THE FUNDAMENTAL GROUP

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Abstract. The focus of this exposition is the fundamental group of a topological space. The paper begins by defining paths and homotopy, and proceeds to construct the fundamental group. Later, attention turns to the consequences of the construction, in particular to the fundamental group of S^1 and Van Kampen's Theorem.

Contents

1. Preliminaries: Concatenation and Homotopy of Paths

Definition 1.1. Let X be a topological space and I be the unit interval $[0, 1]$. A *path* f in X is a continuous map $f : I \to X$. $f(0)$ and $f(1)$ are the *endpoints* of f. When we write $f(t)$ with t in I, we call t the *parameter* of f.

Remark 1.2. Paths are our building blocks, and we want to build a group out of them. So, naturally, we would like to talk about actions and operations on paths that will aid in constructing the elements and operation of the group. This section introduces two such actions on paths. The first one is concatenation, defined below:

Definition 1.3. Let $P(X)$ be the set of all paths in the space X and take f, $g \in P(X)$ such that $f(1) = g(0)$. Then we define:

$$(f \bullet g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} < t \leq 1 \end{cases}$$

Remark 1.4. The above defines the concatenation map $\bullet : P(X) \times P(X) \to P(X)$, which will underly the group operation constructed later.

Remark 1.5. The second useful action to consider is a "smooth deformation", or *homotopy*, of one path into another path. To begin to understand this, consider the following. Take our space X as the empty space in front of you, and imagine a shoelace sitting in X. Secure a vice grip to each end of the lace to keep the ends from moving. A homotopy of the shoelace is an act of physically moving the

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remaining unsecured of the lace over some time frame (allowing the shoelace to pass through itself if necessary). This concept is made rigorous in the definition below. Think of f and g as generalizations of two (possibly different) spatial positions of the same shoelace.

Definition 1.6. Let $f, g : I \to X$ be paths. Then, f is *homotopic* to g if:

1. $f(0) = g(0)$, $f(1) = g(1)$.
2. There exists a family of paths $h_t : I \to X$, $t \in I$, such that:
   (a) $h_0 = f$
   (b) $h_1 = g$
   (c) $h_t(0) = f(0)$ and $h_t(1) = f(1)$ for all t in I.
   (d) The associated map $h(t, s) = h_t(s)$ is continuous.

In this case, the collection h_t is called a *homotopy of paths*, or *homotopy* for short, and we write $f \simeq g$.

Remark 1.7. As you can see, a homotopy can be thought of as a family of paths $h_t : I \to X$, or a single map $h : I \times I \to X$. Both notions are equivalent and useful.

Example 1.8. Let $X = \mathbb{R}^n$ and take paths $f, g : I \to \mathbb{R}^n$ with $f(0) = g(0)$, $f(1) = g(1)$. Then we can consider the *linear homotopy* $h(t, s) = t \cdot g(s) + (1-t) \cdot f(s)$. Note that h satisfies Definition 1.6.

Remark 1.9. Given $f \in P(X)$, it is natural to think about the set of all paths homotopic to f. In fact, this is a useful concept, as the following two propositions show:

Proposition 1.10. *The homotopy relation $\simeq$ in Definition 1.6 is an equivalence relation.*

Proof. We take paths f, g and $r \in P(X)$ to show that $\simeq$ satisfies the three properties of an equivalence relation:

1. Reflexivity: $f \simeq f$ by the identity homotopy $h(t, s) = f(s)$ for all $t \in I$.
2. Symmetry: If $f \simeq g$, then we have continuous $h(t, s)$ satisfying $h(0, s) = f(s)$, $h(1, s) = g(s)$, $h(t, 0) = f(0)$, $h(t, 1) = f(1)$. Define $\bar{h}(t, s) = h(1 - t, s)$. Then $\bar{h}$ is continuous because it is the composition of continuous maps. Also, $\bar{h}(0, s) = h(1, s) = g(s)$, $\bar{h}(1, s) = h(0, s) = f(s)$, $\bar{h}(t, 0) = f(0) = g(0)$, $\bar{h}(t, 1) = f(1) = g(1)$. Hence, we have a homotopy $g \simeq f$.
3. Transitivity: Suppose $f \simeq g$ and $g \simeq r$. Then, we have homotopies h_1 from f to g and h_2 from g to r satisfying Definition 1.6 for their respective paths. Define:

$$H(t, s) = \begin{cases} h_1(2t, s) & 0 \leq t \leq \frac{1}{2} \\ h_2(2t - 1, s) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

   We want H to be a homotopy from f to r. H inherits the essential properties listed in Definition 1.6, including continuity, from h_1 and h_2. Hence, H is a homotopy from f to r, and $f \simeq r$.

