CARDINAL AND ORDINAL NUMBERS

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Abstract. This paper will present a brief set-theoretic construction of the natural numbers before discussing in detail the ordinal and cardinal numbers. It will then investigate the relationship between the two proper classes, in particular the similar difficulties in discussing the size of the classes. We will end with a short section on the cardinalities of well-known infinite sets with which the reader is likely to be familiar.

Contents

1. The Natural Numbers

Although there are several ways to construct the natural numbers, this paper will use a method that defines each natural number as a set which contains each of its predecessors. Before we can make this approach rigorous, we need a definition.

Definition 1.1. For a set x, we define the *successor of* x, x^+, to be the set obtained by adjoining x to the elements of x. In other words, $x^+ = \{x \cup \{x\}\}$.

We can now begin to define the natural numbers. However, we must consider how to start, that is, how to define the first natural number, 0. Since our method is based around defining each natural number with regards to its predecessors, and since 0 has no predecessors in the naturals, we define 0 to be the empty set: $0 = \emptyset$. We then define 1, 2 and 3 in the way alluded earlier:

$$1 = 0^+ = \{0\}$$
$$2 = 1^+ = \{0, 1\}$$
$$3 = 2^+ = \{0, 1, 2\}$$

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This method of defining the natural numbers is useful and consistent with our notation for all finite natural numbers, that is, the set $\mathbb{N}$. However, it is not yet clear that this construction of successors can be carried out in one set indefinitely. That is, it is not clear that there exists a non-empty set which contains the successor of each of its elements. We need a set-theoretic axiom for this.

Axiom 1. *There exists a set containing 0 and containing the successor of each of its elements.*

This statement of existence is often called *The Axiom of Infinity*. Such a set A, defined such that $0 \in A$ and $x^+ \in A$ if $x \in A$, is called a *successor set*. We will next prove that there exists a smallest successor set.

Theorem 1.2. *There exists a smallest successor set.*

Proof. Let ω be the intersection of every successor set. Then ω is a successor set itself. For if not, then for some $x \in \omega$, $x^+ \notin \omega$. But since ω is the intersection of all successor sets, then for some such successor set, $x \in \omega$ but $x^+ \notin \omega$. This is a contradiction of the definition of successor set. Then ω is a successor set and is, by construction, a subset of all successor sets. It is therefore the smallest successor set. □

The reader worried that the intersection of all successor sets might not exist should consider the following, more precise definition of ω. Take a successor set, α, and consider the set of its subsets, $P(\alpha)$. Then look at the set $A_\alpha \subseteq P(\alpha)$ such that every element of A is a successor set. If we look at the intersection of A_α for all successor sets α, then any trouble with dealing with the intersection of all successor sets is alleviated. This comment is only relevant to those very familiar with set theory, in particular with the theory of proper classes. For all other readers, this comment is not worth fretting over.

A natural number is, by definition, an element of ω. This construction of ω makes rigorous the intuitive description of the natural numbers as $\{0, 1, 2, 3, ..\}$, where the ellipsis represent the *so on ad infinitum* normally used to describe the natural numbers.

2. Ordinal Numbers

Before we can begin this new section, we must present an extremely important definition. We assume that the reader is familiar with the concept of a relation and has seen some examples of a relation, such as $<$, $\leq$ and $\in$.

Definition 2.1. A set X is *well-ordered* by the relation R if the following principles hold:

1.) For every x and y in X, if we have xRy then we cannot have yRx. This means that R is *asymmetric* on X.
2.) For every x and y in X, exactly one of xRy, yRx and $x = y$ holds. This means that R is *connected* on X.
3.) For all x, y and z in X, if xRy and yRz, then xRz. This means that R is *transitive* on X.
4.) Every non-empty subset of X has an R-least element.

We call a set W together with a relation that well orders it, $<$, a *well-ordering.* This is often stated by saying the (W, $<$) is a well-ordering.

Our definition of the naturals is ordered by inclusion, since we defined a natural number n as the set of all natural numbers less that n, that is, we defined n=$\{0, 1, 2, ..., n-2, n-1\}$. We now want to use this key property of the natural numbers and ω to define numbers larger than ω. Since we are defining this new type of number by succession as with the natural numbers, we want the set to be well-ordered by inclusion too. Before we can give a precise definition for this new type of number, which we will call an *ordinal number*, or more simply an *ordinal*, we need a couple of definitions.

