SOME BASIC PROPERTIES OF BROWNIAN MOTION

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Abstract. This paper provides a an introduction to some basic properties of Brownian motion. In particular, it shows that Brownian motion exists, that Brownian motion is nowhere differentiability, and that Brownian motion has finite quadratic variation.

Contents

1. Definition of Brownian Motion

Brownian motion plays important role in describing many physical phenomena that exhibit random movement. Brownian motion is defined as follows:

Definition 1.1. A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a *linear Brownian motion* with start in $x \in \mathbb{R}$ if the following holds:

(1) $B(0) = x$,
(2) the process has independent increments, i.e. for all times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ the increments $B(t_n) - B(t_{n-1}),\ B(t_{n-1}) - B(t_{n-2}),\ \ldots,\ B(t_n) - B(t_{n-1})$ are independent random variables,
(3) for all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation zero and variance h,
(4) almost surely, the function $t \mapsto B(t)$ is continuous.

If $x = 0$, $\{B(t) : t \geq 0\}$ is called a *standard Brownian motion.*

2. Brownian Motion Exists

The definition of Brownian motion poses the nontrivial question of whether Brownian motion even exists. While it is rather easy to see that one can construct a stochastic process that fulfills the first three properties of Brownian motion, it is not immediately clear that the fourth property can be fulfilled.

Definition 2.1. A random variable X is normally distributed with mean μ and variance σ^2 if

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$$\mathbf{P}\{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-\frac{(u-\mu)^2}{2\sigma^2}}\, du, \text{ for all } x \in \mathbb{R}.$$

Definition 2.2. A random vector $(X_1, \dots, X_n)$ is called a *Gaussian random vector* if there exists an $n \times m$ matrix A, and an n-dimensional vector b such that $X^T = AY + b$, where Y is an m-dimensional vector with independent standard normal entries.

Lemma 2.3. *Let X_1 and X_2 be independent and normally distributed with expectation 0 and variance $\sigma^2 > 0$. Then $X_1 + X_2$ and $X_1 - X_2$ are independent and normally distributed with expectation 0 and variance $2\sigma^2$.*

Lemma 2.4. *Suppose X is standard normally distributed. Then, for all $x > 0$,*

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \le \mathbf{P}\{X > x\} \le \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Lemma 2.5. *Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of Gaussian random vectors and $\lim_{n\to\infty} X_n = X$, almost surely. If $b := \lim_{n\to\infty} \mathbb{E}[X_n]$ and $C := \lim_{n\to\infty} Cov(X_n)$ exist, then X is Gaussian with mean b and covariance matrix C .*

Lemma 2.6. *Borel-Cantelli Lemma*
Let $\{E_n : n \in \mathbb{N}\}$ be a sequence of events such that $\sum_n \mathbf{P}\{E_n\} < \infty$. Then

$$\mathbf{P}\{\limsup E_n\} = \mathbf{P}\{E_n, i.o.\} = 0.$$

For proofs of 2.3, 2.4, and 2.5, see [1]. For a proof 2.6, see [2].

Theorem 2.7. *Wiener's Theorem: Standard Brownian motion exists.*

Proof. Define

$$\mathcal{D}_n := \left\{ \frac{k}{2^n} : 0 \le k \le 2^n \right\}, \qquad \mathcal{D} := \bigcup_{n=0}^{\infty} \mathcal{D}_n.$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability triple on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independent, standard normally distributed random variables can be defined. For each $n \in \mathbb{N}$ we want to define the random variables $B(d)$, $d \in \mathcal{D}_n$ such that

(1) for all $r < s < t$ in $\mathcal{D}_n$ the random variable $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$, and is independent of $B(s) - B(r)$,
(2) the vectors $\{B(d) : d \in \mathcal{D}_n\}$ and $\{Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n\}$ are independent.

First consider $\mathcal{D}_0 = \{0, 1\}$. If we let $B(0) := 0$ and $B(1) := Z_1$, then this construction adheres to what we wanted for $n = 0$.
Now suppose that we have successfully constructed our $B(d)$ for some $n - 1$.
Define $B(d)$ for $d \in \mathcal{D} \setminus \mathcal{D}_n$ by

$$B(d) := \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}}$$

Since the first summand is the linear interpolation of the values of B at the neighboring points of d in $\mathcal{D}_{n-1}$, $B(d)$ is independent of $(Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n)$ which fulfills the second property.
Clearly, the first and second summands are independent of each other. For the first summand, both terms are normally distributed with mean zero and variance $2^{-(n+1)}$. Thus, by Lemma 2.3, $B(d) - B(d - 2^{-n})$ and $B(d + 2^{-n}) - B(d)$ are independent and normally distributed with mean zero and variance 2^{-n}.

