A COMPARISON OF THE GRAPHS OF THE CHROMATIC AND DIATONIC SCALE

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Abstract. In this paper, we use basic notions of group theory to describe two musical scales, the chromatic and diatonic scales. I will define and explain a Generalized Interval System and a Generalized Tonnetz and then compare the Generalized Tonnetz graphs of the two scales.

Contents

The chromatic scale and diatonic scale are two sets of musical notes which form a basis for much of western music. Another way to understand these structures, besides the classical music theory approach, is to represent them as groups, and then to depict them as graphs called the Generalized Tonnetze. We'll begin with definitions of the basic musical concepts involved.

1. Musical Preliminaries

Here is the set of musical terms necessary to understand the generalized Tonnetz.

Definition 1.1. The *chromatic scale* is the musical scale with twelve pitches that are a half step apart.

Definition 1.2. A *diatonic scale* is a seven-note musical scale with 5 whole steps and 2 half steps, where the half steps have the maximum separation usually 2 or 3 notes apart.

In this paper we will simply consider the major scale, which is one of the diatonic scales.

Definition 1.3. The *major scale* is a diatonic scale with the pattern of whole step, whole, half, whole, whole, whole, and half.

In this paper will also use the notions of *interval* and *triad*.

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Definition 1.4. An *interval* is the distance between two musical notes. Table 1 below gives a complete list of musical intervals and their common names.

Definition 1.5. A *triad* is a set of three notes such that the intervals between consecutive notes are thirds, either major or minor.

TABLE 1. List of Intervals

Number of Half Steps	Interval Name
0	Perfect Unison (PU)
1	Minor Second, or half-step (m2)
2	Major Second, or whole step (M2)
3	Minor Third (m3)
4	Major Third (M3)
5	Perfect Fourth (P4)
6	Tritone (TT)
7	Perfect Fifth (P5)
8	Minor Sixth (m6)
9	Major Sixth (M6)
10	Minor Seventh (m7)
11	Major Seventh (M7)
12	Perfect Octave (PO)

2. Basic Group Theory

In order for one to understand any music in terms of math one must first understand the concept of a *group*.

Definition 2.1. A *group* is a set G with an operation $\cdot : G \times G \to G$ satisfying

(1) (*associativity*) For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(2) (*identity*) There exists $e \in G$ such that for all $x \in G$, $e \cdot x = x \cdot e = x$
(3) (*inverses*) For all $x \in G$ there exists a $y \in G$ such that $x \cdot y = y \cdot x = e$ where e is the identity element.

Now we will define two groups used to understand music, by associating group elements with musical notes. The first group, $\mathbb{Z}_{12}$, gives a numerical representation of the chromatic scale.

Definition 2.2. $\mathbb{Z}_{12}$ is the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ with the operation $\oplus$ defined by $x \oplus y = x + y \pmod{12}$.

Remark 2.3. For any $x \in \mathbb{Z}$, $x \pmod{12}$ is the element of $\{0, 1, 2, \ldots, 11\}$ such that $x \pmod{12} + 12k = x$ for some $k \in \mathbb{Z}$.

We use $\mathbb{Z}_{12}$ to represent the chromatic scale by associating each element to a note, pairing them so that $C = 0$ and $C\sharp = 1$ and following this pattern until $B = 11$.

Proposition 2.4. $(\mathbb{Z}_{12}, \oplus)$ *is a group.*

Proof. $(\mathbb{Z}_{12}, \oplus)$ is closed by Remark 2.3. Associativity holds because it holds in $\mathbb{Z}$. Then we know there is an identity which is 0. Take any $x \in \mathbb{Z}_{12}$ $0 + x = x + 0 = x$ because the same holds for the integers. Finally we can prove that each element in $\mathbb{Z}_{12}$ has a inverse. Take 4 for instance. We want some $x \in \mathbb{Z}_{12}$ and 4 to sum together to be 0. According to modular arithmetic if we add $4+8 = 8+4 = 12 \equiv 0 \pmod{12}$. Thus we know that 8 is the inverse of 4 and similarly we can find any inverse of any element such that the inverse is in $\mathbb{Z}_{12}$. Thus we have that $(\mathbb{Z}_{12}, +)$ is a group. □

More generally we can define $\mathbb{Z}_n$ for any $n \in \mathbb{Z}$.

Definition 2.5. $\mathbb{Z}_n$ is the set $\{0, 1, \ldots, n-1\}$ with the operation $\oplus$ defined by $x \oplus y = x + y \pmod{n}$.

Proposition 2.6. *$(\mathbb{Z}_n, \oplus)$ is a group.*

Proof. Same proof as with $(\mathbb{Z}_{12}, \oplus)$. □

In this paper we will use the group $(\mathbb{Z}_7, \oplus)$ to represent the diatonic scale. For concreteness, we shall use the C-major scale, with notes C, D, E, F, G, A, B as our diatonic scale where $0 = C, 1 = D$, etc. until $6 = B$.

