EQUITABLE PARTITIONS AND ORBIT PARTITIONS

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Abstract. We consider two kinds of partition of a graph, namely orbit partitions and equitable partitions. Although an orbit partition is always an equitable partition, the converse is not true in general. We look at some classes of graphs for which the converse is true.

Contents

1. Some Graph Theoretic Terms

In this section, we collect some graph theoretic terms and fix notations. Let $X = (V, E)$ be an undirected graph with a vertex set $V = V(X)$ and an edge set $E = E(X) \subseteq \binom{V}{2}$, where $\binom{V}{2}$ is the set of subsets of V of size 2. An edge from a vertex to itself is called a *loop*. We say that X is *simple* if it has no loops or multiple edges. We say that u and v are *adjacent* and write $u \sim v$ if there is an edge $\{u, v\} \in E$. If $u \sim v$, we refer to v as a *neighbor* of u and vice versa. The number of neighbors of a vertex v is called the *valency* or the *degree* of v and is denoted by $d(v)$. If every vertex of X has the same valency k, X is said to be *k-regular*.

One important property of a graph is its connectivity, which is related to walks, paths and cycles. For two vertices $u, v \in V$, a *uv-walk* is a sequence of vertices and edges $ue_1x_1e_2x_2 \cdots x_{n-1}e_nv$ where $x_i \in V$ and $e_i \in E$ is an edge from the preceding vertex to the next in the sequence. The vertices x_i's are called *internal vertices*. For a uv-walk, the internal vertices need not be distinct. When internal vertices are distinct, we refer to a uv-walk as a *uv-path*. The *distance* between u and v is the number of edges in the shortest uv-path. If there are k internally disjoint paths between any pair of vertices $u, v \in V$, we say that X is *k-connected*. A path from u to u is called a *cycle*, and the shortest cycle of a graph is called its *girth*.

In an oriented graph X, one instead has an edge set $E \subseteq V \times V$ consisting of ordered pairs (u, v). If $e = (u, v)$ is a directed edge, then we say that u is a *tail* of e and v is a *head* of e.

A graph Y with $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$ is called a *subgraph* of X. Given a set $S \subseteq V(X)$, the *subgraph induced by* S is the subgraph of X with vertex set S and those edges of X whose endpoints lie in S.

An *automorphism* of a graph X is a bijection $\varphi : V \longrightarrow V$ with the property that $u \sim v$ if and only if $\varphi(u) \sim \varphi(v)$. A graph is called *vertex-transitive* if for any $u, v \in V$, there is an automorphism which maps u to v. Note that a vertex-transitive graph is necessarily regular, but a regular graph is not necessarily vertex-transitive.

The *adjacency matrix* A of X is a $|V| \times |V|$ matrix whose rows and columns are indexed by V such that $A_{uv} = 1$ if vertex u is adjacent to v and $A_{uv} = 0$ otherwise. We refer to the characteristic polynomial of A as the characteristic polynomial of X, and similarly for eigenvalues and eigenvectors of A. The set of eigenvalues of X is also called the *spectrum* of X.

2. Orbit Partitions and Equitable Partitions

In general, a graph can be very complicated. To analyze its structure, it is often helpful to partition the graph into more manageable pieces. While there are many different ways to partition a graph, we will focus on two particular partitions, namely orbit partitions and equitable partitions.

Throughout this paper, by a partition of a graph X, we always mean the partition of its vertex set. If π partitions the vertex set of X into $C_1, \ldots, C_n$, then we refer to C_i as a *cell* of π.

Definition 2.1. Let X be a graph. If $H \leq \mathrm{Aut}(X)$ is a group of automorphisms of X, we say that u and v are *similar* under H if there is an automorphism in H which maps u to v. The equivalence classes defined by this similarity are called the *orbits* of the graph by H. The partition of X consisting of the set of orbits by H is called an *orbit partition* of X.

Hence, an orbit partition of a graph is a partition in which cells are orbits. Roughly speaking, the orbit partition groups together those vertices that look the same. Since automorphisms preserve valency, all vertices in a cell have the same valency. Also, if a graph G has an orbit partition with only one cell, then G is vertex-transitive.

We now define and develop the basic properties of equitable partition.

Definition 2.2. A partition π of V with cells $C_1, \ldots, C_r$ is *equitable* if the number of neighbors in C_j of a vertex $v \in C_i$ depends only on the choice of C_i and C_j. In this case, the number of neighbors in C_j of any vertex in C_i is denoted b_{ij}.