Definition 2.2. A set z is *transitive* if whenever x and y are sets such that $x \in y$ and $y \in z$, we have $x \in z$.

Definition 2.3. Let z be a set. We define a relation $\in_z$ by $\in_z = \{(x, y) \in z \times z : x \in y\}$.

We can now define what exactly we mean by an ordinal number and give an example of an ordinal number we have already encountered.

Definition 2.4. An *ordinal* is a set α which is transitive and well-ordered by $\in_\alpha$.

Theorem 2.5. *ω is an ordinal.*

Proof. Theorem 1.3 shows that ω is transitive. To see that ω is well-ordered by $\in_\omega$, let α be a non-empty subset of ω. Then we assert that α has a least element, namely $x = \bigcap_{\beta\in\alpha} \beta$, that is, the intersection of all elements of α. $x \neq \emptyset$, since $0 \in \beta$ for every $\beta \in \alpha$. Now consider γ, the largest element of x. This number must exist, for otherwise every element of α has no largest element, meaning that α cannot be a subset of ω, which consists of only natural numbers, each of which have an $\in$-greatest element. Then by construction, x contains every natural number less than γ. If this were not true, then for some $\beta \in \alpha$, there is some $y \in \beta$ and z such that $z^+ = y$ but $z \notin \beta$, which is absurd based on Definition 1.1. This shows that x is itself a natural number. In fact, it is the natural number $\gamma + 1$, again by Definition 1.1. For each $\beta \in \alpha$, the $\in$-greatest element of β is unique, based on our construction of the naturals. Thus, if $x = \gamma+1$, then there must be $\beta \in \alpha$ such that the $\in$-greatest element of this β is γ. This shows that $\{0, 1, 2, ..., \gamma\} = \gamma+1 = x \in \alpha$. Since $x \in \beta$ for all $\beta \in \alpha$, x is the smallest $\beta \in \alpha$, making x the least element of α. This shows that ω is well-ordered by $\in_\omega$, which completes the proof. □

In the preceding proof, we used the notation ... to indicate a set of natural numbers which includes every natural number in between 2 and γ.

We will now prove a few theorems that characterize ordinal numbers.

Theorem 2.6. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*

Proof. To see that β is transitive, we let x and y be sets with $x \in y$ and $y \in \beta$. Since $y \in \beta$, $\beta \in \alpha$ and α is an ordinal and thus transitive, it follows that $y \in \alpha$. Since $x \in y$ and $y \in \alpha$, it follows that $x \in \alpha$. Now since $x, y, \beta \in \alpha$ $x \in y$, $y \in \beta$ and the relation $\in_\alpha$ is transitive on α, we have $x \in \beta$. Thus, β is transitive.

Notice that $\beta \subseteq \alpha$ because $\beta \in \alpha$ and α is transitive. Therefore, $\in_\beta$ is the restriction of $\in_\alpha$ to the subset $\beta \subseteq \alpha$. Since $\in_\alpha$ is a well-ordering on α, it follows that $\in_\beta$ is a well-ordering on β. Hence, β is an ordinal.

□

Corollary 2.7. *Every $n \in \omega$ is an ordinal.*

Lemma 2.8. *If α is an ordinal, then $\alpha \notin \alpha$.*

Proof. Suppose that α is an ordinal and $\alpha \in \alpha$. Since $\alpha \in \alpha$, $\in_\alpha$ is not asymmetric on α. Thus, $\in_\alpha$ is not a well-ordering on α, so α is not an ordinal, which is a contradiction. □

Theorem 2.9. *Suppose that α and β are ordinals. Then exactly one of the following is true: $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.*

Proof. We will first prove that at least one of $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$ holds. We first claim that $\alpha \cap \beta$ is an ordinal. If $x \in y \in \alpha \cap \beta$, then $x \in y \in \alpha$ and $x \in y \in \beta$, so $x \in \alpha$ and $x \in \beta$, because α and β are ordinals and thus transitive. Thus, $\alpha \cap \beta$ is transitive. Notice that $\in_{\alpha\cap\beta}$ is the restriction of $\in_\alpha$ to the subset $\alpha \cap \beta \subseteq \alpha$. Since $\in_\alpha$ is a well-ordering on α, it follows that $\in_{\alpha\cap\beta}$ is a well-ordering on $\alpha \cap \beta$. Hence, $\alpha \cap \beta$ is an ordinal.

