

# CONFIGURATION SPACES

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ABSTRACT. A wide variety of spaces can be expressed as the configuration space of a linkage or of points on a circle. It is possible to build a linkage whose configuration space includes an open subset of the zero set of any real valued function. The configuration space of  $n$  labeled points on a circle is a  $n$ -torus. It is possible to make the configuration space any Euclidean manifold by identifying some of these points.

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## 1. INTRODUCTION TO LINKAGES AND THEIR CONFIGURATION SPACES

Linkages are machines made of rods connected by hinges. More abstractly, linkages are graphs with a length assigned to each edge. We can ask whether we can actually build a particular abstract linkage in the plane (for example, we cannot build a linkage including a triangle with one side longer than the sum of the other two). We classify the space of all ways to build a particular linkage (realizations), and call this space the configuration space of the linkage. We can also fix certain vertices and construct the configuration space of all realizations that maintain the position of those vertices.

**Definition 1.1.** A *graph*  $G = (\mathcal{V}(G), \mathcal{E}(G))$  is a set of vertices,  $\mathcal{V}(G)$ , and a set of edges,  $\mathcal{E}(G) \subseteq \{\{p, q\} | p \neq q \text{ and } p, q \in \mathcal{V}(G)\}$ .

**Definition 1.2.** A *linkage*  $\mathcal{L} = (L, \ell, W)$  is a graph  $L$ , a function  $\ell : \mathcal{E}(L) \rightarrow \mathbb{R}^+$ , and a set  $W \subseteq \mathcal{E}(L)$  called the marked set. In diagrams, an edge will be labeled  $e^*$  if both endpoints are marked vertices.

**Definition 1.3.** A *planar realization* of a linkage is a map  $\phi : \mathcal{V}(L) \rightarrow \mathbb{R}^2$  where

$$|\phi(v) - \phi(w)| = \ell(\{v, w\}) \quad \forall \{v, w\} \in \mathcal{E}(L).$$

We can think of a planar realization as a way of drawing a linkage in  $\mathbb{R}^2$ , respecting the lengths of the edges.

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**Definition 1.4.** The *configuration space* of a linkage  $\mathcal{L}$  is the set of all realizations of  $\mathcal{L}$  and is denoted  $C(\mathcal{L})$ .  $C(\mathcal{L})$  inherits a metric from  $\mathbb{R}^{2|V(\mathcal{L})|}$ , since each realization can be written

$$\phi = (\phi(v_1), \phi(v_2), \dots, \phi(v_n)) \in \mathbb{R}^{2|V(\mathcal{L})|}.$$

**Definition 1.5.** If  $\mathcal{L} = (L, \ell, W)$  is a linkage and  $Z : W \rightarrow \mathbb{R}^2$  specifies the image of each marked vertex, then the *relative configuration space* of  $\mathcal{L}$  with respect to  $Z$ , denoted  $C(\mathcal{L}, Z)$ , is the set of all realizations  $\phi$  where  $\phi(v) = Z(v)$  for all  $v \in W$ .

**Example 1.6.** Consider a square linkage given by the following:

$$L = (\{A, B, C, D\}, \{\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}\})$$

$$\ell(v) = 1 \quad \forall v \in \mathcal{E}(L)$$

$$W = \{A, B\}, \quad Z(A) = (0, 0), \quad \text{and} \quad Z(B) = (1, 0)$$

Here is a realization of  $(L, \ell, W)$ :

$$\phi_o : \begin{array}{ccc} D & \text{---} & C \\ | & & | \\ A & \text{---} & B \\ & e^* & \end{array}$$

Notice two *degenerate* realization types:

$$\phi_1 : \begin{array}{ccc} D & & \\ || & & \\ A^C & \text{---} & B \\ & e^* & \end{array}$$

$$\phi_2 : \begin{array}{ccc} & & C \\ & & || \\ A & \text{---} & D_B \\ & e^* & \end{array}$$

Also notice three realizations which lie exclusively on the x-axis:

