

RAMSEY THEORY: VAN DER WAERDEN'S THEOREM AND THE HALES-JEWETT THEOREM

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ABSTRACT. We look at the proofs of two fundamental theorems in Ramsey theory, Van der Waerden's Theorem and the Hales-Jewett Theorem. In addition, we study bounds on Van der Waerden numbers.

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1. INTRODUCTION

Ramsey theory is named after Frank Plumpton Ramsey, an English mathematician, philosopher, and economist, who worked at Cambridge and died in 1930. He intended for his original paper proving the infinite version of Ramsey's Theorem, published posthumously in 1930, to have applications to mathematical logic.

After Ramsey, many famous mathematicians worked on what we now call Ramsey theory, including Erdos. Results in Ramsey theory are tied together by what Landman and Robertson describe as, "The study of the preservation of properties under set partition," [2, p. 1]. Another way to visualize partitioning a set into k classes is to think of coloring all the elements of the set with k colors.

The result that is now known as Van der Waerden's Theorem was published in 1927 by B. L. Van der Waerden, and is a fundamental theorem in the Ramsey theory. This theorem was actually conjectured by I. Schur a few years earlier.

The Hales-Jewett Theorem is, in some sense, a more general and powerful theorem than Van der Waerden's Theorem. While Van der Waerden deals with colorings of finite sequences, the Hales-Jewett Theorem looks at colorings of arbitrarily large finite dimensional cubes. The Hales-Jewett Theorem is an important part of Ramsey theory, and many results are based on it.

In fact, Van der Waerden's Theorem can be proved as a corollary of the Hales-Jewett Theorem.

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2. VAN DER WAERDEN'S THEOREM

Definition 2.1. Let $[1, n]$ be the set $\{1, 2, 3, \dots, n\}$.

Definition 2.2. A Van der Waerden number is the least positive integer $w = w(k; r)$ such that for all $n \geq w$, every r -coloring of $[1, n]$ contains a monochromatic arithmetic progression of length k .

Theorem 2.3 (Van der Waerden's Theorem). *For every $r, k \geq 2$, $w(k; r)$ exists.*

Proof. This proof appears in Landman and Robertson [2] in Section 2.6. The overall structure of the proof is an inductive proof hinging on the concept of a refined triple, which introduces more structure via monochromatic equivalence classes of arithmetic progressions. It proceeds by induction on k and r . \square

In addition, Van der Waerden's theorem follows as a corollary of the Hales-Jewett Theorem, which we defer to the last section.

Corollary 2.4. *Any r -coloring of the positive integers must contain arbitrarily long monochromatic arithmetic progressions.*

Proof. For any integer k , there is a monochromatic arithmetic progression of length k in the subset $[1, w(k; r)]$ of the positive integers. \square

Actually, Van der Waerden's Theorem was originally stated in 1927 in the following way: if the positive integers are partitioned into two classes, then at least one of them must contain arbitrarily long arithmetic progressions. That is, a 2-coloring of the positive integers will yield arbitrarily long monochromatic arithmetic progressions in color 1 or color 2.

3. BOUNDS ON VAN DER WAERDEN NUMBERS

Although we know that Van der Waerden numbers exist, their values are only known for the first few small cases. For larger numbers, there are some bounds which are not very good.

Proposition 3.1. *For sufficiently large k , there is a lower bound on Van der Waerden numbers given by $w(k; r) > r^{k/2}$.*

This is a generalization of exercise 2.8 of [2], and we prove it by probabilistic methods. First, we randomly color the integers in $[1, n]$ using r colors. Let A denote an arbitrary arithmetic progression of length k within $[1, n]$. The probability that A is color 1 is $(1/r)^k$ and the probability that A is color 2 is the same, etc. Since there are r colors, the probability that A is monochromatic is $\frac{r}{r^k} = r^{1-k}$.

Now we can bound the probability of getting a monochromatic arithmetic progression within $[1, n]$. First we count the number of k -term arithmetic progressions in $[1, n]$.