Some examples of equitable partitions are shown in Fig. 1. In the figure, the partition is indicated by the vertex coloring. Notice that the partitions in Fig. 1 are also orbit partitions.

2.1. **Combinatorial Results.**
Since each vertex in C_i has precisely b_{ii} neighbors in C_i, the subgraph induced by C_i is regular. However, slightly more is true.

Proposition 2.3. *Let π be an equitable partition with cells $C_1, \ldots, C_r$. Then every vertex in C_i has the same valency.*

Proof. Suppose $v, w \in C_i$. Then, since the partition is equitable, the number of neighbors in C_j of any vertex in C_i is b_{ij}. Hence, we have $d(v) = \sum_{j=1}^{r} b_{ij} = d(w)$. $\square$

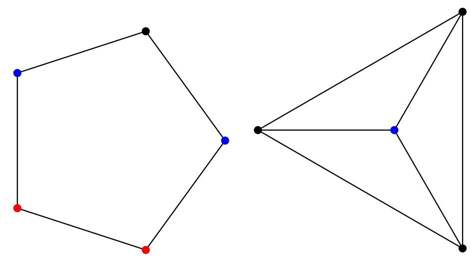

FIGURE 1. Two equitable partitions.

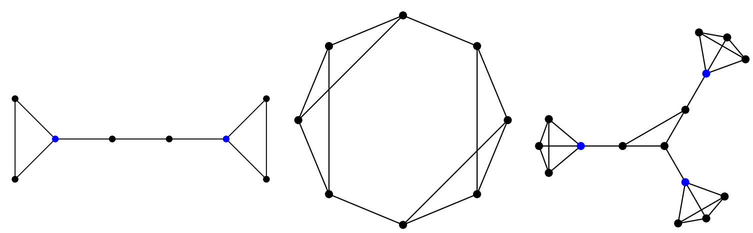

FIGURE 2. Equitable, but not orbit partitions.

So like an orbit partition, an equitable partition groups together vertices of the same valency.

Proposition 2.4. *An orbit partition is an equitable partition.*

Proof. Let $O_1, \dots, O_r$ be an orbit partition of X. Suppose $u, v \in O_i$. Then there is an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(u) = v$. Since φ maps O_j to O_j and preserves valency, u and v must have same number of neighbors in C_j. $\square$

The graphs in Fig. 2 show that the converse is false in general. The first graph in Fig. 2 is called McKay's graph and is probably the most popular graph of this type in the literature. Note that the black cell is not an orbit: no automorphism takes an outer black vertex to an inner black vertex, since automorphisms must preserve cycles.

The second graph is 3-regular, so the trivial partition is an equitable partition. However, since it is not vertex-transitive, the trivial partition is not an orbit. The non-vertex-transitive, regular graphs provide a class of examples of graphs with an equitable partition that is not an orbit partition.

The third graph is a generalization of McKay's graph. The reader can check that the partition is equitable, but nor orbit.

We now look at some consequences of having a special kind of equitable partition. A family of k internally disjoint paths from x to distinct vertices in $P \subseteq V$ is called a *k-fan* from x to P.

Proposition 2.5. *Let X be a connected graph. If X has an equitable partition π consisting of cells $\{u\}$ and $P = V \backslash \{u\}$, then X is a regular graph P with a $|P|$-fan from u to P.*

If in addition, π is the only nondiscrete equitable partition of X and $Aut(X)$ is nontrivial, then P is vertex-transitive.

Proof. Since the partition is equitable, P is regular. Since X is connected, u is not an isolated point. Since X is simple and the partition is equitable, each $v \in V \backslash \{u\}$ has exactly one edge to u. Hence, we have a $|P|$-fan from u to P.

To show the second assertion, suppose that π is the only nondiscrete equitable partition. Then since $\mathrm{Aut}(X)$ is nontrivial, the orbits under $\mathrm{Aut}(X)$ form a nondiscrete equitable partition, which must be π by assumption. Hence, P is an orbit and, in particular, vertex-transitive. □

Proposition 2.6. *Let X be connected and suppose $u \neq v$. Let π_1 be the partition with cells $\{u\}$, $V \backslash \{u\}$ and π_2 be the partition with cells $\{v\}$, $V \backslash \{v\}$. If π_1 and π_2 are both equitable partitions of X, then X is complete.*

Proof. Since X is connected, u is adjacent to some vertex in $V \backslash \{u\}$. Since X is simple and π_1 is equitable, there is an edge from each $w \in V \backslash \{u\}$ to u. Thus the valency of u is $|X| - 1$. Similarly, there is an edge from each $z \in V \backslash \{v\}$ to v. So the valency of v is also $|X| - 1$. Moreover, since $u \in V \backslash \{v\}$ and π_2 is equitable, each vertex in $V \backslash \{v\}$ also has degree $|X| - 1$. Hence, X is complete. □

Since complete graphs have the above partition, this proposition characterizes complete graphs.

