of $x(t)$. This graph defines a solution curve. Proposition 1.3 implies that the solution curves fill the region D of the t, x-plane. This is true because each point in D must lie on at least one solution curve. The solution of the differential equaion is thus a family of solution curves. See below for examples of solution curves

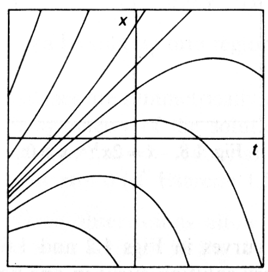

Fig. 1.1. $\dot{x} = x - t$.

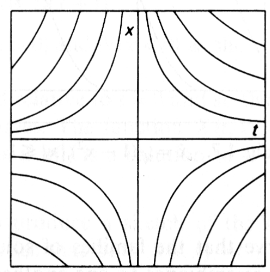

Fig. 1.2. $\dot{x} = -x/t, t \neq 0$.

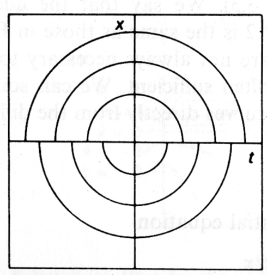

Fig. 1.3. $\dot{x} = -t/x$.

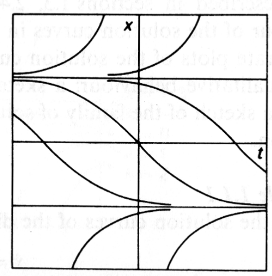

Fig. 1.4. $\dot{x} = \frac{1}{2}(x^2 - 1)$.

If both X and $\frac{\partial X}{\partial x}$ are continuous in D, then Proposition 1.4 implies that there is a unique solution curve passing through every point of D. The solution curves give us the qualitative behavior of the differential equation. However, we can also derive the qualitative behavior from the equation itself.

Example 1.5. Take the differential equation:

$$x' = t + \frac{t}{x}$$

in the region D of the t, x-plane, where $x \neq 0$.

(1) The differential equation gives the slope of the solution curve at each point of the region D. Thus, in particular, the solution curves cross the curve of $t + t/x = k$ for any constant with slope k. This curve is called the **isocline**

of slope k. The set of isoclines, obtained by taking different real values for k, is the set of curves defined by the family of equations

$$x = \frac{t}{k-t}.$$

(2) The sign of x'' determines where in D the solution curves are concave and convex. Thus, the region D can be divided into subsets on which the solution curves are either concave or convex, separated by boundaries where $x'' = 0$.
(3) The isoclines are symmetrically placed relative to $t = 0$ and so the solution curves must also be symmetric. The function $X(t, x) = t + t/x$ satisfies $X(-t, x) = -X(t, x)$.

This allows us to sketch the solution curves for $x' = t + t/x$. The function and its derivative are continuous on D, so there is a unique solution curve passing through each point of D.

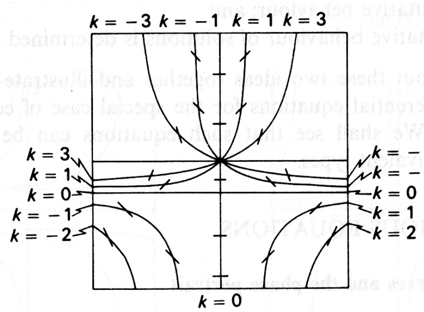

Fig. 1.5. Selected isoclines for the equation $\dot{x} = t + t/x$. The short line segments on the isoclines have slope k and indicate how the solution curves cross them.

Definition 1.6. A differential equation is said to be **autonomous** if x' is determined by x alone and so $X(x, t) = f(x)$ for some function f.

These solutions have an important property. If $\xi(t)$ is a solution to an autonomous differential equation with domain I and range $\xi(I)$, then for any real constant C, $\eta(t) = \xi(t + C)$, is also a solution with the same range, but with domain $\{t : t + C \in I\}$. This follows because

$$\eta'(t) = \xi'(t + C) = X(\xi(t + C)) = X(\eta(t)). \tag{1.7}$$

Furthermore, if there is a unique solution curve passing through each point of $D' = \mathbb{R} \times \xi(I)$ then all solution curves on D' are translations of $x = \xi(t)$.

For families of solution curves related by translations in t, the qualitative behavior of the family of solutions is determined by that of any individual member. The qualitative behavior of such a sample curve is determined by $X(x)$.

When $X(c) = 0$, the solution $x(t) = c$ is represented by the point $x = c$. These solutions are called **fixed points** of the equation. See below for some examples of fixed points.

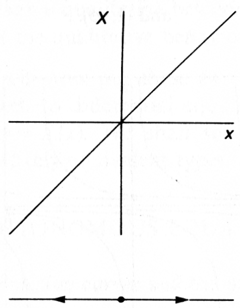

Fig. 1.6. $\dot{x} = x$, $x = 0$ is a fixed point.