Now we have $\alpha \cap \beta \subseteq \alpha$ and $\alpha \cap \beta \subseteq \beta$. If $\alpha \cap \beta \neq \alpha$ and $\alpha \cap \beta \neq \beta$, then $\alpha \cap \beta \in \alpha$ and $\alpha \cap \beta \in \beta$ by Theorem 2.8. Thus, $\alpha \cap \beta \in \alpha \cap \beta$, which contradicts Lemma 2.7. Therefore, either $\alpha \cap \beta = \alpha$ or $\alpha \cap \beta = \beta$. If $\alpha \cap \beta = \alpha$, then $\alpha \subseteq \beta$, and hence either $\alpha = \beta$ or $\alpha \in \beta$ by Theorem 2.8. Similarly, if $\alpha \cap \beta = \beta$, then $\beta \subseteq \alpha$ and by Theorem 2.8, either $\beta = \alpha$ or $\beta \in \alpha$. Thus, in any case, at least one of $\alpha \in \beta$, $\alpha = \beta$ or $\beta \in \alpha$ holds.

All that remains is to show that only one of these three can hold. If $\alpha \in \beta$ and $\alpha = \beta$, then $\alpha \in \alpha$, which is a contradiction. Similarly, if $\alpha = \beta$ and $\beta \in \alpha$, then $\beta \in \beta$. Finally, if $\alpha \in \beta$ and $\beta \in \alpha$, then because α is transitive, $\alpha \in \alpha$. This is another contradiction, so exactly one of $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$ holds. □

Theorem 2.10. *If α and β are ordinals, then $\alpha \subseteq \beta$ if and only if $\alpha{=}\beta$ or $\alpha \in \beta$.*

Proof. ($\Leftarrow$) If $\alpha{=}\beta$, then obviously $\alpha \subseteq \beta$. If $\alpha \in \beta$, then β being transitive implies that $\alpha \subseteq \beta$.

($\Rightarrow$) Suppose that $\alpha \subseteq \beta$ and $\alpha \neq \beta$. Then $\beta \setminus \alpha$ is a non-empty subset of β. Since $\in_\beta$ well-orders β, there exists a $\in_\beta$-least element of $\beta \setminus \alpha$, call it z. We will show that $z = \alpha$, thus proving $\alpha \in \beta$. To see that $z \subseteq \alpha$, let $x \in z$. Since $z \in \beta$ and β is transitive, we have $x \in \beta$. Since $x \in z$, we cannot have $x \in \beta \setminus \alpha$ by our choice of z, so $x \in \alpha$. Thus, $z \subseteq \alpha$, since if all elements of z are in α, then the collection of all elements of z is a subset of α, and this is just z itself. To see that $\alpha \subseteq z$, let $x \in \alpha$. Since $\alpha \subseteq \beta$, we have $x \in \beta$. Since $x, z \in \beta$, x, z are necessarily ordinals, one of $x \in z$, $x = z$ or $z \in x$ holds by Theorem 2.9. We can not have $x = z$, because $x \in \alpha$ and $z \in \beta \setminus \alpha$. Also, we cannot have $z \in x$, because if $z \in x$ is true, then we can also conclude that $z \in \alpha$, because $z \in x \in \alpha$ and α is transitive. This contradicts $z \in \beta \setminus \alpha$. Thus, $x \in z$, so $\alpha \subseteq z$. It follows that $z = \alpha$. □

We can now begin to discuss the collection of all ordinals, which we will call **ORD**. We will prove several theorems that characterize sub-collections of **ORD** with the goal of showing that our choice to speak of the *collection* of ordinal numbers is correct. More precisely, we will prove that the ordinal numbers do not form a set. They are too large. As contrary as this may seem, we assure the reader this crisis will be fully explained by the section's end.