Lemma 3.2. *There are $\frac{n^2 + O(n)}{2(k-1)}$ k -term arithmetic progressions in $[1, n]$.*

Proof. Take any arithmetic progression with initial term a_0 and common difference d . If $a_0 = 1$, then the common difference can take values $d = 1, 2, \dots, \lfloor \frac{n-1}{k-1} \rfloor$. If $a_0 = 2$, then we can have $d = 1, 2, \dots, \lfloor \frac{n-2}{k-1} \rfloor$ and so on. The largest starting point our sequence can take on is $a_0 = n - k + 1$, which corresponds to a common

difference of 1. Therefore, the total number of arithmetic progressions is $\lfloor \frac{n-1}{k-1} \rfloor + \lfloor \frac{n-2}{k-2} \rfloor + \dots + \lfloor \frac{n-(n-k+1)}{k-1} \rfloor$.

To estimate this sum, we take the related sum $S(n) = \frac{n-1}{k-1} + \frac{n-2}{k-2} + \dots + \frac{n-(n-k+1)}{k-1}$. Note that the error for each term is at most 1, so the total error is bounded by $n - k + 1$ and can be absorbed into the $O(n)$ term in our final answer. If we go ahead and calculate $S(n)$ we get

$$\begin{aligned}
 S(n) &= \frac{n(n-k+1) - \sum_{i=1}^{n-k+1} i}{k-1} \\
 &= \frac{n(n-k+1) - \frac{(n-k+1)(n-k+2)}{2}}{k-1} \\
 &= \frac{(n-k+1)(n - \frac{n-k+2}{2})}{k-1} \\
 &= \frac{1}{k-1} (n - (k-1)) \left(\frac{n}{2} + \frac{k-2}{2} \right) \\
 &= \frac{1}{k-1} \cdot \frac{n^2 + O(n)}{2} \\
 &= \frac{n^2 + O(n)}{2(k-1)}.
 \end{aligned}$$

□

Now we can calculate the probability that a k -term monochromatic arithmetic progression exists. Each of our $\frac{n^2 + O(n)}{2(k-1)}$ k -term arithmetic progressions has a r^{1-k} chance of being monochromatic. So the probability that we have a monochromatic k -term arithmetic progression in a coloring is at most

$$(3.3) \quad \sum_{i=1}^{\frac{n^2 + O(n)}{2(k-1)}} r^{1-k} = \frac{n^2 + O(n)}{2(k-1)} \cdot r^{1-k} = \frac{n^2 + O(n)}{2 \cdot r^{k-1}(k-1)}.$$

Using the definition of a Van der Waerden number, we can argue that if $\frac{n^2 + O(n)}{2 \cdot r^{k-1}(k-1)} < 1$, then $w(k; r) > n$. That is, if a r -coloring of $[1, \dots, n]$ does not guarantee us a monochromatic k -term arithmetic progression, then the Van der Waerden number must be greater than n .

Now we can see that this will lead us towards a lower bound. In particular, try $n = r^{k/2}$. We claim that for this value of n with k sufficiently large, $\frac{n^2 + O(n)}{2 \cdot r^{k-1}(k-1)} < 1$. Plugging in, we get

$$(3.4) \quad \frac{r^{(k/2) \cdot 2} + O(r^{k/2})}{2r^{k-1}(k-1)} = \frac{r}{2(k-1)} + \frac{O(r^{k/2})}{2r^{k-1}(k-1)}.$$

Taking the limit as k goes to infinity yields 0, since the limit of the each term individually is 0. So for large enough values of k , we have a lower bound $r^{k/2} < w(k; r)$.

4. A VISUAL PROOF OF THE HALES-JEWETT THEOREM

Definition 4.1. Define C_n^t , the n -cube over t elements, by

$$C_n^t = \{(x_1, x_2, \dots, x_n) \mid x_i \in [0, t-1]\}.$$

Definition 4.2. A line in C_n^t is a set of n points such that each coordinate is fixed or increases from 0 to $t-1$.

Example 4.3. When t is small, we omit parentheses and commas when writing coordinates of points. For example, if $t = 5$ and $n = 3$, $\{1402, 1412, 1422, 1432, 1442\}$ is a line.

Definition 4.4. We define a collection of $n+1$ equivalence classes on $C_{t+1}^n = [0, t]^n$ as follows: The i -th equivalence class has all the points (x_1, x_2, \dots, x_n) such that t appears only in the i rightmost coordinates.

Example 4.5. So for $C_5^2 = [0, 4]^2$, our zeroth equivalence class is C_4^2 , our first equivalence class is $\{(0, 4), (1, 4), (2, 4), (3, 4)\}$ and our second equivalence class is $\{(4, 4)\}$.