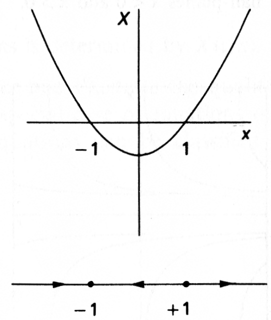

Fig. 1.7. $\dot{x} = \frac{1}{2}(x^2 - 1)$, $x = \pm 1$ are fixed points.

Consider the autonomous system $x' = \frac{dx}{dt} = X(x)$ where $x = (x_1, x_2)$ is a vector in $\mathbb{R}^2$. The solution to this is equivalent to the system of two coupled equations

$$x_1' = X_1(x_1, x_2), \quad x_2' = X_2(x_1, x_2). \tag{1.8}$$

In order to examine the qualitative behavior, we look at the fixed points of the solution. These are solutions of the form $x(t) = c = (c_1, c_2)$, which arise when

$$X_1(c_1, c_2) = 0, \quad X_2(c_1, c_2) = 0. \tag{1.9}$$

The corresponding trajectory is the point (c_1, c_2) in the phase plane.

When calculus fails to give traceable solutions, we can extend the method of isoclines to the plane. The vector field $\mathbf{X} : S \to R^2$ now gives x' at each point of the plane where X is defined, where S is the domain on which X is defined. For qualitative purposes, it is usually sufficient to record the direction of $X(x)$.

If a unique solution $x(t)$ of

$$x' = X(x), \quad x \in S, \quad x(t_0) = x_0 \tag{1.10}$$

exists for any $x_0 \in S$ and $t_0 \in \mathbb{R}$, then each point of S lies on one, and only one, trajectory.

We can look at the differential equation as the velocity of a point on the phase line. The phase line shows the direction and velocity of a point on the plane. Therefore, the differential equation can be thought of defining a flow of phase points along the phase line while the solution to this equation gives the velocity of the flow at each value of $x \in S$. The solution, $x(t)$ that satisfies $x(t_0) = x_0$ gives the past and future positions, or **evolution**, of the phase point which is at x_0 when $t = t_0$. We can formalize this idea by introducing a function $\phi_t : S \to S$ referred to as the **evolution operator**.

The function ϕ_t maps any $x_0 \in S$ onto the point $\phi_t(x_0)$ obtained by evolving for time t along a solution curve through x_0. The point $\phi_t(x_0)$ is equal to $x(t + t_0)$ for any solution $x(t)$ of (1.10). This arises because the solutions of autonomous equations are related by translations in t. Thus, the solution to (1.10) is

$$x(t) = \phi_{t-t_0}(x_0). \tag{1.11}$$

The flow ϕ_t has simple properties that follow from its definition. Uniqueness ensures that

$$\phi_{s+t}(x) = \phi_s(\phi_t(x)) \quad \text{for all } s, t \in \mathbb{R}. \tag{1.12}$$

In particular,

$$\phi_t(\phi_{-t}(x)) = \phi_{-t}(\phi_t(x)) = \phi_0(x) = x. \tag{1.13}$$

and so

$$\phi_t^{-1} = \phi_{-t}. \tag{1.14}$$

In the plane, the autonomous differential equation ensures that solutions are related by translations in t. The mapping ϕ_t maps $x \in \mathbb{R}^2$ to the point obtained by evolving for time t from x according to the differential equation, i.e. $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$. Thus, the orbit or trajectory passing through x is simply $\{\phi_t(x) : t \in \mathbb{R}\}$ oriented by increasing t.

2. Linear Systems

Definition 2.1. A system $x' = X(x)$, where x is a vector in $\mathbb{R}^n$, is called a **linear system** of dimension n, if $X : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping.

If $X : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then it can be written in matrix form as

$$X(x) = \begin{pmatrix} X_1(x_1, \ldots, x_n) \\ \vdots \\ X_n(x_1, \ldots, x_n) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{2.2}$$

so that $x' = X(x)$ becomes

$$x' = X(x) = Ax, \tag{2.3}$$

where A is the coefficient matrix (a_{ij}). We can make a change of variables by expressing each x_i as a function of the new variables. In order to make a change of variables, we must express each $x_i (i = 1, \cdots, n)$ as

$$x_i = \sum_{j=1}^{n} m_{ij} y_j (i = 1, \cdots, n) \text{ i.e. } x = My \tag{2.4}$$

where m_{ij} is a real constant for all i and j. This is a bijection and so M is a non-singular matrix, so the columns, m_i, of M are linearly independent. In other words,

$$x = \sum_{i=1}^{n} y_i m_i. \tag{2.5}$$

In terms of these new variables, (2.3) becomes

$$x' = My' = AMy, \tag{2.6}$$

and thus

$$y' = By \tag{2.7}$$

for $B = M^{-1}AM$. We say that matrices A and B related by this sort of equation are **similar**.