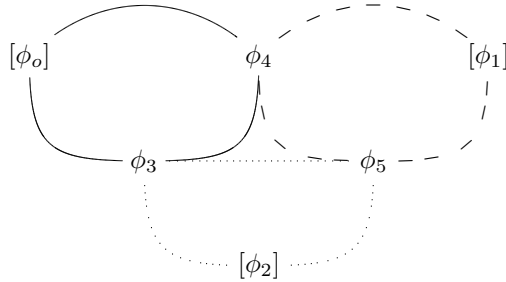
$$\phi_3 : \quad A \text{---} e^* B^D \text{---} C$$

$$\phi_4 : \quad D \text{---} A^C \text{---} e^* B$$

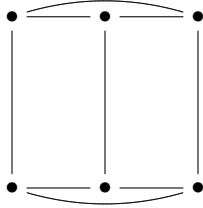
$$\phi_5 : \quad A_C \text{---} e^* B_D$$

Each realization type (the two degenerate types,  $[\phi_1]$  and  $[\phi_2]$ , and one non-degenerate type,  $[\phi_o]$ ) allows for a circle's worth of motion. There are two positions on each circle where the entire realization lies on the x-axis, permitting a switch to one other type. Thus the configuration space is three circles, joined pairwise at

points:



*Remark 1.7.* Such degenerate forms of parallelograms and the singularities they introduce (in this case  $\phi_3, \phi_4$  and  $\phi_5$ ) complicate attempts to create a linkage whose configuration space is a manifold. However, we can eliminate the singularities by constructing *rigidified* parallelograms using degenerate triangles:



## 2. FUNCTIONAL LINKAGES AND FIBER SUMS

We can treat linkages as “functional linkages” by designating some vertices as inputs and some others as outputs, so long as fixing a realization on the inputs specifies the positions of the output vertices. Functional linkages were particularly interesting to early linkage studiers who wanted to solve engineering problems, like how to convert linear motion of a piston to circular motion of a railroad wheel (Kempe, Muirhead). The fiber sum is a way of identifying vertices of two linkages to create a new linkage.

**Definition 2.1.** If  $\mathcal{L}$  is a linkage and  $S \subseteq V(L)$ , then realizations  $\phi_1, \phi_2$  are *equivalent with respect to  $S$*  (denoted  $\phi_1 \sim_S \phi_2$ ) if

$$\phi_1(s) = \phi_2(s) \quad \forall s \in S.$$

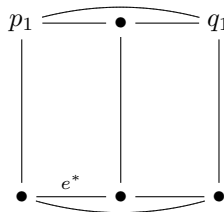
**Definition 2.2.** A *functional linkage* is a linkage with designated input vertices  $P \subseteq V(L)$  and output vertices  $Q \subseteq V(L)$  such that:

$$\phi_1 \sim_P \phi_2 \implies \phi_1 \sim_Q \phi_2 \quad \forall \phi_1, \phi_2 \in C(\mathcal{L}, Z).$$

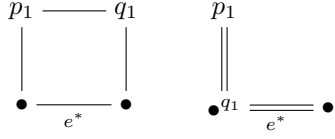
*Remark 2.3.* This allows us to associate a function  $F : O \rightarrow \mathbb{R}^{2|Q|}$  to  $\mathcal{L}$ , where the *input domain*  $O$  is given by:

$$O = \{(\phi(p_1), \phi(p_2), \dots, \phi(p_n)) \in \mathbb{R}^{2|P|} \mid \phi \in C(\mathcal{L}, Z)\}.$$

**Example 2.4.** The following rigidified square is a functional linkage:



**Example 2.5.** However, the following non-rigidified square is not, because the position of  $p_1$  does not uniquely specify the position of  $q_1$ :



**Definition 2.6.** Given two linkages  $\mathcal{L}' = (L', \ell', W')$  and  $\mathcal{L}'' = (L'', \ell'', W'')$ , with fixing functions  $Z'$  and  $Z''$  respectively, a map  $\beta : S' \subseteq \mathcal{V}(L') \rightarrow \mathcal{V}(L'')$  is *compatible* if:

$$Z'(w') = Z''(\beta(w')) \quad \forall w' \in W' \cap S',$$

and for all  $(v, u) \in (S' \times S')$  with  $\{v, u\} \in \mathcal{E}(L')$ ,

$$\{\beta(v), \beta(u)\} \in \mathcal{E}(L'') \implies \ell'(\{v, u\}) = \ell''(\{\beta(v), \beta(u)\}).$$