Theorem 4.6 (Hales-Jewett Theorem). *For any r, t , there exists an integer $\text{HJ}(r, t)$ such that for all $N \geq \text{HJ}(r, t)$, an r -coloring of C_t^N contains a monochromatic line.*

Proof. This proof appears in [1]. It is a two-part inductive proof that uses the idea of layering, which we now define.

Definition 4.7. The cube C_{t+1}^n is layered if each of its $n+1$ equivalence classes are monochromatic.

Definition 4.8. A k -dimensional subspace of C_{t+1}^n is a k -dimensional cube. While the subspace is not actually equal to C_{t+1}^k , because points in it have n coordinates, the points in the subspace can be identified with points in C_{t+1}^k .

Definition 4.9. A k -dimensional subspace is layered if the coloration is layered when the subspace is identified with C_{t+1}^k .

Example 4.10. A line in C_{t+1}^k is layered if the first t points of the line are monochromatic. The last point may be any color.

So the proof proceeds by induction in two parts using two statements:

- $\text{HJ}(t)$: For this value of t , any value of r will produce a number $\text{HJ}(r, t)$ such that for any $N \geq \text{HJ}(r, t)$, an r -coloring of C_t^N yields a monochromatic line.
- $\text{LHJ}(t)$: For all r, k we have a number $\text{CHJ}(r, t, k)$ such that for all $M \geq \text{CHJ}(r, t, k)$ an r -coloring of C_{t+1}^M has a layered k -dimensional subspace.

We use the base case $t = 1$. We have that $\text{HJ}(r, 1)$ exists for any r because C_1^N is a point for any N and a line in this N -dimensional cube is a point. So we would, without much effort, have a monochromatic line.

Lemma 4.11 will show that $\text{HJ}(t)$ implies $\text{LHJ}(t)$ and Lemma 4.12 will show that $\text{LHJ}(t)$ implies $\text{HJ}(t+1)$. \square

Lemma 4.11. $\text{HJ}(t) \Rightarrow \text{LHJ}(t)$

Proof. We use $k = 1$ as our base case. A 1-dimensional subspace is a line. We already know that $HJ(r, t)$ exists by the statement $HJ(t)$ so take $M > HJ(r, t)$. If we take an r -coloring of C_{t+1}^M , we will have a monochromatic line in C_t^M and thus a layered line in C_{t+1}^M .

Now, we induct on k to prove $LHJ(t)$. To go from k to $k + 1$, we assume that $LHJ(r, t, k)$ exists and show that $LHJ(r, t, k + 1)$ exists. Let $LHJ(r, t, k) = m$ and $LHJ(r^{(t+1)^m}, t, 1) = m'$ and we claim that $LHJ(r, t, k + 1) \leq m + m'$. We know that $LHJ(r^{(t+1)^m}, t, 1) = HJ(r^{(t+1)^m}, t)$ exists by assumption.

Let χ be an r -coloring on $C_{t+1}^{m+m'}$.

Define a coloring χ^* on $C_{t+1}^{m'}$ whose values are r -colorings of C_{t+1}^m , and is based on the coloring χ by

$$\chi^*(x) = \chi((x, \cdot)).$$

Since χ^* is an $r^{(t+1)^m}$ -coloring on $C_{t+1}^{m'}$, where $m' = HJ(r^{(t+1)^m}, t)$ we are guaranteed a layered line $x_0, x_1, x_2, \dots, x_t \in C_{t+1}^{m'}$. Note that the first t points are monochromatic.

We can now r -color C_{t+1}^m by χ^{**} based on how the points relate to points in our layered line by

$$\chi^{**}(y) = \chi((x_i, y)) \text{ for any } i \in [0, t - 1],$$

since x_0, x_1, \dots, x_t being layered under χ^* implies $\chi((x_0, y)) = \chi((x_1, y)) = \dots = \chi((x_{t-1}, y))$. The coloring χ^{**} has only r colors because coloring the point y under χ^{**} is equivalent to coloring the point (x_0, y) under χ and χ is an r -coloring. By our definition of m as $LHJ(r, t, k)$ we have a k -dimensional layered subspace $S \subset C_{t+1}^m$ under χ^{**} . Since S is layered, we have monochromatic equivalence classes S_0, S_1, \dots, S_k .