3. The Generalized Interval System

In order to understand the generalized Tonnetz one must understand the generalized interval system.

Definition 3.1. A *generalized interval system* is defined by an ordered triple (S, G, int). S is a set called the musical space of the system. G is a group called the group of intervals. int is a function $int : S \times S \to G$. This is called the interval function and must satisfy the following

1. For all $x, y, z \in S$, $int(x, y) \cdot int(y, z) = int(x, z)$
2. For all $x \in S$ and all $g \in G$, there exists a unique $y \in S$ such that $int(x, y) = g$. [1]

Definition 3.2. A generalized interval system (S, G, int) is called *commutative* if the group of intervals G is commutative.

Note that we make no additional assumptions about the interval function.

Example 3.3. Consider the chromatic GIS given by $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int)$ where int is defined such that for all $x, y \in \mathbb{Z}_{12}$ $int(x, y) = y - x \bmod 12$. This is a GIS because int satisfies the two necessary properties.

$$\begin{aligned} int(x, y) + int(y, z) &= (y - x) + (z - y) \\ &= y - x + z - y \\ &= z - x \\ &= int(x, z) \end{aligned}$$

(all operations above being mod 12). For any $x \in \mathbb{Z}_{12}$ and $g \in \mathbb{Z}_{12}$ there exists one unique $y \in \mathbb{Z}_{12}$ such that $y - x = g$. Thus there exists only one $y \in \mathbb{Z}_{12}$ such that $int(x, y) = g$ based on the definition of int. Therefore we can concluded that $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int)$ is a commutative generalized interval system.

Example 3.4. We can also define the diatonic GIS $(\mathbb{Z}_7, \mathbb{Z}_7, int)$ with $int(x,y) = y - x \bmod 7$. The function int satisfies the two properties because.

$$\begin{aligned} int(x,y) + int(y,z) &= (y-x) + (z-y) \\ &= y - x + z - y \\ &= z - x \\ &= int(x,z). \end{aligned}$$

(all operations above being mod 7). Also for any $x \in \mathbb{Z}_7$ and $g \in \mathbb{Z}_7$ there exists one unique $y \in \mathbb{Z}_7$ such that $y - x = g$. Thus there exists only one $y \in \mathbb{Z}_7$ such that $int(x,y) = g$ by the definition of int. Thus $(\mathbb{Z}_7, \mathbb{Z}_7, int)$ is a commutative generalized interval system.

Remark 3.5. The function int defined in the GIS for $\mathbb{Z}_{12}$ is the chromatic interval, that is to say, $int(x,y)$ gives the number of half steps between the notes x and y, when we identify notes with elements of $\mathbb{Z}_{12}$. However, the function int for the GIS of $\mathbb{Z}_7$ does not represent the chromatic intervals, but the number of scale steps between two notes.

4. The Generalized Tonnetz

In this section we will introduce the generalized Tonnetz.

Definition 4.1. Let $G_0 = (S, G, int^*)$ be a commutative GIS and assume that the group G is generated by a finite subset B. A *generalized Tonnetz* $T(G;B)$ is the directed graph (S, A, B, int) where A is the complete inverse image of B under int^* and int is the restriction of int^* to A. The group G is the set of vertices of this graph, and two vertices x and y are connected by an edge directed from x to y whenever $int(x,y)$ is an element of B.[2]

The complete inverse image of B under $int^* : S \times S \to G$ is the set of all order pairs of vertices (x,y) such that their image $int^*(x,y)$ is in B. Next are two examples of generalized Tonnetze.

Example 4.2. The first example of a GIS is based on the chromatic scale. In this example the GIS is $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int^*)$. Define the generators as the set $\{3,4,7\}$. The set $\mathbb{Z}_{12}$ is generated by the set $\{3,4,7\}$ because any number can be written as a sum of multiples of $3, 4$ and 7 mod 12. The graph has the set of vertices which is the set $\mathbb{Z}_{12}$ and the set of the edges listed in Table 2. The graph of the generalized Tonnetz of the chromatic scale is embedded on the torus or the donut and not the plane.

For a visual representation see Figure 1.

The main idea of this paper is the graph or generalized Tonnetz of the diatonic scale.

Example 4.3. Start with the GIS $(\mathbb{Z}_7, \mathbb{Z}_7, int^*)$. Define the generators of the tonnetz from this GIS as the set $\{2,4\}$. The set $\mathbb{Z}_7$ is generated by the set $\{2,4\}$ because if we start with the 0 element and add the generator 2 repeatedly we will obtain all the different elements of the set. The set of vertices is $\mathbb{Z}_7$ and the set of edges is as follows.