Theorem 2.11. *If A is a non-empty subset of **ORD**, then A has a least element. In particular, this least element is $\bigcap A$.*

Proof. Since $A \neq \emptyset$, we may fix an ordinal $\alpha \in A$. If $A \cap \alpha = \emptyset$, then for any $\beta \in A$, we cannot have $\beta \in \alpha$. Hence either $\alpha = \beta$ or $\alpha \in \beta$ by Theorem 2.9. Suppose $A \cap \alpha \neq \emptyset$. Since $A \cap \alpha \subseteq \alpha$ is non-empty, it has an $\in_\alpha$-least element, call it x. Let $\beta \in A$ and notice that β is an ordinal. Then by Theorem 2.9, either $\beta \in \alpha$, $\beta = \alpha$, or $\alpha \in \beta$. If $\beta \in \alpha$, then $\beta \in A \cap \alpha$, so either $x = \beta$, or $x \in \beta$, based on our choice of x. If $\beta = \alpha$, then $x \in \beta$ because $x \in \alpha$. If $\alpha \in \beta$, then $x \in \alpha \in \beta$, so β being transitive gives $x \in \beta$. It follows that x is the least element of A.

Thus A has a least element, x. Since $x \in A$, we have $\bigcap A \subseteq x$. For all $\alpha \in A$, we then have $x = \alpha$ or $x \in \alpha$, so $x \subseteq \alpha$ by Theorem 2.8. Thus, $x \subseteq \bigcap A$. It follows that $x = \bigcap A$. □

Theorem 2.12. *If $A \subset$ **ORD**, and A is a set, then $\bigcup A$ is an ordinal. Furthermore, $\bigcup A =$ sup$A$.*

Proof. First we will show that $\bigcup A$ is transitive. Suppose that $x \in y \in \bigcup A$. Since $y \in \bigcup A$, there is an ordinal $\alpha \in A$ such that $y \in \alpha \in A$. Since α is transitive and $x \in y \in \alpha$, we have $x \in \alpha$. Thus, $x \in \bigcup A$, so $\bigcup A$ is transitive.

We will now show that $\in_{\bigcup A}$ well-orders $\bigcup A$. First we will show that $\in_{\bigcup A}$ is transitive on $\bigcup A$. Let $x, y, z \in \bigcup A$, where $x \in y \in z$. Since $z \in \bigcup A$, there is some $\alpha \in A$, necessarily an ordinal, such that $z \in \alpha \in A$. Since $z \in \alpha$ and α is an ordinal, we may use Theorem 2.6 to conclude that z is an ordinal. Thus, because z is transitive, we recall that $x \in y \in z$ to conclude that $x \in z$.

We next show that $\in_{\bigcup A}$ is asymmetric on $\bigcup A$. Take $x \in \bigcup A$ and fix $\alpha \in A$, an ordinal, such that $x \in \alpha \in A$. Then Theorem 2.6 shows that x is an ordinal, and by Lemma 2.8, $x \notin x$.

We must now show that $\in_{\bigcup A}$ is connected on $\bigcup A$. Let $x, y \in \bigcup A$. Fix ordinals α, $\beta \in A$, such that $x \in \alpha \in A$ and $y \in \beta \in A$. By Theorem 2.6, we can conclude that x, y are ordinals, so either $x \in y$, $x = y$ or $y \in x$, by Theorem 2.9.

Finally, suppose the $X \subseteq \bigcup A$ and $X \neq \emptyset$. Then notice that for any $y \in X$, there exists $\alpha \in A$, necessarily an ordinal, such that $y \in \alpha \in A$, so by Theorem 2.6, y is an ordinal. Therefore, X is a non-empty subset of **ORD**, so by Theorem 2.11, X has a least element, with respect to $\in_{\bigcup A}$. This shows that $\bigcup A$ is well-ordered by $\in_{\bigcup A}$ and hence is an ordinal.

We must now show that $\bigcup A = \sup A$. Suppose that $\alpha \in A$. For any $\beta \in \alpha$, we have $\beta \in \alpha \in A$, hence $\beta \in \bigcup A$. It follows that $\alpha \subseteq \bigcup A$, so Theorem 2.8 gives $\alpha \leq \bigcup A$. Thus, $\bigcup A$ is an upper bound for A. Suppose that γ is an upper bound for A, that is, γ is an ordinal and $\alpha \leq \gamma$ for all $\alpha \in A$. For any $\beta \in \bigcup A$, we can fix $\alpha \in A$ such that $\beta \in \alpha$ and notice that $\beta \in \alpha \subseteq \gamma$, so $\beta \in \gamma$. It follows that $\bigcup A \subseteq \gamma$, and hence $\bigcup A \leq \gamma$ by Theorem 2.8. Thus, $\bigcup A = \sup A$. □