**Definition 2.7.** Given two linkages  $\mathcal{L}'$  and  $\mathcal{L}''$ , fixing functions  $Z'$  and  $Z''$  and a compatible map  $\beta$ , the *fiber sum* of  $\mathcal{L}'$  and  $\mathcal{L}''$  with respect to  $\beta$ ,  $\mathcal{L}' *_\beta \mathcal{L}''$ , is a linkage  $\mathcal{L} = (L, \ell, W)$  such that:

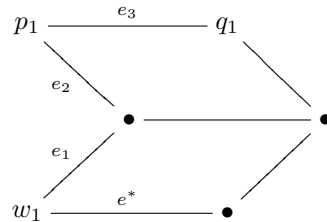
$$\begin{aligned} L &= (L' \sqcup L'') / (\beta(v) = v) \\ W &= (W' \sqcup W'') / (\beta(v) = v) \\ Z(w) &= \begin{cases} Z'(w) & w \in W' \\ Z''(w) & w \in W'' \end{cases} \quad \forall w \in W \\ \ell(\{v, w\}) &= \begin{cases} \ell'(\{v, w\}) & \{v, w\} \in \mathcal{E}(L') \\ \ell''(\{v, w\}) & \{v, w\} \in \mathcal{E}(L'') \end{cases} \end{aligned}$$

*Remark 2.8.* Note that  $\beta$  must be compatible for  $\mathcal{L}$  to be well-defined.

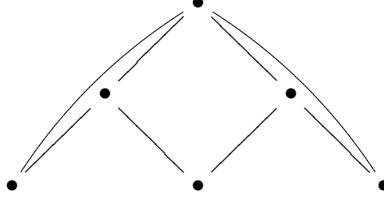
### 3. ELEMENTARY LINKAGES

This section includes constructions of elementary (translation, addition, multiplication, etc.) functional linkages. The purpose is to exhibit that with these elementary functional linkages, using the fiber sum to compose functional linkages, we can create functional linkages for any polynomial. By fixing the output at zero, we can create linkages whose configuration space contains an open subset of the zero-set of any polynomial.

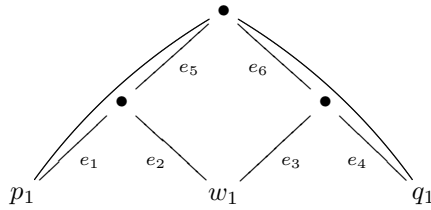
**Construction 3.1.** This *translator* takes an input value,  $\phi(p_1)$ , in the input domain  $O = \{(x, y) \mid |(x, y) - Z(w_1)| \leq \ell(e_1) + \ell(e_2)\}$  and returns  $\phi(q_1) = \phi(p_1) + \ell(e_3)$ . Alternatively, switch the roles of  $p_1$  and  $q_1$  for subtraction. Assume both parallelograms in the translator are rigidified. By adjusting the lengths of  $e_1, e_2$  and  $e_3$ , we can construct a translator with any desired input domain and translation distance.



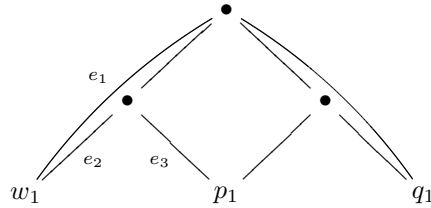
**Construction 3.2.** The *pantograph* can be used to construct several functional linkages. Assume the parallelogram is rigidified, and note that no vertices have been fixed yet.



**Example 3.3.** *Negation:* We can use the pantograph to rotate around the origin,  $Z(w_1)$ . By setting  $\ell(e_1) = \ell(e_2) = \dots = \ell(e_6)$  and treating  $\phi(q_1), \phi(p_1) \in \mathbb{C}$ , we get  $\phi(q_1) = -\phi(p_1)$ . We can adjust the input domain  $O = \{(x, y) \mid |(x, y)| \leq 2\ell(e_1)\}$  to be include anything we wish by choosing an appropriate value for  $\ell(e_1)$ .