We now want to show that $B(d) - B(d-2^{-n})$ are independent for $d \in \mathcal{D} \setminus \{0\}$. It suffices to show that the $B(d) - B(d-2^{-n})$ are pairwise independent because the vector containing $B(d) - B(d-2^{-n})$ for $d \in \mathcal{D} \setminus \{0\}$ is Gaussian.
We already know that $B(d) - B(d-2^{-n})$ and $B(d+2^{-n}) - B(d)$ with $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ are independent. Thus, we need to consider the case when the increments are over intervals separated by some $d \in \mathcal{D}_{n-1}$. Choose $d \in \mathcal{D}_j$ with this property and minimal j , so that the two intervals are contained in $[d-d^{-j}, d]$ and $[d, d+2^{-j}]$. The increments over these two intervals of length 2^{-j} are independent, and the increments over the intervals of length 2^{-n} are constructed from the independent increments $B(d) - B(d-2^{-j})$ and $B(d+2^{-j}) - B(d)$, using a disjoint set of variables $(Z_t : t \in \mathcal{D}_n)$. This implies they are independent which verifies the first property.

Define

$$F_0(t) = \begin{cases} Z_1 & \text{for } t = 1 \\ 0 & \text{for } t = 0 \\ \text{linear in between.} \end{cases}$$

and, for each $n \geq 0$,

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & \text{for } t \in \mathcal{D} \setminus \mathcal{D}_n \\ 0 & \text{for } t \in \mathcal{D}_{n-1} \\ \text{linear between consecutive points in } \mathcal{D}_n. \end{cases}$$

Clearly, these functions are continuous on $[0,1]$. Furthermore, we claim that for all n and $d \in \mathcal{D}_n$,

$$B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d). \tag{2.8}$$

For $n = 0$, $B(0) = 0$ and $B(1) = Z_1$ which is what we wanted to show.
Now suppose that (2.7) holds for $n-1$. Let $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Since for $0 \leq i \leq n-1$ the function F_i is linear on $[d-2^{-n}, d+2^{-n}]$, we get

$$\sum_{i=0}^{n-1} F_i(d) = \sum_{i=0}^{n-1} \frac{F_i(d-2^{-n}) + F_i(d+2^{-n})}{2} = \frac{B(d-2^{-n}) + B(d+2^{-n})}{2}$$

Since $F_n(d) = 2^{-(n+1)/2} Z_d$, this gives (2.7).

On the other hand, we have, by definition of Z_d and by Lemma 2.4, for $c > 0$ and large n,

$$\mathbf{P}\{|Z_d| \geq c\sqrt{n}\} \leq \exp\left(\frac{-c^2 n}{2}\right),$$

so that the series

$$\sum_{n=0}^{\infty} \mathbf{P}\{\text{there exists } d \in \mathcal{D}_n \text{ with } |Z_d| \geq c\sqrt{n}\} \leq \sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} \mathbf{P}\{|Z_d| \geq c\sqrt{n}\}$$
$$\leq \sum_{n=0}^{\infty} \exp\left(\frac{-c^2 n}{2}\right),$$

converges as soon as $c > \sqrt{2\log 2}$. Fix such a c. By the Borel-Cantelli lemma there exists a random and almost surely finite N such that for all $n \geq N$ and $d \in \mathcal{D}_n$ we have $|Z_d| < c\sqrt{n}$. Hence, for all $n \geq N$, $||F_n||_\infty < c\sqrt{n}2^{-n/2}$, which implies that almost surely, $B(t) = \sum_{n=0}^{\infty} F_n(t)$ converges uniformly on $[0,1]$. Denote the continuous limit by $\{B(t) : t \in [0,1]\}$.

We need to show that the increments of this process have the right marginal distributions. Suppose that $t_1 < t_2 < \cdots < t_n$ are in $[0,1]$. Since $\mathcal{D}$ is a dense set, we can find $t_{1,k} < t_{2,k} < \cdots < t_{n,k}$ in $\mathcal{D}$ with $\lim_{k\to\infty} t_{i,k} = t_k$. And since B is continuous on $[0,1]$, for $1 \leq i \leq n-1$,

$$B(t_{i+1}) - B(t_i) = \lim_{k\to\infty} B(t_{i+1,k}) - B(t_{i,k}).$$

As $\lim_{k\to\infty} \mathbb{E}[B(t_{i+1,k}) - B(t_{i,k})] = 0$ and

$$\lim_{k\to\infty} \mathrm{Cov}\left(B(t_{i+1,k}) - B(t_{i,k}), B(t_{j+1,k}) - B(t_{j,k})\right)$$

$$= \lim_{k\to\infty} \mathbb{I}_{\{i=j\}}(t_{i+1,k} - t_{i,k}) = \mathbb{I}_{\{i=j\}}(t_{i+1} - t_i),$$

the increments $B(t_{i+1}) - B(t_i)$ are, by Lemma 2.5, independent Gaussian random variables with mean 0 and variance $t_{i+1} - t_i$, which is what we wanted to show.