2.2. **Algebraic Results.** We next look at some algebraic properties of equitable partitions. Every result here comes from Godsil's text [3].

To use linear algebra, we will encode the information about a partition in a matrix.

Definition 2.7. Given a partition π with cells $C_1, \ldots, C_n$, the *characteristic matrix* P is a $|V| \times n$ matrix with $p_{ij} = 1$ if a vertex i belongs to the cell C_i and $p_{ij} = 0$ otherwise.

The j^{th} column of P is called the *characteristic vector* for C_j. It has nonzero entry at the i^{th} position whenever the i^{th} vertex belongs to C_j. Note that $P^t P$ is a diagonal matrix with j^{th} diagonal entry equal to the number of vertices in the cell C_j. Since the diagonal entries are nonzero, $P^t P$ is invertible.

As in the case for groups, we can extract some information about a graph by studying its quotient.

Definition 2.8. Given an equitable partition π, the directed graph with vertices $C_1, \ldots, C_r$ with b_{ij} arcs from C_i to C_j is called the *quotient* of X over π and is denoted X/π.

The adjacency matrices for X and its quotient X/π relate to each other in the following way.

Proposition 2.9. *Let π be an equitable partition and let P be the corresponding characteristic matrix. Then $A(X)P = PA(X/\pi)$. Hence, we have $A(X/\pi) = (P^t P)^{-1} P^t A(X)P$.*

Proof. Suppose without loss of generality that $u \in C_i$ and consider $(A(X)P)_{uj}$. Note that a summand in $(A(X)P)_{uj} = \sum_{k=1} A(X)_{uk} P_{kj}$ is 1 if the vertex u is adjacent to a vertex in the cell C_j and is 0 otherwise. So $(A(X)P)_{uj}$ counts the number of neighbors of the vertex u in C_j, and this number is b_{ij}.

Likewise, since u belongs to exactly one of the cells, namely C_i, the only nonzero entry in the row u of P is at the i^{th} column. Thus, we have $(PA(X/\pi))_{uj} = b_{ij}$, so $A(X)P = PA(X/\pi)$.

By the previous part, we have $P^tA(X)P = P^tPA(X/\pi)$. Since P^tP is invertible by the remark above, we have $A(X/\pi) = (P^tP)^{-1}P^tA(X)P$. □

From the above proposition, we see that the equitable partition can be characterized by the following property of its characteristic matrix P.

Corollary 2.10. *A partition π is equitable if and only if the column space of the characteristic matrix P is invariant under $A(X)$.*

Proof. Note that the column space of P is invariant under $A(X)$ if and only if there is some B such that $A(X)P = PB$. □

Here we see that the spectrum of the quotient X/π partially determines the spectrum of X.

Theorem 2.11. *Let π be an equitable partition and let P be the corresponding characteristic matrix. The characteristic polynomial of X/π divides the characteristic polynomial of X.*

In particular, θ is an eigenvalue of X/π with multiplicity n, then θ is also an eigenvalue of X with multiplicity $\geq n$. Moreover, the following also holds.

Proposition 2.12. *Let π be an equitable partition and let P be the corresponding characteristic matrix. If v is an eigenvector with eigenvalue θ for X/π, then Pv is an eigenvector with eigenvalue θ for X.*

Proof. We have $\theta Pv = P\theta v = PA(X/\pi)v = A(X)Pv$. Since P^tP is invertible and $v \neq 0$, $Pv \neq 0$. So Pv is an eigenvector of X with eigenvalue θ. □

For orbit partitions, we have a partial converse for the Theorem 2.11.

Theorem 2.13. *Let X be a vertex-transitive graph. Let π be an orbit partition of some nontrivial subgroup of $Aut(G)$. If π has a singleton cell $\{u\}$, then every eigenvalue of X is an eigenvalue of X/π.*

Hence, if a vertex-transitive graph has a non-identity automorphism with a fixed point, then we can completely determine the spectrum for X from its quotient X/π.

In the next two sections, we investigate the following problem.

Problem 2.14. *Characterize those graphs with the property that every equitable partition is an orbit partition.*

3. Sufficient Conditions

One known sufficient condition for every equitable partition to be an orbit partition is compactness. The definition of a compact graph is rather involved. We first define some terms.