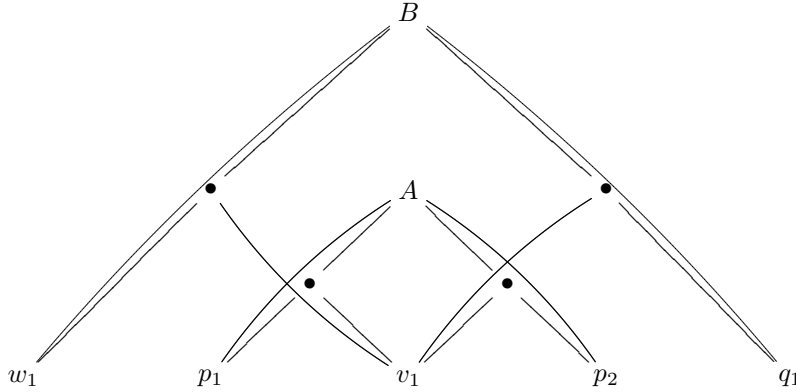


**Example 3.4.** *Scalar multiplication:* The pantograph with input value  $\phi(p_1)$  and  $Z(w_1) = (0, 0)$  returns  $\phi(q_1) = \frac{\ell(e_1)}{\ell(e_2)}\phi(p_1)$ . It is a functional linkage for scalar multiplication, and adjusting the relative and absolute lengths of  $e_1, e_2$  and  $e_3$  (assume the pantograph is symmetric) allows us to build a linkage for multiplication by any  $\lambda > 1$  with any input domain  $O = \{(x, y) \mid |\ell(e_3) - \ell(e_2)| \leq |(x, y)| \leq \ell(e_2) + \ell(e_3)\}$ . We can switch the roles of  $p_1$  and  $q_1$  to get multiplication by a scalar  $\frac{1}{\lambda} < 1$ .



**Example 3.5.** *Addition:* The following is a functional linkage that takes inputs  $\phi(p_1), \phi(p_2)$  and returns  $\phi(q_1) = \phi(p_1) + \phi(p_2)$ . This is the fiber sum of two pantographs:  $A$ , which takes inputs  $\phi(p_1), \phi(p_2)$  and returns their average,  $\phi(v_1) = \frac{\phi(p_1) + \phi(p_2)}{2}$ ; and  $B$ , which has fixed point  $Z(w_1) = (0, 0)$ , takes input  $\phi(v_1)$  and returns  $\phi(q_1) = 2\phi(v_1)$ . Their fiber product is the composition of the two – it takes inputs  $\phi(p_1), \phi(p_2)$  and returns  $\phi(q_1) = \phi(p_1) + \phi(p_2)$ . The input domain of the fiber product is  $O = O_B \cap A(O_A)$ . Making  $A$  large lets us average numbers that are far apart, and making  $B$  large lets us double large averages. Therefore, we can build a functional linkage for addition whose input domain is an arbitrarily

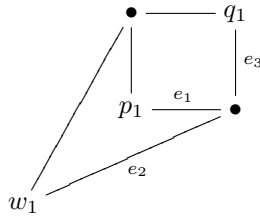
large region around  $(0, 0)$ .



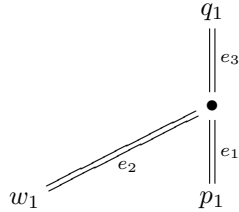
**Construction 3.6.** The *Peaucellier inversor*, with  $Z(w_1) = (0, 0)$ ,  $\ell(e_1) = \ell(e_3) < \ell(e_2)$ , and symmetry of edge lengths across the line containing  $w_1, p_1$  and  $q_1$  is meant to output:

$$\phi(q_1) = \frac{\ell(e_2)^2 - \ell(e_1)^2}{\overline{\phi(p_1)}}$$

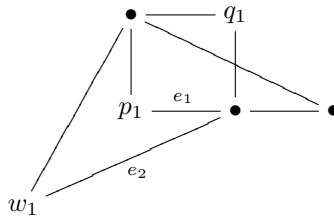
where  $\overline{\phi(p_1)}$  is the complex conjugate of  $\phi(p_1)$ .