Proposition 2.8. ***(AP 2.2.1)*** *Let A be a real 2×2. Then there is a real, non-singular matrix M such that $J = M^{-1}AM$ is one of the types:*

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

The matrix J is said to be the **Jordan form** of A. The eigenvalues of the matrices A and J are the values for λ for which

$$p_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0 \tag{2.9}$$

so the eigenvalues of A are

$$\lambda_1 = \tfrac{1}{2}(\mathrm{tr}(A) + \sqrt{\Delta}) \quad \lambda_2 = \tfrac{1}{2}(\mathrm{tr}(A) - \sqrt{\Delta}) \tag{2.10}$$

where

$$\Delta = (\mathrm{tr}(A))^2 - 4\det(A)$$

The nature of the eigenvalues determines the type of the Jordan form of A.

The Jordan form can help determine the phase diagrams of the differential equation. In particular, it gives us information about the nature of the fixed points of the differential system.

Example 2.11. Suppose the system has real, distinct eigenvalues. Then J has the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.12}$$

Then we have the system given by

$$y_1' = \lambda_1 y_1, \quad y_2' = \lambda_2 y_2. \tag{2.13}$$

If λ_1 and $\lambda_2 \neq 0$ have the same sign, the phase diagrams have a single fixed point at the origin. Thus, the origin of the y_1, y_2-plane is a fixed point, or **node**. A **stable node** is one in which all the trajectories are oriented towards the origin, where $\lambda_1, \lambda_2 < 0$. If the trajectories are oriented away from the node, in which $\lambda_1, \lambda_2 > 0$, then the origin is said to be **unstable**.

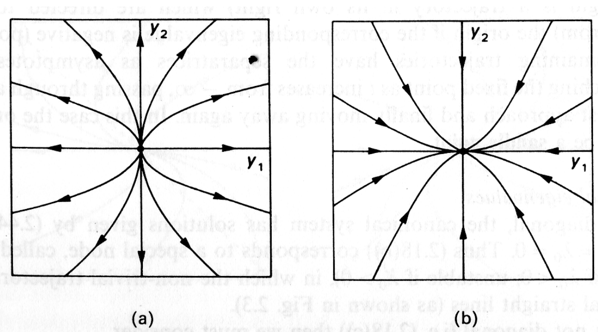

Fig. 2.1. Real distinct eigenvalues of the same sign give rise to nodes: (a) unstable ($\lambda_1 > \lambda_2 > 0$); (b) stable ($\lambda_2 < \lambda_1 < 0$).

The shape of the trajectories is determined by the ratio $\gamma = \lambda_1/\lambda_2$. If the eigenvalues have opposite signs, then we get a saddle point.

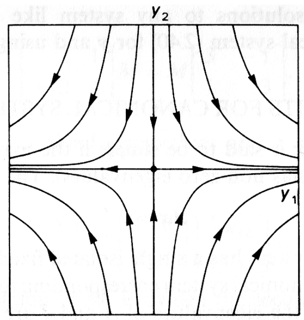

Fig. 2.2. Real eigenvalues of opposite sign ($\lambda_2 < 0 < \lambda_1$) give rise to saddle points.

Now suppose the system has equal eigenvalues. If J is diagonal, the system has solutions given by

$$y_1(t) = C_1 e^{\lambda_0 t} \quad y_2(t) = C_2 e^{\lambda_0 t}$$

so J takes on the form

$$\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}. \tag{2.14}$$

This is a special node, called a star node, in which the non-trivial trajectories are all radial straight lines.

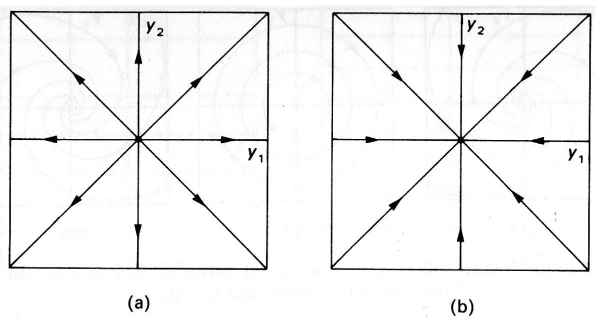

Fig. 2.3. Equal eigenvalues ($\lambda_1 = \lambda_2 = \lambda_0$) give rise to star nodes: (a) unstable; (b) stable; when A is diagonal.

If J has the form

$$\begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} \tag{2.15}$$

and so is not diagnoal, then we have to consider

$$y_1' = \lambda_0 y_1 + y_2 \quad y_2' = \lambda_0 y_2$$

which has solutions

$$y_1(t) = (C_1 + tC_2)e^{\lambda_0 t} \quad y_2(t) = C_2 e^{\lambda_0 t}.$$

In this system, equal eigenvalues indicate that the origin is an improper node which is stable if $\lambda_0 < 0$ and unstable if $\lambda > 0$. It resembles the star node in that the trajectories radiate inward (outward if unstable), but these trajectories are no longer straight, radial lines.