□

Definition 1.11. For $f \in P(X)$, the *homotopy class* $[f]$ of f is the equivalence class of f under the equivalence relation $\simeq$.

Remark 1.12. By the symmetry of $\simeq$, we can now say "f and g are homotopic" instead of the one-sided statement, "f is homotopic to g".

Definition 1.13. Let $A \subset X$. A *deformation retraction* is a homotopy $f_t : X \to X$, $t \in I$, such that:

(1) $f_0 = id_X$, the identity map on X.
(2) $f_1(X) = A$
(3) $f_t|A = id_X$ for all $t \in I$

2. Loops and the Fundamental Group

Definition 2.1. For a path $f : I \to X$, f is a *loop* if $f(0) = f(1)$. We denote by $\pi_1(X, x_0)$ the set of all homotopy classes $[f]$ of loops f with basepoint $x_0 = f(0)$.

Now, we are ready to construct the fundamental group.

Theorem 2.2. *$\pi_1(X, x_0)$ is a group under the operation $[f] \cdot [g] \equiv [f \bullet g]$ for loops $f,\ g \in \pi_1(X, x_0)$.*

Proof. We must check that $\pi_1(X, x_0)$ obeys the group axioms:

(1) The operation $\cdot$ on $[f]$ and $[g]$ is a well-defined internal law of composition: $[f \bullet g] \in \pi_1(X, x_0)$.
(2) $[f] \cdot ([g] \cdot [r]) = ([f] \cdot [g]) \cdot [r]$: This amounts to taking representatives f, g, and r from $[f]$, $[g]$, and $[r]$ and showing that the loops $f \bullet (g \bullet r)$ and $(f \bullet g) \bullet r$ are homotopic. Let's write out their equations:

$$f \bullet (g \bullet r)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(4s-2) & \frac{1}{2} \le s \le \frac{3}{4} \\ r(4s-3) & \frac{3}{4} \le s \le 1 \end{cases}$$

$$(f \bullet g) \bullet r(s) = \begin{cases} f(4s) & 0 \le s \le \frac{1}{4} \\ g(4s-2) & \frac{1}{4} \le s \le \frac{1}{2} \\ r(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Our homotopy from $f \bullet (g \bullet r)$ to $(f \bullet g) \bullet r$, then, is the following:

$$h(t,s) = \begin{cases} f((2s)t + (4s)(1-t)) & 0 \le s \le (\frac{t}{2} + \frac{1-t}{4}) \\ g(4s-2) & (\frac{t}{2} + \frac{1-t}{4}) \le s \le (\frac{3t}{4} + \frac{1-t}{2}) \\ r((4s-3)t + (2s-1)(1-t)) & (\frac{3t}{4} + \frac{(1-t)}{2}) \le s \le 1 \end{cases}$$

Since h satisfies Definition 1.6, we indeed have associativity.

(3) There exists $id \in \pi_1(X, x_0)$ such that $[f] \cdot id = id \cdot [f] = [f]$: Let $id_{x_0}(s) = f(1) = x_0$ for all $s \in I$. Then, $[id_{x_0}] \in \pi_1(X, x_0)$. $[f] \cdot [id_{x_0}] = [f \bullet id_{x_0}]$. Now, $f \bullet id_{x_0} \simeq f$ via the homotopy,

$$h(t,s) = \begin{cases} f((2-t)s) & 0 \le t \le \frac{(1+t)}{2} \\ id_{x_0}(s) & \frac{1+t}{2} \le t \le 1 \end{cases}$$

Showing that $id_{x_0} \bullet f \simeq f$ is similar. Hence, $[id_{x_0}] = id$ is the identity in $\pi_1(X, x_0)$.

(4) For all $[f] \in \pi_1(X, x_0)$ there exists $[f]^{-1}$ such that $[f] \cdot [f]^{-1} = [f]^{-1} \cdot [f] = id$: Take a representative of $[f]$ and call it f. Let $f^{-1}(s) = f(1-s)$ for all $s \in I$. We need to show that $f \bullet f^{-1} \simeq id_{x_0}$ for some constant loop $id_{x_0} \in id$. For this, we take $x_0 = f(0)$ and use the homotopy:

$$h(t,s) = \begin{cases} f(s) & 0 \leq s \leq \frac{1-t}{2} \\ f(\frac{1-t}{2}) & \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ f^{-1}(s) & \frac{1+t}{2} \leq s \leq 1 \end{cases}$$

This shows that $f \bullet f^{-1} \simeq id_{x_0}$, which means $f \cdot f^{-1} \simeq id_{x_0}$. Showing $f^{-1} \cdot f \simeq id_{x_0}$ is similar. Hence, we can take $[f^{-1}]$ as our $[f]^{-1}$.