However, even rigidifying the parallelogram allows for some degenerate configurations such as:



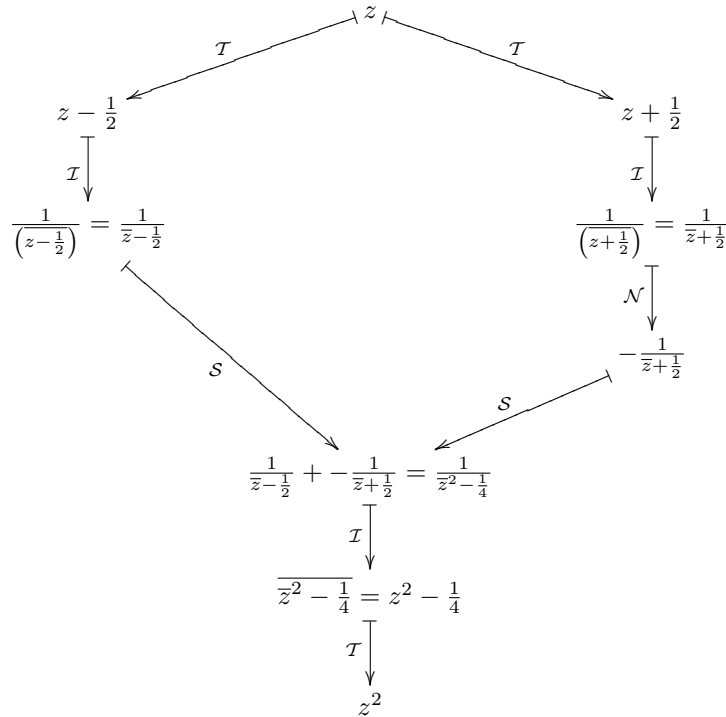
Kapovich and Millson suggest adding a “hook,” which restricts the image domain in order to eliminate these degeneracies:



By adjusting  $\ell(e_1), \ell(e_2)$ , and the size of the hook, we can make an inversor whose input domain is an arbitrarily large washer about  $(0, 0)$ .

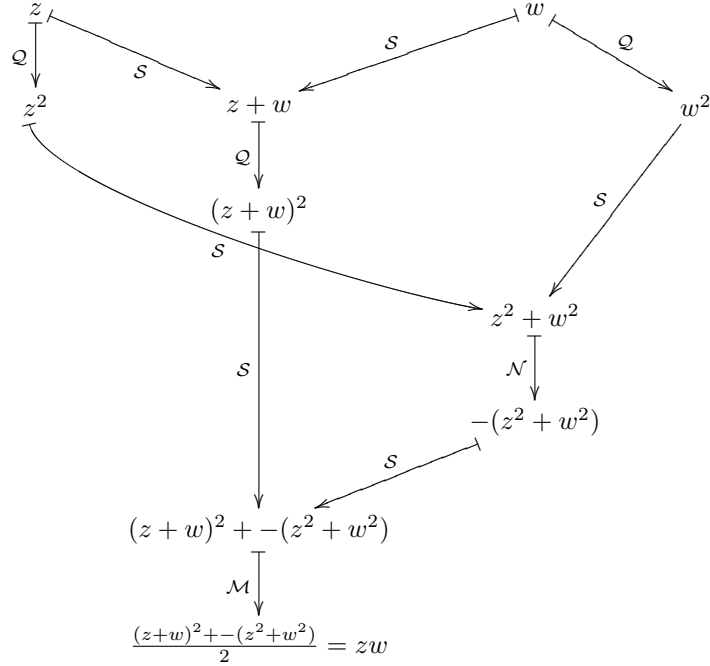
**Construction 3.7. *Multiplication:*** Kapovich and Millson use algebraic tricks to build a linkage for multiplication. Connecting the outputs of functional linkages to the inputs of another functional linkage allows for function composition. The following diagram illustrates how to create a linkage for squaring using our previous constructions – functional linkages for translation ( $\mathcal{T}$ ), negation ( $\mathcal{N}$ ), inversion of conjugates ( $\mathcal{I}$ ), scalar multiplication ( $\mathcal{M}$ ), and addition ( $\mathcal{S}$ ):