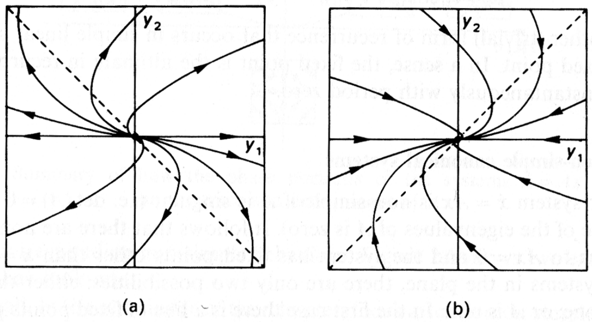

Fig. 2.4. When A is not diagonal, equal eigenvalues indicate that the origin is an improper node: (a) unstable ($\lambda_0 > 0$); (b) stable ($\lambda_0 < 0$).

Now suppose the eigenvalues are complex, so the Jordan matrix is given by

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{2.16}$$

so the system is given by

$$y_1' = \alpha y_1 - \beta y_2 \quad y_2' = \beta y_1 + \alpha y_2.$$

The solution to this system can be found by introducing plane polar coordinates such that $y_1 = r\cos\theta,\ y_2 = r\sin\theta$ and we obtain $r' = \alpha r,\ \theta' = \beta$ with solutions

$$r(t) = r_0 e^{\alpha t} \quad \theta(t) = \beta t + \theta_0.$$

If $\alpha \neq 0$, the orgin is said to be a **focus**. The phase portrait is often said to consist of an attracting or repelling spirals, depending on the sign of α. The parameter $\beta \geq 0$ determines the angular speed of the spiral. If $\alpha = 0$, then the orgin is said to be a center and the phase portrait is a continuum of concentric circles.

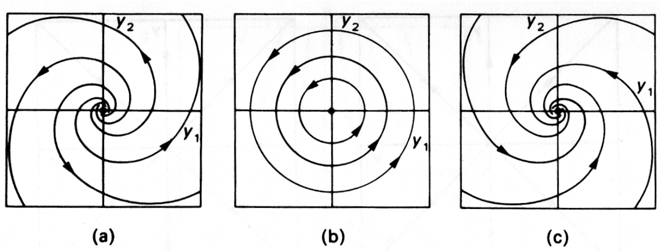

Fig. 2.5. Complex eigenvalues give rise to (a) unstable foci ($\alpha > 0$), (b) centres ($\alpha = 0$) and (c) stable foci ($\alpha < 0$).

3. Non-linear systems in the plane

Definition 3.1. (AP 3.1.1) A **neighbourhood**, N, of a point $x_0 \in \mathbb{R}^2$ is a subset of $\mathbb{R}^2$ containing a disc $\{x \mid |x - x_0| < r\}$ for some $r > 0$.

Definition 3.2. (AP 3.1.2) The part of the phase portrait of a system that occurs in a neighbourhood N of x_0 is called the **restriction** of the phase portrait to N.

These definitions allow us to examine a phase portrait in terms of both its local and its global behavior. Consider the restriction of a simple linear system to a neighbourhood N of the origin. There is a neighbourhood $N' \subseteq N$ such that the restriction of this phase portrait to N' is qualitatively equivalent to the global phase portrait of the simle linear system itself. In other words, there is a continuous bijection between N' and $\mathbb{R}^2$ which maps the phase portrait restricted to N' onto the complete phase portrait.

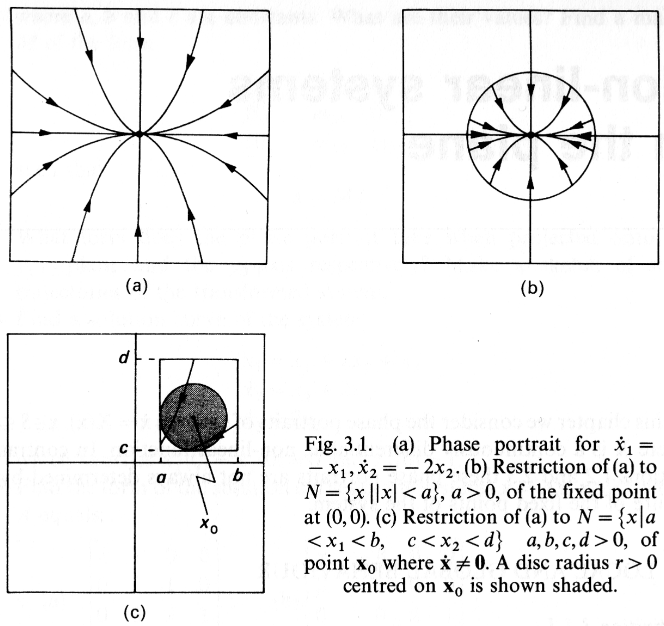

Fig. 3.1. (a) Phase portrait for $\dot{x}_1 = -x_1, \dot{x}_2 = -2x_2$. (b) Restriction of (a) to $N = \{x \,||x| < a\}$, $a > 0$, of the fixed point at $(0, 0)$. (c) Restriction of (a) to $N = \{x | a < x_1 < b, \quad c < x_2 < d\} \quad a, b, c, d > 0$, of point $\mathbf{x}_0$ where $\dot{\mathbf{x}} \neq \mathbf{0}$. A disc radius $r > 0$ centred on $\mathbf{x}_0$ is shown shaded.