□

Remark 2.3. We would like to talk about $\pi_1(X)$ independently of a choice of basepoint x_0. We can do this because of the following:

Proposition 2.4. *Let $f \in \pi_1(X, x_1)$. Let $h \in P(X)$, $h(0) = x_0$, $h(1) = x_1$. Then, the map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $\beta_h([f]) = [h \bullet f \bullet h^{-1}]$ is an isomorphism.*

Proof. We must check the following:

(1) β_h is well-defined: f is a loop with basepoint x_1. So by the definition of h, $h \bullet f \bullet h^{-1}$ is well-defined and a loop with basepoint x_0. Hence, β_h is well-defined.
(2) β_h is a homomorphism: Let $g \in \pi_1(X, x_1)$. Then, $\beta_h([f] \cdot [g]) = \beta_h([f \bullet g]) = [h \bullet f \bullet g \bullet h^{-1}] = [h \bullet f \bullet h^{-1} \bullet h \bullet g \bullet h^{-1}] = [h \bullet f \bullet h^{-1}] \cdot [\bullet h \bullet g \bullet h^{-1}] = \beta_h([f]) \cdot \beta_h([g])$.
(3) β_h is bijective: This amounts to producing an inverse of β_h. Our choice is $\beta_{h^{-1}}$ because $\beta_h \beta_{h^{-1}}([f]) = \beta_h([h^{-1} \bullet f \bullet h]) = [h \bullet h^{-1} \bullet f \bullet h \bullet h^{-1}] = [f]$. Showing that $\beta_{h^{-1}} \beta_h([f]) = [f]$ is similar.

□

Definition 2.5. If $\pi_1(X) = 0$, we say that the fundamental group of X is *trivial.* If in addition X is path-connected, then we say that X is *simply-connected.*

Proposition 2.6. *If the spaces X and Y are path-connected, then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Proof. Define $f : I \to X \times Y$. f is continuous iff the associated coordinate maps $g : Z \to X$ and $h : Z \to Y$ defined by $f(s) = ((g(s), h(s))$ are continuous. Also, the basepoint of f is (x_0, y_0) iff g and h have basepoints of x_0 and y_0, respectively. Hence, f is a loop iff g and h are loops. In fact, for every loop f in $X \times Y$ we get a unique pair of loops g in X and $h \in Y$, and vice versa. Similarly, there exists one and only one homotopy f_t of loops in $X \times Y$ for every pair of loop homotopies g_t in X and h_t in Y, and vice versa. Thus, we have a bijection between $\pi_1(X \times Y, (x_0, y_0)$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. Next, note that the map $[f] \mapsto ([g], [h])$ is a homomorphism. Hence, this map is an isomorphism. □

Proposition 2.7. *If X is path-connected, then $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.*

Proof. First, assume $\pi_1(X)$ is abelian. Consider a loop γ with basepoint x_0. Suppose we have two distinct paths $f,\ g : I \to X$ such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = a \in X \backslash \{x_0\}$. Then, $\beta_f[\gamma] = [f^{-1} \cdot \gamma \cdot f]$ and $\beta_g[\gamma] = [g^{-1} \cdot \gamma \cdot g]$. Hence,

$$[f] \cdot \beta_f[\gamma] \cdot [f^{-1}] = [f] \cdot [f^{-1} \cdot \gamma \cdot f] \cdot [f^{-1}] = [f \cdot f^{-1} \cdot \gamma \cdot f \cdot f^{-1}] = [\gamma]$$
$$[g] \cdot \beta_g[\gamma] \cdot [g^{-1}] = [g] \cdot [g^{-1} \cdot \gamma \cdot g] \cdot [g^{-1}] = [g \cdot g^{-1} \cdot \gamma \cdot g \cdot g^{-1}] = [\gamma]$$

Thus,

$$[f] \cdot \beta_f[\gamma] \cdot [f^{-1}] = [g] \cdot \beta_g[\gamma] \cdot [g^{-1}]$$
$$\beta_f[\gamma] = [f^{-1} \cdot g] \cdot \beta_g[\gamma] \cdot [g^{-1} \cdot f]$$