A square matrix $S = (s_{ij})$ is *doubly-stochastic* if each entry is nonnegative and $S\mathbf{1} = \mathbf{1} = S^t\mathbf{1}$ where $\mathbf{1}$ is the vector whose entries are 1. Note that the set of doubly stochastic matrices forms a convex set in $\mathbb{R}^{n^2}$. If V is a closed, convex set, a point $p \in V$ is called an *extreme point* of V if $p \neq \frac{x+y}{2}$ for any $x, y \in V$.

Definition 3.1. Let A be the adjacency matrix of a graph X. Define $\mathcal{S}(A)$ to be the set of all doubly stochastic matrices that commute with A. If every extreme point of $\mathcal{S}(A)$ is a permutation matrix, then the graph X is called *compact*.

Note that $\mathcal{S}(A)$ is a convex set and contains all of the permutation matrices that commute with A, i.e., all automorphisms of X. Hence, if X is compact, then the automorphisms of X are precisely the extreme points of $\mathcal{S}(A)$.

One of the motivations for looking at compact graphs is that the characterization of automorphisms as the extreme points of a convex set allows us to find the automorphisms in polynomial time. However, we are interested in compact graphs because all of their equitable partitions are orbit partitions.

First, recall the following results from linear algebra.

Theorem 3.2 (König, Birkhoff)**.** *The permutation matrices are the extreme points of the set of doubly stochastic matrices.*

Theorem 3.3 (Carathéodory)**.** *Let K be a nonempty, closed, bounded, convex set in a vector space X with $\dim X = n$. Then every point of K can be represented as a convex combination of at most $n+1$ extreme points of K.*

Hence, any doubly stochastic matrix can be written as a convex combination of permutation matrices. We will also need a following technical lemma.

Lemma 3.4. *The partition π is equitable if and only if $Q = P(P^tP)^{-1}P^t \in \mathcal{S}(A)$, where P is the characteristic matrix of π.*

Proposition 3.5. *If X is compact, then every equitable partition is an orbit partition.*

Proof. Suppose X is compact. Then, by Lemma 3.4, $Q = P(P^tP)^{-1}P \in \mathcal{S}(A)$. So by compactness, we have $Q = \sum a_k R_k$, where $a_k \geq 0$ with $\sum a_k = 1$ and R_k are the permutation matrices that define automorphisms of X.

Now, consider the entries of $Q = P(P^tP)^{-1}P^t$. Since P^tP is a diagonal matrix whose j^{th} diagonal entry is equal to the number of vertices in the cell C_j, the diagonal entries of P^tP, and therefore of $(P^tP)^{-1}$ are in particular positive.

Hence, the ij-entry of $Q = P(P^tP)^{-1}P^t$ is nonzero if and only if vertex i and j are in the same cell. To see this, note that the ij-entry of PP^t is 1 if and only if vertex i and j are in the same cell and is zero otherwise. By the above remark, inserting the factor $(P^tP)^{-1}$ does not change which entries of PP^t are nonzero.

But since $Q = \sum a_k R_k$, the ij-entry of Q is nonzero if and only if there is a permutation matrix in $\mathcal{S}(A)$ whose ij-entry is nonzero, i.e., if and only if there is an automorphism of A taking vertex i to vertex j. Hence, vertex i and vertex j are in the same cell if and only if there is an automorphism of A taking vertex i to vertex j. Thus, every cell of π is an orbit. □

It is not clear from the above definition what kinds of graphs are compact. In fact, finding an alternative characterization of compact graphs is still an open problem. Here, we content ourselves by giving some examples.

Proposition 3.6. *The complement of a compact graph is compact.*

Proof. Let X be a graph with adjacency matrix A. Then the complement $\overline{X}$ has the adjacency matrix $J - I - A$ where J is the square matrix with all entries equal to 1. Since any doubly stochastic matrix that commutes with A commutes with $J - I - A$, we have $\mathcal{S}(A) = \mathcal{S}(J - I - A)$. So $\overline{X}$ is compact. □

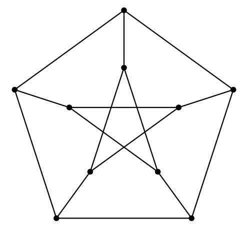

FIGURE 3. The Petersen Graph.

We next show that complete graphs are compact.