However, non-linear systems can have more than one fixed point and we can often obtain the local phase portraits at all of them, but they do not always determine the global phase portrait. We begin by examining non-linear systems with a fixed point at the origin.

Definition 3.3. (AP 3.2.1) Suppose the system $y' = Y(y)$ can be written in the form:

$$y_1' = ay_1 + by_2 + g_1(y_1, y_2) \tag{3.4}$$

$$y_2' = cy_1 + dy_2 + g_2(y_1, y_2), \tag{3.5}$$

where $[g_i(y_1, y_2)/r] \to 0$ as $r = (y_1^2 + y_2^2)^{1/2} \to 0$. Essentially, this is a remainder term in the system that disappears as the system approaches 0.

The linear system

$$y_1' = ay_1 + by_2 \quad y_2' = cy_1 + dy_2 \tag{3.6}$$

is said to be the **linearization** (linearized system) of (3.4) and (3.5) at the origin. The components of this linear vector field are said to form the **linear part** of Y.

This can be applied to fixed points that are not located at the origin by simply introducing local coordinates. Suppose (ξ, η) is a fixed point of the non-linear system $x' = X(x)$, $x = (x_1, x_2)$. Then the variables

$$y_1 = x_1 - \xi \quad y_2 = x_2 - \eta \tag{3.7}$$

are a set of coordinates for the phase plane. Therefore,

$$y_i' = x_i' = X_i(y_1 + \xi, y_2 + \eta) \tag{3.8}$$

and, through this change of variables, the system has a fixed point of interest at the origin of its phase plane.

Example 3.9. We will show that the system

$$x_1' = e^{x_1+x_2} - x_2 \quad x_2' = -x_1 + x_1x_2$$

has only one fixed point and we will find the linearization of the system at this point.

The fixed points of the system must satisfy

$$\begin{aligned} e^{x_1+x_2} - x_2 &= 0, \\ -x_1 + x_1x_2 &= 0. \end{aligned}$$

The second equation is only satisfied by $x_1 = 0$ or $x_2 = 1$. However, $x_1 = 0$ does not give a real solution to the first equation, so no fixed point has $x_1 = 0$. If $x_2 = 1$, then substitution into the first equation gives us $e^{x_1+1} = 1$ which has one real solution: $x_1 = -1$. Thus, (-1,1) is the only fixed point of the system.

To find the linearized system at this fixed point, we introduce local coordinates $y_1 = x_1 + 1$ and $y_2 = x_2 - 1$. We find

$$y_1' = e^{y_1+y_2} - y_2 - 1 \quad y_2' = -y_2 + y_1y_2.$$

We can write this in the form given by (3.4) and (3.5) by using the power series expansion of $e^{y_1+y_2}$,

$$\begin{aligned} y_1' &= y_1 + \tfrac{(y_1+y_2)^2}{2!} + \tfrac{(y_1+y_2)^3}{3!} + \cdots \\ y_2' &= -y_2 + y_1y_2. \end{aligned}$$

So the linearization is given by

$$y_1' = y_1 \quad y_2' = -y_2.$$

This suggests a systematic way of obtaining linearizations. We can obtain these linearlizations by using Taylor expansions. If the component functions $X_i(x_1, x_2)$ for $i = 1, 2$ are continuously differentiable in some neighbourhood of the point (ξ, η) then for each i,

$$X_i(x_1, x_2) = X_i(\xi, \eta) + (x_1 - \xi)\frac{\partial X_i}{\partial x_1}(\xi, \eta) + (x_2 - \eta)\frac{\partial X_i}{\partial x_2}(\xi, \eta) + R_i(x_1, x_2), \tag{3.10}$$

where $R_i(x_1, x_2)$ must satisfy

$$\lim_{r\to\infty}[R_i(x_1, x_2)/r] = 0$$

where $r = \left\{(x_1 - \xi)^2 + (x_2 - \eta)^2\right\}^{1/2}$. If (ξ, η) is a fixed point, then $X_i(\xi, \eta) = 0$ and we obtain

$$\begin{aligned} y_1' &= y_1\tfrac{\partial X_1}{\partial x_1}(\xi, \eta) + y_2\tfrac{\partial X_1}{\partial x_2}(\xi, \eta) + R_1(y_1 + \xi, y_2 + \eta), \\ y_2' &= y_2\tfrac{\partial X_2}{\partial x_1}(\xi, \eta) + y_2\tfrac{\partial X_2}{\partial x_2}(\xi, \eta) + R_1(y_1 + \xi, y_2 + \eta). \end{aligned} \tag{3.11}$$