All that remains is to glue together a sequence $B_1, B_2, \ldots$ of independent $\mathcal{C}[0,1]$-valued random variables with the distribution of the process $B : [0,1] \to \mathbb{R}$ that we have found. Define

$$B(t) := B_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} B_i(1), \text{ for all } t \geq 0.$$

This gives us a continuous random function $B : [0,\infty] \to \mathbb{R}$ that meets the requirements of a Brownian motion. □

3. Brownian Motion is Nowhere Differentiable

Even though Brownian motion is everywhere continuous, the randomness allows Brownian motion to also be nowhere differentiable.

Lemma 3.1. *(Scaling Invariance). Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion and let $a > 0$. Then the process $\{X(t) : t \geq 0\}$ defined by $X(t) = \frac{1}{a}B(a^2 t)$ is also a standard Brownian motion.*

For a proof of 3.1, see [1].

Theorem 3.2. *Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all t,*

$$\text{either} \quad \limsup_{h\to 0} \frac{B(t+h) - B(t)}{h} = \infty \quad \text{or} \quad \limsup_{h\to 0} \frac{B(t+h) - B(t)}{h} = -\infty \quad \text{or both.}$$

Proof. Suppose $\exists\ t_0 \in [0,1]$ such that

$$\limsup_{h\to 0} \frac{|B(t_0+h) - B(t)|}{h} < \infty.$$

Since Brownian motion is bounded on $[0,2]$, for some finite constant M, there exists t_0 with

$$\sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t)|}{h} \leq M.$$

We want to show that this event has probability zero for any M. Fix M. If $t_0 \in [(k-1)/2^n, k/2^n]$ for $n > 2$, then for all $1 \leq j \leq 2^n - k$ the triangle inequality gives

$$\begin{aligned}
&|B((k+j)/2^n) - B((k+j-1)/2^n)| \\
&\qquad \leq |B((k+j)/2^n) - B(t_0)| + |B(t_0) - B((k+j-1)/2^n)| \\
&\qquad \leq M(2j+1)/2^n
\end{aligned}$$

Define events

$$\Omega_{n,k} := \{|B((k+j)/2^n) - B((k+j-1)/2^n)| \leq M(2j+1)/2^n \text{ for } j = 1,2,3\}$$

Then by the fact that Brownian motion has independent increments and Lemma 3.1, for $1 \leq k \leq 2^n - 3$,

$$\begin{aligned}
\mathbf{P}\{\Omega_{n,k}\} &\leq \prod_{j=1}^{3} \mathbf{P}\left\{|B((k+j)/2^n) - B((k+j-1)/2^n)| \leq M(2j+1)/2^n\right\} \\
&\leq \mathbf{P}\left\{|B(1)| \leq 7M/\sqrt{2^n}\right\}^3,
\end{aligned}$$

which is at most $(7M2^{-n/2})^3$, since the normal density is bounded by $1/2$. Hence

$$\mathbf{P}\left(\bigcup_{k=1}^{2^n-3} \Omega_{n,k}\right) \leq 2^n(7M2^{-n/2})^3 = (7M)^3 2^{-n/2}$$

which is summable over all n. Hence, by the Borel-Cantelli lemma,

$$\begin{aligned}
&\mathbf{P}\left\{\text{there is } t_0 \in [0,1] \text{ with } \sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t)|}{h} \leq M\right\} \\
&\qquad \leq \mathbf{P}\left(\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \text{ for infinitely many } n\right) = 0
\end{aligned}$$

□

4. Brownian Motion has Finite Quadratic Variation

As we have seen, even though Brownian motion is everywhere continuous, it is nowhere differentiable. The randomness of Brownian motion means that it does not behave well enough to be integrated by traditional methods. However, because Brownian motion has finite quadratic variation, it can be integrated with Stochastic calculus.