But $f^{-1} \cdot g$ and $g^{-1} \cdot f$ are loops because f and g share the same starting and ending points. Hence, by the hypothesis,

$$\beta_f[\gamma] = [f^{-1} \cdot g] \cdot \beta_g[\gamma] \cdot [g^{-1} \cdot f] = \beta_g[\gamma] \cdot [f^{-1} \cdot g] \cdot [g^{-1} \cdot f] = \beta_g[\gamma]$$

This proves that for any two basepoint-change homomorphisms β_f and β_g, we have $\beta_f = \beta_g$ if the corresponding paths $f$and g share endpoints. Hence, if $\pi_1(X)$ is abelian, β_h depends only on the endpoints of h.

Next, assume each basepoint-change homomorphism β_h depends only on the endpoints of the corresponding path h. Take two arbitrary nonconstant loops, $f$and g, which share the same basepoint x_0. Decompose f into $f_1 : [0, 1/2] \to X$, $f_2 : [1/2, 1]$ such that $f_1 \bullet f_2 = f$. Decompose g into analogous paths g_1 and g_2. By the assumption that $\beta_{f_1}[g] = \beta_{f_2}[g]$, we have:

$$[f_1^{-1} \cdot g \cdot f_1] = [f_2 \cdot g \cdot f_2^{-1}]$$

Multiplying by $[f_1]$ on the left and by $[f_2]$ on the right:

$$[f_1] \cdot [f_1^{-1} \cdot g \cdot f_1] \cdot [f_2] = [f_1] \cdot [f_2 \cdot g \cdot f_2^{-1}] \cdot [f_2]$$

which simplifies down to:

$$[g \cdot f_1 \cdot f_2] = [f_1 \cdot f_2 \cdot g]$$

and hence:

$$[g] \cdot [f] = [f] \cdot [g]$$

Thus, $\pi_1(X)$ is abelian.

□

Remark 2.8. We would like to describe $\pi_1(S^1)$, where S^1 is the unit circle in $\mathbb{R}^2$. This will help us describe more complicated fundamental groups. To describe $\pi_1(S^1)$, we will navigate back and forth between three sets: I, $\mathbb{R}$, and S^1. Specifically, will consider loops from I to S^1, and factor them through $\mathbb{R}$ to count the number of times they wind around S^1. This results in the following commutative diagram for a loop f:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
I \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & S^1
\end{tikzcd}$$

Here, $p(s) = (\cos(2\pi s), \sin(2\pi s))$. The map $\widetilde{f}$ is called a *lift* of f. We will state without proof:

Theorem 2.9. *Let X be a path-connected space. For each $F : X \times I \to S^1$ and lift $\widetilde{f} : X \times \{0\} \to \mathbb{R}$ of $F|X \times \{0\}$, there is a unique lift $\widetilde{F} : X \times I \to \mathbb{R}$ of F such that $\widetilde{F}|X \times \{0\} = \widetilde{f}$.*

Corollary 2.10. *Given a path $f \colon I \to S^1$ with $f(0) = x_0 \in S^1$ and a chosen point $\widetilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\widetilde{f} : I \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{x}_0$.*

Proof. Take X to be a point x in the statement of Theorem 2.9. Then the theorem reads: For each $f : \{x\} \times I \to S^1$ and lift $\widetilde{f} : \{x\} \times \{0\} \to \mathbb{R}$ of $f(\{x\} \times \{0\})$, there is a unique lift $\widetilde{f} : \{x\} \times I \to \mathbb{R}$ of f such that $\widetilde{f}(\{x\} \times \{0\}) = \widetilde{f}$. But we can ignore $\{x\}$ here, and the claim immediately follows. □

Corollary 2.11. *For each homotopy $f_t : I \to S^1$ with all $f_t(0) = x_0$, and each $\widetilde{x_0} \in p^{-1}(x_0)$, there exists a unique "lifted homotopy" $\widetilde{f}_t : I \to \mathbb{R}$ of paths with each $\widetilde{f}(0) = \widetilde{x}_0$.*

Proof. Look at f as a map from $I \times I \to S^1$. Apply Corollary 2.10 to get a unique lift $\widetilde{f} : I \times \{0\} \to \mathbb{R}$. Then, letting $X = I$ in the statement of Theorem 2.9, we have a unique lift $\widetilde{f} : I \times I \to \mathbb{R}$. $\widetilde{f}(0, \cdot)$ and $\widetilde{f}(1, \cdot)$ are paths lifting the constant path at x_0, so the uniqueness part of Corollary 2.10 requires that $\widetilde{f}(0, \cdot)$ and $\widetilde{f}(1, \cdot)$ are constant. Hence, $\widetilde{f}$ is a homotopy lifting f. □