Theorem 2.13. *There is no set which contains exactly all of the ordinal numbers. In other words, **ORD** is not a set.*

Proof. Suppose that **ORD** is a set, so there is a set O such that α is an ordinal if and only if $\alpha \in O$. If this is the case, then by Theorem 2.6, O is a transitive set that is well-ordered by $\in_O$. To see that O is well-ordered by $\in_O$, notice that transitivity follows from the fact that ordinals are transitive sets, asymmetry follows from Lemma 2.8, connectedness follows from Theorem 2.9 and the fact that every non-empty subset has a least element is given by Theorem 2.10. Therefore, O is itself an ordinal, and so it follows that $O \in O$, contrary to Lemma 2.8. Hence, **ORD** is not a set. □

This final theorem seems quite paradoxical, and is appropriately called the *Burali-Forti Paradox*. Although we had to prove several theorems to prove this paradox rigorously, it is based around the less complicated idea that if there were a set of all ordinals, it would be well-ordered and hence an ordinal itself, making this ordinal an element of itself. Regardless of how the proof of the paradox is treated, we are still left in a rather difficult situation: if **ORD** is not a set, then what is it? We must introduce new terminology to encompass this collection. This terminology is quite loose however, and will not be given a formal definition. We call some collection a *class* if it has properties we can write down to determine the collection. That is, if we have a formula or statement to determine what elements are in the collection, it is a class. Then while **ORD** is not a set, it is a class, because it is defined by the properties that the collection is transitive and well-ordered by the relation $\in$. More generally, all sets are classes, but the informal definition of class avoids the restrictions of the definition of a set. A *proper class* is a class that is not also a set. **ORD** is a proper class, but the collection $\{0, 1, 2, 3\}$ is not, because it is clearly a set. A *subclass* is a collection of elements that are elements of the class under discussion. It is assumed axiomatically that any subclass of a set is a set (The Axiom of Separation) and that if $\mathbf{F}$ is a function on classes, often called a *class function*, and A is a set, then there is a set containing the image of A under $\mathbf{F}$ (The Axiom of Collection).

The reader may still be unclear on why **ORD** is not a set. It is important to remember that **ORD** is well-ordered, which we showed characterizes **ORD** in very particular ways. In particular, it implies that if **ORD** were in fact a set, it would be a well-ordered set. This, combined with the basic construction of **ORD** means that the hypothetical set of all ordinals would need to contain *itself* as an element, which makes it impossible to be understood as a set. This answer may still be inadequate, but any further investigation of this topic is far out of the scope of this paper. We direct the curious reader to Joseph Mileti's notes, which are cited in this paper's bibliography.

We will now prove some theorems about the proper class **ORD**. Mainly, we will prove that some of our results about subsets of **ORD** carry over to our new definition of subclass.

Theorem 2.14. *If $\mathbf{C}$ is a non-empty subclass of* ***ORD****, then $\mathbf{C}$ has a least element.*

Proof. Since $\mathbf{C}$ is non-empty, we may fix an ordinal $\alpha \in \mathbf{C}$. If $\mathbf{C} \cap \alpha = \emptyset$, then for any $\beta \in \mathbf{C}$, we cannot have $\beta \in \alpha$. Hence Theorem 2.10 gives that either $\alpha = \beta$ or $\alpha \in \beta$, meaning α is the least element of $\mathbf{C}$. Suppose $\mathbf{C} \cap \alpha \neq \emptyset$. In this case, $\mathbf{C} \cap \alpha$ is a non-empty subset of ordinals, and hence it has a least element δ by Theorem 2.11. It follows that δ is the least element of $\mathbf{C}$. □

Theorem 2.15. *(Induction on* ***ORD****) Suppose that $\mathbf{C} \subseteq \mathbf{ORD}$ and that for all ordinals α, if $\beta \in \mathbf{C}$ for all $\beta < \alpha$, then $\alpha \in \mathbf{C}$. Then $\mathbf{C} = \mathbf{ORD}$.*

Proof. Suppose that $\mathbf{C} \subset \mathbf{ORD}$. Let $\mathbf{B} = \mathbf{ORD} \setminus \mathbf{C}$ and notice that $\mathbf{B}$ is a non-empty class of ordinals. Then by Theorem 2.14, $\mathbf{B}$ has a least element, call it α. For all $\beta < \alpha$, we then have $\beta \notin \mathbf{B}$, hence $\beta \in \mathbf{C}$. By assumption, thus implies that $\alpha \in \mathbf{C}$, which is a contradiction. Hence $\mathbf{C} = \mathbf{ORD}$. □