Proposition 3.7. *The complete graphs on n vertices are compact.*

Proof. Note that if J is the square matrix with all entries equal to 1 and S is a stochastic matrix, then $JS = J = SJ$. Since the complete graph has adjacency matrix $J - I$ and every doubly stochastic matrix commutes with $J - I$, we see that $\mathcal{S}(J - I)$ is precisely the set of all doubly stochastic matrices. Hence, by Theorem 3.2, the complete graph is compact. □

We note that Tinhofer showed in [7] that trees and cycles are also compact. The proofs are considerably harder than the proofs for complete graphs and for complements of compact graphs. Currently, no other graph theoretic property is known to imply compactness. However, if we require a graph to be regular, we obtain the following.

Proposition 3.8. *A compact, connected, regular graph is vertex-transitive.*

To prove the proposition, we need the following lemma by Hoffman. The proof can be found on p. 15 of [1].

Lemma 3.9 (Hoffman)**.** *Let X be a graph on n vertices with adjacency matrix A. Then, the matrix $\frac{1}{n}J$, where J is the all 1 matrix, is in $\mathcal{S}(A)$ if and only if X is connected and regular.*

Proof of Proposition 3.8. Let X be compact, connected and regular. Then $J \in \mathcal{S}(A)$. Since X is compact, we have $J = \sum a_k P_k$ where $a_k \geq 0$ with $\sum a_k = 1$ and P_k is a permutation matrix that commutes with A by Theorem 3.3. Now, since the ij-entry of J is 1, there must be at least one matrix in the sum with nonzero ij-entry. Hence, there must be at least one permutation matrix that commutes with A whose ij-entry is 1, i.e., there is an automorphism taking vertex i to j. Hence, X is vertex-transitive. □

Note that the converse is false. Yet again, the Petersen graph is a counterexample. Here is an alternate proof of the previous proposition.

Proposition 3.10. *If X is regular and every equitable partition is an orbit partition, then X is vertex-transitive. In particular, a compact, conncted, regular graph is vertex-transitive.*

Proof. If X is regular and every equitable partition is an orbit partition, then the trivial partition is an orbit partition. Hence, X is vertex-transitive. A compact graph satisfies the hypotheses by Proposition 3.5. □

It turns out that Proposition 3.8 can be strengthened. A compact, connected, regular graph is in fact *generously* vertex-transitive, i.e., for every pair of vertices v, w, there is an automorphism φ such that $\varphi(v) = w$ and $\varphi(w) = v$. The proof of this fact is rather lengthy, but can be found in either [4] or [5]. However, the following weaker result is easy to prove.

Proposition 3.11. *Let X be a compact, regular graph. If $v \sim w$, then there is an automorphism φ such that $\varphi(v) = w$.*

Proof. Suppose X is compact and k-regular. Then $\frac{1}{k}A$ is doubly stochastic and commutes with A, so $\frac{1}{k}A \in \mathcal{S}(A)$. Then by compactness, we have $\frac{1}{k}A = \sum a_i P_i$. So whenever the ij-entry of A is nonzero, there must be a permutation matrix that commutes with A whose ij-entry is nonzero. Hence, we have an automorphism taking vertex i to j. □

We note that Wang and Li give some characterizations of 3-regular, compact graphs in [8]. In the same paper, Wang showed that if G is compact and regular, then $G - v$ is also compact for any v. Here, $G - v$ is the graph with vertex v and all the edges incident to v removed.

Though much weaker, the following is a sort of reverse of Wang's result.

Proposition 3.12. *Suppose P is l-regular and compact, and let X be a graph consisting of P together with a k-fan from u to P. If $k, k+1 \neq l$, then every equitable partition of X is an orbit partition.*

Proof. Suppose π is an equitable partition of X with cells $C_1, \ldots, C_n$. By Proposition 2.3, we must have $C_1 = \{u\}$, say. Since u must be a fixed point of every automorphism of X, the cells $C_2, \ldots, C_n$ forms an equitable partition of P. Since P is compact, $C_2, \ldots, C_n$ are orbits by Proposition 3.5. Hence, $C_1, \ldots, C_n$ forms an orbit partition of X. □

It is not known whether the graph described is actually compact.

We end with the following quesiton: what can we say about a graph in which every equitable partition is an orbit partition? No published results are known. This condition is fairly weak. It is satisfied by any graph whose only equitable partition is the discrete partition. It is satisfied by any compact graphs, including trees and cycles, and the complement of any compact graph.

Acknowledgments. It is a pleasure to thank my mentors Spencer Dowdall and Ilya Gekhtman for providing guidance and reading the drafts of this paper. We also thank Peter May for organizing the REU programs.

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