Theorem 2.12. *$\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$.*

We would like to identify every integer n with exactly one homotopy class $[f]$ in $\pi_1(S^1)$. This n represents the "number of times" $[f]$ winds counterclockwise around S^1 (a negative n represents $|n|$ clockwise turns). This notion is made rigorous in the following proof:

Proof. We define a map $\Phi : \mathbb{Z} \to \pi_1(S^1)$, $\Phi(n) = [\omega_n]$. Here,

$$\begin{aligned} \omega_n(s) &= (\cos(2\pi n s), \sin(2\pi n s)) \\ &= p \circ \widetilde{\omega}_n(s) \end{aligned}$$

where

$$\widetilde{\omega}_n(s) = n \cdot s$$

Observe that $\widetilde{\omega}_n$ is a lift of ω_n.

This is perhaps the most concrete definition of Φ, but it will be useful to generalize: $\Phi(n) = [p \circ \widetilde{f}]$ for any path $\widetilde{f} : I \to \mathbb{R}$ from 0 to n. This is because $\widetilde{f} \simeq \widetilde{\omega}_n$ by the homotopy $(1-t)\widetilde{f} + t\widetilde{\omega}_n$, which makes $p\widetilde{f} \simeq \omega_n$ and so $\phi(n) = [\omega_n] = [p \circ \widetilde{f}]$.

Now, must show that Φ is an isomorphism by verifying the following properties:

(1) $\Phi(m+n) = \Phi(m)\cdot\Phi(n)$ for $m,\, n \in \mathbb{Z}$ (i.e., Φ is a homomorphism): Let $T_m : \mathbb{R} \to \mathbb{R}$ be defined by $T_m(s) = m + s$. Note that $\widetilde{\omega}_m \bullet (T_m \circ \widetilde{\omega}_n : I \to \mathbb{R}$ is a path from 0 to $m+n$, so $[p(\widetilde{\omega}_m \bullet (T_m \circ \widetilde{\omega}_n)] = \Phi(n+m)$. But $p(\widetilde{\omega}_m \bullet (T_m \circ \widetilde{\omega}_n) = \omega_m \bullet \omega_n$, so $[p \circ (\widetilde{\omega}_m \bullet (T_m \circ \widetilde{\omega}_n)] = [\omega_m]\cdot[\omega_n] = \Phi_m \cdot \Phi_n$. Hence, we have $\Phi(m+n) = \Phi(m)\cdot\Phi(n)$.

(2) For all loops f in S^1 there exists $n \in \mathbb{Z}$ such that $\Phi(n) = [f]$ (Φ is surjective): Take a loop $f \in \pi_1(S^1)$ with basepoint $(1,0)$. By Corollary 2.10, f has a lift $\widetilde{f}$ with $\widetilde{f}(0) = 0$. Also, $\widetilde{f}(1) = n$ for some $n \in \mathbb{Z}$ because $p \circ \widetilde{f}(1) = f(1) = (1,0)$ and $p^{-1}(1,0) = \mathbb{Z} \subset \mathbb{R}$ (otherwise, f wouldn't be a loop). Hence, by the second definition of Φ, we have $\Phi(n) = [f]$.

(3) $\Phi(m) = \Phi(n)$ implies that $m = n$ (Φ is injective): Suppose $\Phi(m) = \Phi(n)$. By the original definition of Φ, we have $\omega_m \simeq \omega_n$. Next, take a homotopy $f : I \times X \to S^1$ such that $f(0,x) = \omega_m(x)$ and $f(1,x) = \omega_n(x)$. By Corollary 2.11, f induces a homotopy $\widetilde{f} : I \times X \to \mathbb{R}$ such that $\widetilde{f}(t,0) = 0$ for all $t \in I$. $\widetilde{f}(0,\cdot)$ and $\widetilde{f}(1,\cdot)$ are unique by Corollary 2.10, and so $\widetilde{f}(0,x) = \widetilde{\omega}_m$ and $\widetilde{f}(1,x) = \widetilde{\omega}_n$. Since $\widetilde{f}$ is a homotopy, $\widetilde{f}(0,0) = \widetilde{f}(1,0)$. But $\widetilde{f}(0,0) = m$ and $\widetilde{f}(1,0) = n$, so $m = n$.