Let  $z = a + bi$



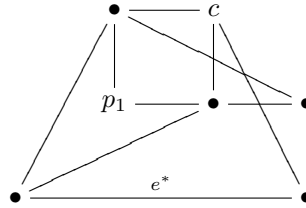
Using this construction ( $\mathcal{Q}$ ) and more algebraic tricks, we can create a linkage for multiplication:

Let  $z = a + bi$  and  $w = c + di$



The input domain of this multiplication functional linkage is restricted by the input domains of each of its parts. The input domains of  $\mathcal{T}$ ,  $\mathcal{N}$ ,  $\mathcal{M}$ , and  $\mathcal{S}$  can be arbitrarily large regions about zero, and the input domain of  $\mathcal{I}$  can be an arbitrarily large washer about zero. Note that the images of open sets under these functions contain open sets. The multiplier's input domain contains an open set, because it contains intersections of large open sets.

**Construction 3.8.** The *Peaucellier straight-line motion linkage* restricts motion of the input,  $p_1$ , which in this case is also the output, to a straight line. It will be useful in restricting the complex-valued functions our linkages represent to form real-valued functions. Its original construction was inspired by engineering questions about using linkages to turn linear motion ( $p_1$ ) into circular motion ( $c$ ).



**Theorem 3.9.** Functional linkages for arbitrary polynomials: *By composing our elementary linkages for translation, scalar multiplication, addition, negation and multiplication, we can create functional linkages for arbitrary complex polynomials. We can also use the straight-line motion linkage to create functional linkages for arbitrary real polynomials by restricting inputs to the real axis.*

**Proposition 3.10.** Expansion of domain: *When composing functional linkages (with the fiber sum operation), we must be careful about the image domains. Suppose*



we want to compose polynomial linkages  $\mathcal{L}$  and  $\mathcal{M}$ , with image domains  $O_{\mathcal{L}}$  and  $O_{\mathcal{M}}$  respectively. For our composition to make sense, we need  $F_{\mathcal{L}}(O_{\mathcal{L}}) \subseteq O_{\mathcal{M}}$ , where  $F_{\mathcal{L}}$  is the function associated with  $\mathcal{L}$ .

Assuming  $O_{\mathcal{M}}$  contains a region around the origin and  $\mathcal{M}$  is a homogeneous polynomial of degree  $n$ , we can expand the image domain of  $\mathcal{M}$  by a factor of  $\lambda \in \mathbb{R}$  by building the composition  $\mathcal{S}_{\lambda^n} \circ \mathcal{M} \circ \mathcal{S}_{\lambda^{-1}}$ , where  $\mathcal{S}_{\lambda^n}$  and  $\mathcal{S}_{\lambda^{-1}}$  are scalar-multiplication linkages with arbitrarily large input domains for multiplication by  $\lambda^n$  and  $\lambda^{-1}$  respectively.

But all polynomials are sums of homogeneous polynomials, and we can make our summing machine have as large a domain as we wish. And if  $O_{\mathcal{M}}$  does not contain a region around the origin, as long as it contains a region around some  $p \in \mathbb{R}^2$ , we can translate it by  $-p$  so it does, do the same process, then translate back, since we can make negators and translators with arbitrarily large domains.

Since we can expand the domain of any polynomial linkage  $\mathcal{M}$ , we can compose any two polynomial linkages, and build up from our elementary linkages to arbitrary polynomials.

**Theorem 3.11.** *Given a functional linkage with one output, we can create a linkage whose configuration space of inputs  $(C(\mathcal{L}, Z)/\sim_P)$  contains an arbitrary large open subset of the zero-set of that function simply by fixing the output of the functional linkage to zero.*

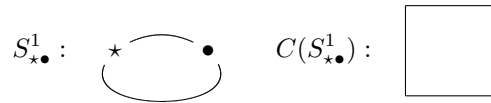
Since we can construct functional linkages for any real-valued polynomial, this gives us a method for obtaining configuration spaces that are zero-sets of any real-valued polynomial.