Now we will extend this layered k -dimensional subspace of C_{t+1}^m under χ^{**} to a $(k + 1)$ -dimensional subspace by sticking on a last point. We define T_j by concatenating the x_i from our layered line with elements in each equivalence class of S ,

$$T_j = \{(x_i, s) | 0 \leq i \leq t - 1, s \in S_j\}.$$

Finally, notice that if we take two elements (x_i, s) and $(x_{i'}, s')$ of the same equivalence class T_j then s and s' have the same color under χ^{**} so the points will be the same color under χ

$$\chi((x_i, s)) = \chi^{**}(s) = \chi^{**}(s') = \chi((x_{i'}, s')).$$

Thus we have a $k + 1$ -dimensional layered subspace with equivalence classes T_0, T_1, \dots, T_k and then take T_{k+1} to be the point beginning with x_t and ending in a string with all t 's. \square

Lemma 4.12. $LHJ(t) \Rightarrow HJ(t + 1)$

Proof. We will prove this for a given r . Since we know that $LHJ(t)$ holds, take $M > CHJ(r, t, r)$ such that an r -coloring of C_{t+1}^M yields a layered r -dimensional subspace.

Now, we need to show that an r -dimensional space, that is, C_{t+1}^r with at most r colors contains a monochromatic line.

The case of a 2 coloring of C_4^2 can be easily illustrated by drawing the three equivalence classes and noticing that two equivalence classes of the same color will create a line of five elements.

In general, we can choose the bottom left-most element of each of the $r + 1$ equivalence classes to identify the class. Under an r -coloring, we must have two equivalence classes with the same color. More specifically, we chose our $r + 1$ representative points as such

$$x_i = (x_{i1}, \dots, x_{ik}), \text{ and } x_{ij} = \begin{cases} t & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

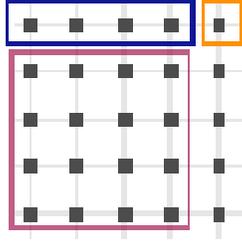
So, in our example, $x_0 = (0, 0)$, $x_1 = (4, 0)$ and $x_2 = (4, 4)$.

Then let x_u and x_v be the same color where $u < v$. We define a line y_0, \dots, y_t that will be monochromatic, by

$$y_s = (y_{s1}, y_{s2}, \dots, y_{sk}) \text{ and } y_{si} = \begin{cases} t & \text{if } i \leq u, \\ s & \text{if } u < i \leq v, \\ 0 & \text{if } u < i. \end{cases}$$

In our example, in the case where the first and second equivalence classes are monochromatic, we get the monochromatic line $(4, 0), (4, 1), (4, 2), (4, 3), (4, 4)$. \square

The diagram below illustrates the three equivalence classes of C_4^2 , the zeroth equivalence class is magenta, the first equivalence class is blue, and the second equivalence class is orange.



5. PROVING THE VAN DER WAERDEN THEOREM FROM THE HALES-JEWETT THEOREM

This proof appears on page 38 of Graham, Rothschild, and Spencer [1]. The trick to proving Van der Waerden's Theorem as a corollary of the Hales-Jewett Theorem is to be able to translate between a coloring of $[0, 1, 2, \dots, t^N - 1]$ and an n -dimensional cube with t elements.

We claim that $w(t; r) \leq t^N - 1$, where N is the number that is large enough so that an r -coloring of an N -dimensional cube on t elements must have a monochromatic line. We know that such an N must exist because of the Hales-Jewett Theorem.

Take any $a \in [0, 1, 2, \dots, t^N - 1]$ and write it as in base- t as $(a_0, a_1, \dots, a_{N-1})$, where $a = a_0 + a_1t + \dots + a_{N-1}t^{N-1}$.

We have just associated $[0, t^N - 1]$ with an N -dimensional cube of t elements, so Hales-Jewett gives us a monochromatic line of length t .

Now, based on the definition of a monochromatic line, each coordinate is either constant throughout or increasing by one each time, so the points on the monochromatic line translate directly back into a t -term arithmetic progression with difference of the sum of certain powers of t .

REFERENCES

- [1] Graham, Ronald L, Rothschild, Bruce L, and Spencer, Joel H. Ramsey Theory. John Wiley & Sons, 1980.
- [2] Landman, Bruce M. and Robertson, Aaron. Ramsey Theory on the Integers. American Mathematical Society, 2004.

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