□

Proposition 2.13. $\pi_1(S^n) = 0$ *if* $n \neq 1$.

Proof. If $n = 0$ then $S^0 = \{-1, 1\}$, which makes $\pi_1(S^n) = 0$. So consider just $n > 1$. First note that $S^n \backslash \{x\}$ is homotopy equivalent to $\mathbb{R}^n$ for any point $x \in S^n$. (Just imagine pulling $S^2\backslash\{x\}$ away from where x used to be, flattening the result onto the plane, and stretching it out infinitely far to cover the plane.) Since $\mathbb{R}^n$ is simply-connected, this implies that $S^n\backslash\{x\}$ is simply-connected. So if we show that any loop f in S^n is homotopic to some loop g in $S^n\backslash\{x\}$, then f will be null-homotopic since g is in a simply-connected space.

To show $f \simeq g$ for some g, consider some point x of S^n that is not the basepoint of f. Take an open ball B with x in B. Then, $f^{-1}(B)$ is open in $[0,1]$ since f is continuous. This implies $f^{-1}(B) = \cup_{j \in J}(a_j, b_j)$ for some index set J. Since f is continuous and $\{x\}$ is compact in B, $f^{-1}(x)$ is compact in $f^{-1}(B)$. Hence, there is a finite subcover $\{(a_i, b_i) \mid i = 1, \dots, m\} \subset \{(a_j, b_j) \mid j \in J\}$ of $f^{-1}(x)$.

Note that we could have picked our (a_j, b_j) such that each $f(a_j, b_j)$ is path-connected in B and $f(a_j)$, $f(b_j)$ on the boundary of $\bar{B}$. So assume this is true for each (a_i, b_i). Define $f_i : [a_i, b_i] \to \bar{B}$ to be the path segment of f corresponding to $[a_i, b_i]$. Also define a path segment g_i from $[a_i, b_i]$ to the boundary of $\bar{B}$, and note that $f_i \simeq g_i$ and g_i does not intercept x ($x \in \bar{B}^\circ$). Form the path g by replacing all the f_i with g_i in f, and note that $f \simeq g$ and g does not cross x. □

Proposition 2.14 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial has at least one complex root.*

Proof. It suffices to prove the contrapositive: namely, that if a polynomial has no complex root, then it must be constant. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$ be our nonconstant polynomial. Since p has no complex root, we can define for

$s \in [0,1]$:

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|} \qquad r \geq 0$$

For each real $r \geq 0$, f_r is a loop with basepoint $f_r(1) = 1$ so that the collection of all f_r is a homotopy of loops in S^1. And since f_0 is the constant loop, we have reached the first important checkpoint of this proof: that $[f_r] = 0$ for all r.

Fix $r > \max\{|a_0| + \cdots + |a_{n-1}|, 1\}$. Then, for a complex number z with norm $|z| = r$,

$$\begin{aligned}
|z^n| &= r^n \\
&= r \cdot |z^{n-1}| \\
&> (|a_0| + \cdots + |a_{n-1}|) \cdot |z^{n-1}| \\
&> |a_0| + |a_1| \cdot |z| + \cdots + |a_{n-1}| \cdot |z^{n-1}| \qquad \text{(because } |z| = r > 1) \\
&\geq |a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|
\end{aligned}$$

So $|z^n| > |p(z)|$, which means that the polynomial $p_t(z) = z^n + t(a_0 + a_1 z + \cdots + a_{n-1}z^{n-1})$ has no roots z satisfying $|z| = r$ when $0 \leq t \leq 1$. Now, we can define:

$$g_t(s) = \frac{p_t(re^{2\pi is})/p(r)}{|p_t(re^{2\pi is})/p(r)|},\ 0 \leq t \leq 1$$

which is a homotopy from $\omega_n = e^{2\pi ins}$ to f_r. Hence, we have second checkpoint of the proof: $[f_r] = [\omega_n]$.

Combining the first and second checkpoints, we have $[\omega_n] = [f_r] = 0$. But since $[\omega_n] \in \pi_1(S^1) \simeq \mathbb{Z}$, we have that $[\omega_n] = 0$ iff $n = 0$. Hence, $n = 0$ and our original polynomial p is constant.

$\square$

3. Free Groups and Van Kampen's Theorem

Definition 3.1. Let S be a nonempty set. A *word* of S is a formal string formed by formally concatenating a countable number of elements of S. The *empty word* e is also considered a word, and we write $we = ew = w$ for all words w of S.