Theorem 2.16. *(Limit Induction of* $\boldsymbol{ORD}$*) Suppose that* $\boldsymbol{C} \subseteq \boldsymbol{ORD}$ *and that the following propositions hold:*

1.) $0 \in \boldsymbol{C}$

2.) Whenever $\alpha \in \boldsymbol{C}$, $\alpha^+ \in \boldsymbol{C}$

3.) Whenever α *is a limit ordinal and* $\beta \in \boldsymbol{C}$ *for all* $\beta < \alpha$*, we have* $\alpha \in \boldsymbol{C}$

We then have $\boldsymbol{C} = \boldsymbol{ORD}$*.*

Proof. Suppose that $\mathbf{C} \subset \mathbf{ORD}$. Let $\mathbf{B} = \mathbf{ORD} \setminus \mathbf{C}$ and notice that $\mathbf{B}$ is a non-empty class of ordinals. By Theorem 2.14, $\mathbf{B}$ has a least element, call it α. We cannot have $\alpha = 0$, because $0 \in \mathbf{C}$. Also, it is not possible that α is a successor ordinal, because if $\alpha = \beta^+$, then $\beta \notin \mathbf{B}$, because $\beta < \alpha$. This would imply that $\beta \in \mathbf{C}$, and hence $\alpha = \beta^+ \in \mathbf{C}$, which is a contradiction. Then suppose α is a limit ordinal. Then for all $\beta < \alpha$, we have $\beta \notin \mathbf{B}$, implying that $\beta \in \mathbf{C}$. By assumption, this implies that $\alpha \in \mathbf{C}$, which is a contradiction. Hence, $\mathbf{B}$ is empty, which means $\mathbf{C} = \mathbf{ORD}$. □

We will now give two theorems without proof. Although the proofs for these theorems are not particularly difficult, they require definitions in mathematical logic that are beyond the scope of this paper. We again direct the reader to Mileti's notes for further explanation of these ideas.

Theorem 2.17. *(Recursive Definitions of* $\boldsymbol{ORD}$*) Let* $\boldsymbol{G} : \boldsymbol{V} \to \boldsymbol{V}$ *be a class function. Then there exists a unique class function* $\boldsymbol{F} : \boldsymbol{ORD} \to \boldsymbol{V}$ *such that* $\boldsymbol{F}(\alpha) = \boldsymbol{G}(\boldsymbol{F} \mid \alpha)$ *for all* $\alpha \in \boldsymbol{ORD}$*.*

Theorem 2.18. *(Recursive Definitions with Parameters on* $\boldsymbol{ORD}$*) Let* $\boldsymbol{P}$ *be a class and let* $\boldsymbol{G} : \boldsymbol{P} \times \boldsymbol{V} \to \boldsymbol{V}$ *be a class function. Then there exists a unique class function* $\boldsymbol{F} : \boldsymbol{P} \times \boldsymbol{ORD} \to \boldsymbol{V}$ *such that* $\boldsymbol{F}(p, \alpha) = \boldsymbol{G}(\boldsymbol{F}_p|\alpha)$ *for all* $p \in \boldsymbol{P}$ *and all* $\alpha \in \boldsymbol{ORD}$*.*

In these theorems and those that appear in Section 4, the notation $(\boldsymbol{F}|\alpha)$ *means the function* $\boldsymbol{F}$ *over all* $\alpha \in \boldsymbol{ORD}$*.*

As arcane and tedious as these theorems appear, what they mean for our purposes is not so difficult. Simply stated, these theorems mean that if we have a function $\mathbf{F}$ whose domain is a proper class $\mathbf{V}$, then there is a function, $\mathbf{G}$, with domain $\mathbf{ORD}$ and which maps onto $\mathbf{V}$ such that $\mathbf{F}$ can be written in terms of $\mathbf{G}$. This allows us to use ordinals for the domain of any class function. To see an example, we encourage the reader to look ahead at Definition 4.11.

With these theorems, we are now ready to prove a theorem demonstrating how useful ordinals can be. First we need a definition.