Proposition 4.1. *If $\alpha < 1/2$, then, almost surely, Brownian motion is everywhere locally α-Hölder continuous.*

Lemma 4.2. *If X, Z are independent, symmetric random variables in L^2, then*

$$\mathbb{E}\left[(X+Z)^2 | X^2 + Z^2\right] = X^2 + Z^2$$

Theorem 4.3. *(Lévy's Downward Theorem). Suppose that* $\{\mathcal{G}_n : n \in \mathbb{N}\}$ *is a collection of* σ*-algebras such that*

$$\mathcal{G}_\infty := \bigcap_{k=1}^{\infty} \mathcal{G}_k \subset \cdots \subset \mathcal{G}_{n+1} \subset \mathcal{G}_n \subset \cdots \subset \mathcal{G}_1.$$

An integrable process $\{X_n : n \in \mathbb{N}\}$ *is reverse martingale if almost surely,* $X_n = \mathbb{E}[X_{n-1}|\mathcal{G}_n]$ *for all* $n \geq 2$*. Then* $\lim_{n\to\infty} X_n = \mathbb{E}[X_1|\mathcal{G}_\infty]$ *almost surely.*

Lemma 4.4. *Fatou's Lemma.*

$$\mathbf{P}\{\liminf E_n\} \leq \liminf \mathbf{P}\{E_n\}$$

For proofs of 4.1, 4.2, and 4.3, see [1]. For a proof of 4.4, see [2].

Definition 4.5. A right-continuous function $f : [0, t] \to \mathbb{R}$ is a function of *bounded variation* if

$$V_f^{(1)}(t) := \sup \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| < \infty$$

where the supremum is over all $k \in \mathbb{N}$ and partitions $0 = t_0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k = t$. If the supremum is infinite f is said to be *unbounded variation.*

Theorem 4.6. *Suppose that the sequence of partitions*

$$0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_{k(n)-1}^{(n)} \leq t_{k(n)}^{(n)} = t$$

is nested, i.e. at each step one or more partition points are added, and the mesh

$$\triangle(n) := \sup_{1\leq j\leq k(n)} \left\{ t_j^{(n)} - t_{j-1}^{(n)} \right\}$$

converges to zero. Then, almost surely,

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2 = t$$

and therefore Brownian motion is of unbounded variation.

Proof. By Proposition 4.1, for any $\alpha \in (0, 1/2)$, $\exists\, n$ such that $|B(a)-B(b)| \leq |a-b|^\alpha$ for all $a, b \in [0, t]$ with $|a - b| \leq \triangle(n)$, which means that

$$\sum_{j=1}^{k(n)} \left| B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right| \geq \triangle(n)^{-\alpha} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2.$$

If we can show that the random variables

$$X_n := \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2$$

converge almost surely to a positive random variable, then we will have shown that Brownian motion is almost surely of unbounded variation. We can insert elements in the sequence when necessary to ensure that at each step exactly one point is added to the partition.

Let $\mathcal{G}_n$ be the σ-algebra generated by the random variables $X_n, X_{n+1}, \ldots$. Then

$$\mathcal{G}_\infty := \bigcap_{k=1}^{\infty} \mathcal{G}_k \subset \cdots \subset \mathcal{G}_{n+1} \subset \mathcal{G}_n \subset \cdots \subset \mathcal{G}_1.$$

We now want to show that $\{X_n : n \in \mathbb{N}\}$ is a reverse martingale. Suppose $s \in (t_1, t_2)$ is the inserted point. $B(s) - B(t_1)$ and $B(t_2) - B(s)$ are symmetric independent random variables. Let $\mathcal{F}$ be the σ-algebra generated by $(B(s) - B(t_1))^2 + (B(t_2) - B(s))^2$. Then by lemma 4.2,

$$\mathbb{E}[(B(t_2) - B(t_1))^2 | \mathcal{F}] = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2,$$

which shows that

$$\mathbb{E}[(B(t_2) - B(t_1))^2 - (B(s) - B(t_1))^2 - (B(t_2) - B(s))^2 | \mathcal{F}] = 0,$$

which implies that $\{X_n : n \in \mathbb{N}\}$ is a reverse martingale.

By the Lévy Downward Theorem, $\lim_{n\to\infty} X_n = \mathbb{E}[X_1 | \mathcal{G}_\infty]$ almost surely. The limit has expectation $\mathbb{E}[X_1] = t$ and, by Fatou's lemma, its variance is bounded by

$$\liminf_{n\to\infty} \mathbb{E}[(X_n - \mathbb{E}[X_n])^2] = \liminf_{n\to\infty} 3 \sum_{j=1}^{k(n)} \left(t_j^{(n)} - t_{j-1}^{(n)} \right)^2 \leq 3 \liminf_{n\to\infty} \triangle(n) = 0.$$

Hence, $\mathbb{E}[X_1 | \mathcal{G}_\infty] = t$ almost surely, which is what we wanted to show. □

Definition 4.7. For a sequence of partitions as above, we define

$$V^{(2)}(t) := \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2$$

to be the *quadratic variation* of Brownian motion.

Remark 4.8. From theorem 4.5, we see that Brownian motion has finite quadratic variation.

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References

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