Example 3.2. Let S = $\{s_1,\ s_2,\ s_3\}$. Then some examples of words of S are s_1, $s_3 s_2$, and $s_1 s_2 s_3$.

Definition 3.3. Let G_1, G_2 be groups. Then their free product $G_1 * G_2$ is the set of all words formed by formally concatenating a finite number of elements of $G_1 \cup G_2$, and then performing the usual group operations of G_1 and G_2 to adjacent pairs of elements belonging to the same group. The free product forms a group under concatenation with the empty word as the identity element.

Definition 3.4. A *presentation* of a group G is an isomorphism $G \cong \langle S\rangle/\langle f_1, \ldots, f_m\rangle$ of G with the free group generated by S modulo the normal subgroup $\langle f_1, \ldots, f_m\rangle$ generated by a finite collection of words $f_1, \ldots, f_m$ formed from S. Note that concatenation in the free group $\langle S\rangle$ descends to the group operation on G in the quotient. We think of the presentation as giving a way to multiply elements of G

freely, subject to the relations $f_1 = \cdots = f_m = 0$. We write $P_G = \langle s_1, \ldots, s_n \mid f_1, \ldots, f_m \rangle$.

Definition 3.5. For elements a and b of a free group F, we say that the word $[a, b] = aba^{-1}b^{-1}$ is the *commutator* of a and b. The *commutator subgroup* $[F, F]$ of F is the subgroup of F generated by all the commutators of F. The *abelianization* F^{Ab} of F is the abelian group $F/[F, F]$.

Remark 3.6. Commutators are a measure of the "abelian-ness" of a group: $[a, b]$ is the identity for all a, b iff the group is abelian. Hence, F^{Ab} is abelian.

Remark 3.7. Now that we know $\pi_1(S^n)$, we can describe the fundamental groups of spaces that look like multiple S^n's "glued together". Van Kampen's Theorem, stated below but not proven (See [1] for the proof), is useful for this purpose.

Theorem 3.8 (Van Kampen)**.**

(1) *Suppose $X = \cup_\alpha A_\alpha$ and that for all α:*
   (a) *A_α is open.*
   (b) *A_α is path-connected*
   (c) *basepoint x_0 is in A_α*
   Then, the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.

(2) *Now suppose we also have that each three-way intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected. Also, define the following homomorphisms:*

$j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ *induced by the inclusion map,* $A_\alpha \hookrightarrow X$

$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ *induced by the inclusion map,* $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$

Then the kernel of Φ is the normal subgroup N generated by all elements in $*_\alpha \pi_1(A_\alpha)$ of the form, $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$, where $\omega \in A_\alpha \cap A_\beta$. In this case, Φ induces the isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.

Remark 3.9. Part (1) simply says that given a decomposition of X into the appropriate A_α, we can decompose any loop in X into the free product of loops in A_α. Part (2) says that if we have a nicely-behaved space X, all loops of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ are null-homotopic. Hence, we should quotient out these loops from $*_\alpha \pi_1(A_\alpha)$ to obtain a group isomorphic to $\pi_1(X)$.

Example 3.10 (Wedge Sums)**.** Let X_α be a collection of spaces with basepoints x_α. Define $\vee_\alpha = \amalg_\alpha X_\alpha / \sim$, where the equivalence relation $\sim$ regards any two basepoints equivalent and all other points pairwise distinct. Think of $\vee$ as the thing that "glues all the X_α together" at the basepoints such that the two spaces don't touch anywhere else, similar to the construction of a cell-complex. If each of x_α is the result of a deformation retract of open neighborhoods $U_\alpha \subset X_\alpha$, then X_α is the result of some deformation retract of a corresponding open neighborhood $A_\alpha = X_\alpha \vee_{\beta \neq \alpha} U_\beta$. If we intersect two or more distinct A_α, we always get $\vee_\alpha U_\alpha$. Hence, we can apply Van Kampen's Theorem to deduce that $\Phi : *_\alpha \pi_1(X_\alpha) \to \pi_1(\vee_\alpha X_\alpha)$ is an isomorphism.