Therefore, the linearization at (ξ, η) is given by

$$a = \tfrac{\partial X_1}{\partial x_1} \quad b = \tfrac{\partial X_1}{\partial x_2} \quad c = \tfrac{\partial X_2}{\partial x_1} \quad d = \tfrac{\partial X_2}{\partial x_2} \tag{3.12}$$

evaluated at (ξ, η). Thus, in matrix form, the linearization is $y' = Ay$ where

$$A = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{bmatrix}.$$

3.1. **The Linearization Theorem.**

Definition 3.13. (AP 3.3.1) A fixed point at the origin of a non-linear system on a plane is said to be **simple** if its linearized system has a single solution, $x = 0$, to $Ax = 0$.

Theorem 3.14. *(AP 3.3.1) Let the non-linear system*

$$y' = Y(y)$$

have a simple fixed point at $y = 0$. Then, in a neighbourhood of the origin the phase portraits of the system and its linearization are qualitatively equivalent provided the linearized system is not a center.

Thus, provided the eigenvalues of the linearized system have a non-zero real part, the phase portraits of the non-linear system and its linearization are qualitatively equivalent in the neighbourhood of the fixed point. Such fixed points are said to be **hyperbolic**.

A fixed point on a non-linear system is said to be **non-simple** if the corresponding linearized system is non-simple. These linear systems contain a straight line, or possible a whole plane, of fixed points. Therefore, the local phase portrait is now determined by non-linear terms, so there are now infinitely many different types of local phase portraits.

3.2. **Stability.**

Definition 3.15. (AP 3.5.1) A fixed point x_0 of the system $x' = X(x)$ is said to be **stable** if, for every neighbourhood N of x_0, there is a smaller neighbourhood $N' \subseteq X$ of x_0 such that every trajectory that passes through N' remains in N as t increases.

Definition 3.16. (AP 3.5.2) A fixed point x_0 of the system $x' = X(x)$ is said to be **asympotically stable** if it is stable and there is a neighbourhood N of x_0 such that every trajectory passing through N approaches x_0 as t approaches infinity.

Definition 3.17. (AP 3.5.3) A fixed point x_0 of the system $x' = X(x)$ is said to be **neutrally stable** if it is stable but not asymptotically stable.

Definition 3.18. (AP 3.5.4) A fixed point which is not stable is said to be **unstable**.

This means that there is a neighbourhood N of the fixed point such that for every neighbourhood $N' \subseteq N$ there is at least one trajectory which passes through N' and does not remain in N. For example, consider a saddle point.

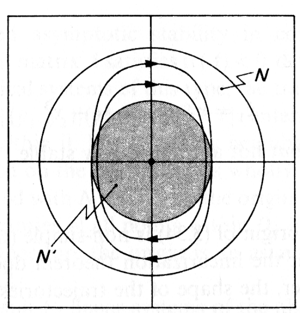

Fig. 3.2 Typical neighbourhoods N and N' (shaded) or Definition 3.15. Observe all trajectories passing through N' remain in N.

In order to determine the stability type of any given fixed point, we can find a Liapunov function for the system. In order to develop this idea, we need the following definitions.

Definition 3.19. **(AP 3.5.5)** A real-valued function $V : N \subseteq \mathbb{R}^2 \to \mathbb{R}$, where N is a neighbourhood of $0 \in \mathbb{R}^2$ is said to be **positive (respectively, negative) definite** in N if $V(x) > 0$ (respectively, $V(x) < 0$) for $x \in N \setminus \{0\}$ and $V(0) = 0$.

Definition 3.20. **(AP 3.5.6)** A real-valued function $V : N \subseteq \mathbb{R}^2 \to \mathbb{R}$, where N is a neighbourhood of $0 \in \mathbb{R}^2$ is said to be **positive (respectively, negative) semi-definite** in N if $V(x) \leq 0$ (respectively, $V(x) \geq 0$) for $x \in N \setminus \{0\}$ and $V(0) = 0$.

Definition 3.21. **(AP 3.5.7)** The derivative of $V : N \subseteq \mathbb{R}^2 \to \mathbb{R}$ along a parameterized curve given by $x(t) = (x_1(t), x_2(t))$ is defined by

$$\frac{d}{dt}V(x(t)) = \frac{\partial V(x(t))}{x_1}x_1'(t) + \frac{\partial V(x(t))}{x_2}x_2'(t).$$

With these definitions, we can now formulate a theorem that will allow us to determine the type of stability of a fixed point.