**Theorem 3.12.** *Since any smooth curve can be approximated by a polynomial, we can create functional linkages whose outputs approximate arbitrary smooth curves.*

#### 4. POINTS ON A CIRCLE

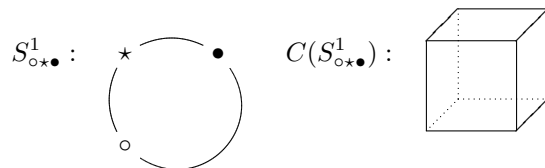
We can also talk about the configuration spaces of collections of points on a circle. The configuration space of  $n$  distinct points is a  $n$ -torus, and we can identify our points to get configuration spaces of moduli spaces of  $n$ -tori.

**Construction 4.1.** Consider a circle with two points on it,  $\star$  and  $\bullet$ :



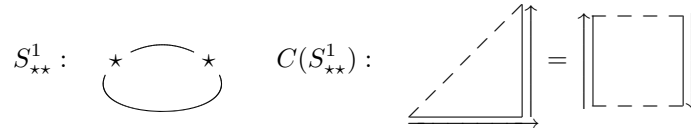
The configuration space of this circle is a square with opposite sides identified (a 2-torus), because each configuration of  $\star$  and  $\bullet$  defines an ordered pair of angles between 0 and  $2\pi$ , with 0 and  $2\pi$  identified.

**Construction 4.2.** Now consider a circle with three points on it,  $\circ$ ,  $\star$  and  $\bullet$ :



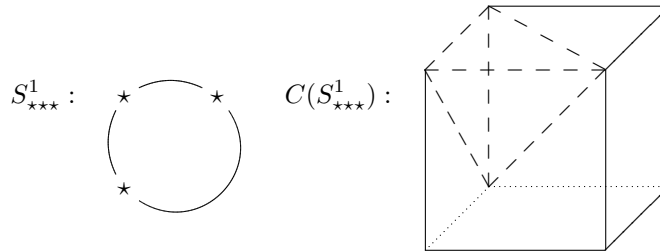
The configuration space is a cube with opposite faces identified (a 3-torus), since configurations of  $\circ$ ,  $\star$  and  $\bullet$  define ordered triples of angles between 0 and  $2\pi$ , with 0 and  $2\pi$  identified. Likewise, the configuration space of a collection of  $n$  distinct points on a circle is the  $n$ -torus.

**Construction 4.3.** We can get a different configuration space by identifying points on our circle. For example:



Since we have identified the two points, a configuration defines an *unordered* pair of points in  $[0, 2\pi]/(0 = 2\pi)$ . Thus each point  $(a, b)$  on the square is identified with its reflection along the dotted line,  $(b, a)$ .  $C(S_{**}^1)$  is a triangle with two consecutive sides identified, in other words a Möbius strip.

**Construction 4.4.** Let's explore what happens to the configuration space of the circle with three points when we identify all three points. When the points were distinct, the configuration space was a cube with opposite faces identified. Now the ordered triples become unordered triples. All edges of our cube become identified, and all corners. If we graph triples  $\{a, b, c \mid a \leq b \leq c\}$  as  $(a, b, c)$ , all points lie in the tetrahedron surrounded by dashed lines:



The back left side of this tetrahedron is identified with the top, with a twist. The front and right sides, where two coordinates are equal, form the boundary, which is a 2-torus.

*Remark 4.5.* We can construct any Euclidean manifold by identifying some points on our circle, and taking the configuration space.

Euclidean manifolds can be classified by their fundamental groups, and every  $n$ -dimensional Euclidean manifold can be covered by an  $n$ -torus (Thurston 125, 222).

The fundamental group of an  $n$ -torus is the free abelian group with  $n$  generators, which correspond to  $n$  points on a circle. The configurations space of a circle with points identified can be covered by the configuration space of a circle with points distinct. Thus, any Euclidean manifold is the configuration space of some collection of points on a circle.

**Acknowledgments.** It is a pleasure to thank my mentors, Jim Fowler and Strom Borman, for helping me understand configuration spaces and so many other fascinating topics.

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