One immediate application of this fact is that $\pi_1(\vee_\alpha S^1)$ is isomorphic to the free product, $*_\alpha \mathbb{Z}$. For example $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

Example 3.11 (Hawaiian Earring Group)**.** With the help of Van Kampen, we can begin to describe loops in the following space $X \subset \mathbb{R}^2$. Let C_n be the circle in $\mathbb{R}^2$ with radius $\frac{1}{n}$ and apply the axiom of choice to pick a point c_n from each C_n. Then, let $X = \amalg_{n=1}^{\infty} C_n / \sim$, where $\sim$ identifies $c_n \sim c_m$ for all $n, m \in \mathbb{N}$. Note that X is a wedge sum of countably many circles, and hence $\pi_1(X)$, dubbed the *Hawaiian Earring Group* should be countable. However, there is a contradiction:

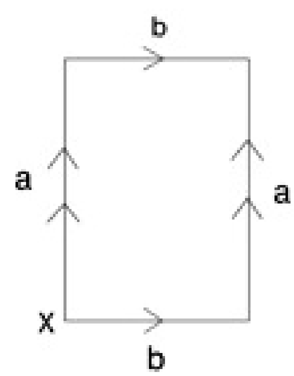
Consider the family of retractions $r_n : X \to C_n$, where $r_n | C_n$ is the identity and $r_n | C_i = [c_n]$ for all $i \neq n$. By Van Kampen's Theorem, each r_n induces a surjection $\rho_n : \pi_1(X) \to \pi_1(C_n)$, which is the same as $\rho_n : \pi_1(X) \to \mathbb{Z}$ up to isomorphism. In these maps, we define $\rho_n([c_n]) = 0$. Now, take the product of all the ρ_n's to make the homomorphism $\rho : \pi_1(X) \to \amalg_\infty \mathbb{Z}$. ρ is surjective: for every sequence $\{a_n\}$ of integers, there exists a loop $f : I \to X$ that winds a_n times around C_n on $[1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$. This implies that since $\amalg_\infty \mathbb{Z}$ is uncountable, $\pi_1(X)$ is uncountable.

Proposition 3.12. *Define the following:*

(1) X*, a path-connected space.*
(2) $\phi_\alpha : S^1 \to X$*, a collection of paths from some* x_α *back to* x_α*, each of which attaches a 2-cell* e_α^2 *to X.*
(3) γ_α*, a collection of paths from some basepoint* $x_0 \in X$ *to* x_α
(4) Y*, the space formed by attaching the the* e_α^2 *to* X *with the* ϕ_α*.*
(5) N*, the normal subgroup of* $\pi_1(X, x_0)$ *generated by the loops of the form,* $\gamma_\alpha \phi_\alpha \gamma_\alpha^{-1}$.

Then, the inclusion map $i : X \hookrightarrow Y$ *induces a surjection* $i^* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ *with kernel N. Hence,* $\pi_1(Y)$ *is isomorphic to* $\pi_1(X)/N$.

Example 3.13 (M_g Surfaces)**.** We can use a corollary Van Kampen's Theorem to describe the fundamental group of M_g, a surface of genus g. The corollary is the following: Let's start with the surface M_1, the torus. Recall that the torus can be constructed by attaching the sides labeled "a" together and the sides labeled "b" together in the following diagram. Note that attachment identifies all vertices to x so that a and b are loops.



In this way, we can view M_1 as a cell-complex and apply the above proposition: think of a and b as 1-cells attached at the vertices of the rectangle, and call the resulting space X. Then,"fill it in" by attaching a 2-cell by identifying the boundary of the boundary of its closure with the rectangle's boundary. Call the result Y. By the above proposition, $\pi_1(Y)$ is isomorphic to $\pi_1(X)/N$.

To see what this means, let a and b be paths in the directions given in the diagram above. Since $\pi_1(X) = \langle a, b\rangle$ and N is the subgroup generated by $[a, b]$, we have $\pi_1(Y) = \langle a, b \mid [a, b]\rangle = (\mathbb{Z} * \mathbb{Z})^{Ab} = \mathbb{Z}^2$.

Now, one can informally think of M_1 as a balloon with a single "donut hole" in the center. Similarly, M_g is a balloon with g "donut holes". We say that M_g is the 2-dimensional surface with genus g. With arguments similar to the above, we can show that $\pi_1(M_g)$ is isomorphic to $\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1]\cdots[a_g, b_g]\rangle$. Hence, the abelianization $\pi_1(M_g)^{Ab}$ is the direct sum of $2g$ copies of $\mathbb{Z}$. It follows that for $g \neq h$, $\pi_1(M_g)^{Ab}$ is not isomorphic to $\pi_1(M_h)^{Ab}$, so M_g and M_h are not homotopy equivalent.

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References

[1] Hatcher, Allen. Algebraic Topology. Cambridge of Chicago Press. 2002.
[2] J Cannon, G. Connor. The combinatorial structure of the Hawaiian earring group. Topology and its Applications 106 (2000), 225-271.