Theorem 3.22. ***(AP 3.5.1)*** *Suppose the system $x' = X(x)$, $x \in S \subseteq \mathbb{R}^2$ has a fixed point at the origin. If there exists a real-valued function V in a neighbourhood N of the origin such that*

(1) the partial derivatives $\frac{\partial V(x(t))}{\partial x_1}$ and $\frac{\partial V(x(t))}{\partial x_2}$ exist and are continuous,
(2) V is positive definite, and
(3) V' is negative semi-definite along solution curves,

then the origin is a stable fixed point of the system. If (3) is replaced by the stronger condition (4) V' is negative definite, then the origin is an asymptotically stable fixed point.

Proof. Properties 1 and 2 imply that the level curves of V form a continuum of closed curves around the origin. Thus, there is a positive k such that $N_1 = \{x \mid V(x) < k\}$ is a neighbourhood of the origin contained in N. If $x_0 \in N_1 \setminus \{0\}$, then $V(\phi_t(x_0)) \leq 0$ for all $t \geq 0$ by (3) and $V(\phi_t(x_0))$ is a non-increasing function of t. Therefore, $V(\phi_t(x_0)) < k$ for all $t \geq 0$, and so $\phi_t(x_0) \in N_1$ for all $t \geq 0$. Consequently, by Definition 3.15, the fixed point is stable.

If (3) is replaced by condition (4) then we obtain the asymptotic stability by the following argument. The function $V(\phi_t(x_0))$ is strictly decreasing in t and $V(\phi_{t_2}(x_0)) - V(\phi_{t_1}(x_0)) < k$ for all $t_2 > t_1 \geq 0$. The mean value theorem gives the existence of a sequence $\{\tau_i\}_{i=1}^{\infty}$ such that $V(\phi_{\tau_i}(x_0)) \to 0$ as $\tau_i \to \infty$. This, in turn, implies that $\phi_t(x_0) \to 0$ as $\tau_i \to \infty$ because V' is negative definite. Now, $V(\phi_t(x_0)) < V(\phi_{t_i}(x_0))$ for all $t > \tau_i$ because $V(\phi_t(x_0))$ is decreasing. However, V is positive definite and therefore $\{\phi_t(x_0) \mid t > \tau_i\}$ lies inside the level curve of V containing $\phi_{\tau_i}(x_0)$. This is true for every τ_i. This argument is valid for all x_0 in N_1 and therefore $x = 0$ is an asymptotically stable fixed point. $\square$

4. Applications

4.1. **A Model of Animal Conflict.** Suppose we wish to model the conflicts that occur within a species when, for example, there is competition for mates, territory, etc. Conflict occurs when two individual confront one another and we will suppose that there are three possible actions:

(1) Display
(2) Escalation of a fight
(3) Running Away

The population to be modelled is taken to consist of individuals who respond to confrontation in one of a finite number of ways. Suppose that each individual adopts one of the stratagies given in the following table. An individual playing

Index i	Strategy	Initial Tactic	Tactic if opponent escalates
1	Hawk (H)	Escalate	Escalate
2	Dove (D)	Display	Run Away
3	Bully (B)	Escalate	Run Away

Table 1. Three different animal strategies.

strategy i against an opponent playing j receives a 'payoff' a_{ij}. This payoff is taken to be related to the individual's capability to reproduce. Assuming that only pure strategies are played and that individuals breed true, the model is able determine the evolution of the three sections of the populations.

Let x_i be the proportion of the population playing strategy i. It follows that $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0$. The payoff to an individual playing i against the rest of the population is

$$\sum_j a_{ij} x_j = (Ax)_i$$

where A is the payoff matrix. The average payoff to an individual is

$$\sum_i x_i (Ax)_i = x^T Ax.$$

The advantage of playing i is therefore

$$(Ax)_i - x^T Ax.$$

The growth rate of the section of the population playing strategy i can be assumed to be proportional to this advantage. By choosing a suitable unit of time then,

$$x_i' = x_i((Ax)_i - x^T Ax). \tag{4.1}$$

We can obtain a payoff matrix by assigning scores at each confrontation. The actual values chosen are not important—it is their signs and magnitudes that are important.

Example 4.2. We will show that the dynamical equations (4.1), with a payoff matrix A such that

$$A = \begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -3 \\ 2 & 4 & 0 \end{bmatrix}$$

has a fixed point at $Q = (x_1, x_2, x_3) = (3/5, 0, 2/5)$. We will use the function

$$V(x) = x_1^{3/5} x_3^{2/5}$$

to show that this fixed point is asymptotically stable on

$$\Delta = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1; x_1, x_2, x_3 > 0\}.$$