See Figure 2 for a better image of the graph.

TABLE 2. Set of edges for $T(\mathbb{Z}_{12}, \{3, 4, 7\})$

(0, 3)	(0, 4)	(0,7)
(1, 4)	(1, 5)	(1, 8)
(2, 5)	(2, 6)	(2, 9)
(3, 6)	(3,7)	(3, 10)
(4, 7)	(4, 8)	(4, 11)
(5, 8)	(5, 9)	(5, 0)
(6, 9)	(6, 10)	(6, 1)
(7, 10)	(7, 11)	(7, 2)
(8, 11)	(8, 0)	(8, 3)
(9, 0)	(9, 1)	(9, 4)
(10, 1)	(10, 2)	(10, 5)
(11, 2)	(11, 3)	(11, 6)

Tonnetz of Chromatic Scale.pdf

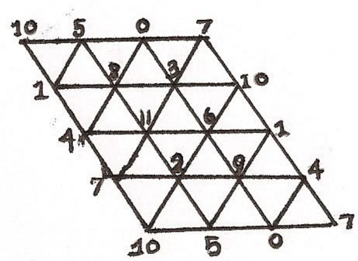

FIGURE 1. The lattice structure graph of the chromatic scale

This graph becomes a band known as the *harmonic band*. The *harmonic band* is a graph such that if we pick any two edges that share a vertex one can find an edge that contains the other two vertices making a triangle which is a type of triad in musical terms.[3] The graph of the generalized Tonnetz of the diatonic scale can be embedded on the *Möbius strip* and not the plane nor the torus.

To see how this generalized Tonnetz of the diatonic scale embed on the *Möbius strip* see Figure 3.

Table 3. The set of edges for $T(\mathbb{Z}_7, \{2, 4\})$

(0, 2)	(0, 4)
(1, 3)	(1, 5)
(2, 4)	(2, 6)
(3, 5)	(3, 0)
(4, 6)	(4, 1)
(5, 0)	(5, 2)
(6, 1)	(6, 3)

Tonnetz of the Diatonic Scale.pdf

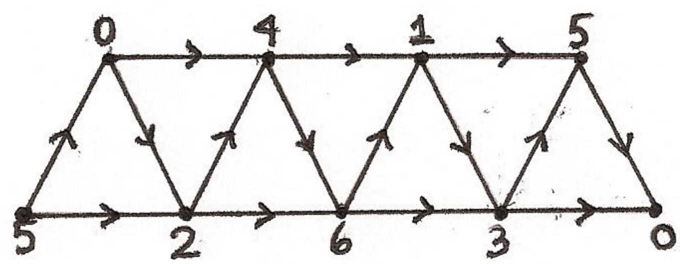

Figure 2. The lattice structure graph of the diatonic scale.

5. Comparison of the Chromatic Scale and Diatonic Scale

The reason that the graph of the chromatic scale generalized Tonnetz embeds on a torus is based on the fact that each vertex has degree 6, which means that the graph is $6-regular$. On the other hand the graph of the diatonic scale generalized Tonnetz embeds on a *Möbius strip* because each vertex is of degree 4, or the graph is $4-regular$. That is why the graph based on the diatonic scale does not embed on the same space as the chromatic scale. The differences in these graphs indicate the differences in musical structure of the two scales.

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Strip.pdf

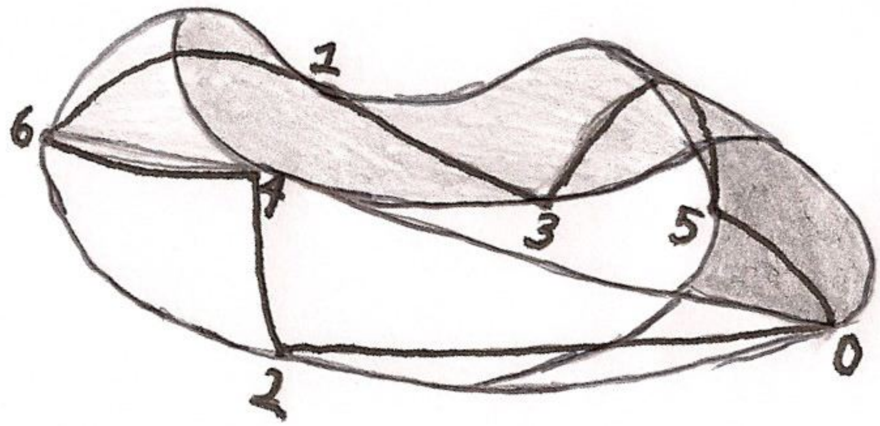

FIGURE 3. A Great Picture of the Mobius Strip!

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