Proof. To check that $x = (\frac{3}{5}, 0, \frac{2}{5})$ is a fixed point, we note that $x^T Ax = \frac{6}{5}$. For $i = 1, 3$, $(Ax)_i = \frac{6}{5}$ and hence $x_1' = x_3' = 0$. When $i = 2$, $x_2' = 0$ since $x_2 = 0$. We show that the point $(\frac{3}{5}, 0, \frac{2}{5})$ is asymptotically stable by using an argument of the Liapunov type. The level surfaces $V(x_1, x_2, x_3)$ cut the $x_2 = 0$ plane in hyperbolae and are invariant under translation parallel to the x_2-axis. On Δ, the derivative of V along the trajectories is:

$$\begin{aligned} V' &= V(x)(\tfrac{3x_1'}{5x_1} + \tfrac{2x_3'}{5x_3}) \\ &= V(x)\left[(\tfrac{3}{5}, 0, \tfrac{2}{5})Ax - x^T Ax\right] \\ &= V(x)\left[(1 - x_1 - x_3)(\tfrac{11}{5} - x_1 - x_3) + 5(x_1 - \tfrac{3}{5})^2\right]. \end{aligned}$$

Hence, $V'(x)$ is positive for $x \in \Delta$ and V increases along the trajectories as t increases. ☐

4.2. **Bifurcations.** The dynamical equations of a model frequently involve time-independent quantities in addition to the dynamical variables. There are circumstances in which it is advantageous to think of a parameter as a continuous variable that is independent of time. The result is then a family of differential equations indexed by the parameter.

An analysis of a family of differential equations involves recognizing the topologically distinct types of phase portraits exhibited by its members. The parameter values at which changes of type take place are called **bifurcation points** of the family. The characteristic feature of a bifurcation point is that every neighbourhood of it in parameter space contains points giving rise to topologically distinct phase portraits.

In order to find the bifurcation points, we consider their nature. The phase portrait at a bifurcation point must be such that an arbitrarily small change in the parameters can result in qualitatively distinct behavior; in other words, it must be **structurally unstable**. It follows that any structurally unstable feature of a phase portrait can be a bifurcation point. The two main types of bifurcations are saddle-node bifurcations and Hopf bifurcations. The one we shall examine more in-depth is the saddle-node bifurcation.

4.2.1. *Saddle-node bifurcation.* Consider the one-parameter family of planar systems given by

$$x_1' = x_1, x_2' = \mu - x_2^2 \tag{4.3}$$

where μ is real. Setting $\mu = 0$ gives a fixed point at $(x_1, x_2) = (0, 0)$. For $\mu < 0$ there is no fixed point; at $\mu = 0$, a non-hyperbolic fixed point appears at the origin and; as μ increases above 0, this separates into two fixed points: a saddle and a node.

In a saddle-node bifurcation, either a single fixed point appears and separates into two fixed points which move apart, or two fixed points move together, coalesce into one, and disappear. Technically, the distinguishing feature is the nature of the non-hyperbolic fixed point that occurs at the bifurcation point. The linearized system at this fixed point must have one zero and one non-zero eigenvalue. Thus, $\det(A) = 0$ and $\operatorname{tr}(A) \neq 0$.

However, the saddle-node bifurcation is characterized by the appearance of quadratic terms in the expression for x_2' in (4.3). It is unlikely that the dynamical equations of a model will fail to contain the necessary quadratic terms. We say that the occurrence of a saddle-node bifurcation is a generic property of families of differential equations exhibiting the symptoms as described above.

Example 4.4. Consider the dynamical equations of the form

$$\begin{aligned} a' &= a\left[(K-a) + \tfrac{p}{1+p}\right] \\ p' &= -\tfrac{p}{2} + \tfrac{ap}{1+p} \end{aligned}$$

where K is a positive parameter. The fixed points other than $(0,0)$ of (4.5) lie at the intersection of the curves

$$\begin{aligned} p &= \frac{a-K}{K+1-a} \\ p &= 2a-1 \end{aligned}$$

on which, respectively, a' and p' are zero. Substitution gives

$$2a^2 - 2(K+1)a + 1 = 0$$

with solutions

$$a = \frac{1}{2}\left[(K+1) \pm \sqrt{(K+1)^2 - 2}\right].$$

These solutions are complex for $K < K^* = \sqrt{2} - 1$, so we see that there are no non-trivial fixed points when $K < K^*$, one when $K = K^*$, and two when $K > K^*$. From the solution for a, we see that the single non-trivial solution is $a = a^* = \frac{1}{\sqrt{2}}$. This gives us $p = p^* = \sqrt{2} - 1$. The linearization of the system at this fixed point has a coefficient matrix A given by

$$\begin{bmatrix} K^* - 2a + p(1+p)^{-1} & a(1+p)^{-2} \\ p(1+p)^{-1} & -\frac{1}{2} + a(1+p)^{-2} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 & 1 \\ 2K^* & -K^* \end{bmatrix}.$$

This gives $\det(A) = 0$ and $\operatorname{tr}(A) \neq 0$. Thus, this system will undergo a saddle-node bifurcation at $(a,p) = (a^*, p^*)$.

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