Definition 2.19. If two well ordered sets W_1 and W_2 are isomorphic, then we write $W_1 \cong W_2$. Here, W_1 and W_2 are isomorphic if there exists a bijection f such that $f : W_1 \to W_2$ and $f^{-1} : W_2 \to W_1$ are both order-preserving maps.

Theorem 2.20. *Let (W, <) be a well-ordering. There exists a unique ordinal α such that $W \cong \alpha$.*

Proof. Fix a set a such that $a \notin W$. We define a class function $\mathbf{F} : \mathbf{ORD} \to W \cup \{a\}$ recursively as follows. If $a \in ran(\mathbf{F}|\alpha)$ or $ran(\mathbf{F}|\alpha) = W$, let $\mathbf{F}(\alpha) = a$. Otherwise, $\text{ran}(\mathbf{F}|\alpha) \subset W$, and we let $\mathbf{F}(\alpha)$ be the least element of $W\setminus$ ran $(\mathbf{F}|\alpha)$.

Since **ORD** is a proper class, we have that $\mathbf{F}$ is not injective. From this it follow that $a \in \text{ran}(\mathbf{F})$, for otherwise induction shows that $\mathbf{F}$ is injective. Let α be the least ordinal such that $\mathbf{F}(\alpha) = a$. Now it follows that $\mathbf{F}|\alpha : \alpha \to W$ is an isomorphism, and uniqueness is given by the fact that if $\alpha \cong \beta$, then we must have $\alpha = \beta$. □

In this proof, we used the fact that a function from a proper class to a proper set cannot be injective. This proof requires mathematical logic outside the scope of this paper, but the idea should seem intuitively correct, given our understanding of how large a class is when compared to a set. This theorem is an important one, and will be quite useful later in the paper.

Definition 2.21. Let (W, <) be a well-ordering. The unique ordinal α such that $W \cong \alpha$ is called the *order type* of (W, <).

We will now discuss ordinal arithmetic. Although this section might seem trivial compared to the previous results, it is quite necessary, particularly in characterizing a special type of ordinal called a *limit ordinal*:

Definition 2.22. A *limit ordinal* is an ordinal with no immediate predecessor.

3. Ordinal Arithmetic

Definition 3.1. We define ordinal *addition* recursively as a function $+ :$ **ORD**$\times$ **ORD**$\longrightarrow$ **ORD** with the following properties for ordinals α and β:
1.) $\alpha + 0 = \alpha$
2.) $\alpha + \beta^+ = (\alpha + \beta)^+$
3.) If β is a limit ordinal, $\alpha + \beta = \bigcup\{\alpha + \gamma : \gamma < \beta\}$

Definition 3.2. We define ordinal *multiplication* recursively as a function $\cdot$: **ORD**$\times$ **ORD**$\longrightarrow$ **ORD** with the following properties for ordinals α and β:
1.) $\alpha \cdot 0 = 0$
2.) $\alpha \cdot \beta^+ = \alpha \cdot \beta + \alpha$
3.) If β is a limit ordinal, $\alpha \cdot \beta = \bigcup\{\alpha \cdot \gamma : \gamma < \beta\}$

Definition 3.3. We define ordinal *exponentiation* recursively as follows:
1.) $\alpha^0 = 1$
2.) $\alpha^{\beta^+} = \alpha^\beta \cdot \alpha$
3.) If β is a limit ordinal, $\alpha^\beta = \bigcup\{\alpha^\gamma : \gamma < \beta\}$

Having defined our basic operations of **ORD**, we can now prove some elementary properties of arithmetic on ordinals.

Theorem 3.4. *Let α, β, γ be ordinals. If $\beta \leq \gamma$, then $\alpha + \beta \leq \alpha + \gamma$.*

Proof. Fix ordinals α and β. We will prove by induction on γ that if $\beta \leq \gamma$, then $\alpha + \beta \leq \alpha + \gamma$. If $\gamma = \beta$, this is trivial. Suppose $\beta \leq \gamma$ and we know the result holds for γ. Then we have

$$\alpha + \beta \leq \alpha + \gamma < (\alpha + \gamma)^+ = \alpha + \gamma^+$$

This shows inductively that the theorem holds for γ with an immediate predecessor. Suppose now that γ is a limit ordinal and that $\gamma > \beta$. We then have

$$\alpha + \beta \leq \bigcup\{\alpha + \delta : \delta < \gamma\} = \alpha + \gamma$$

This proves the theorem for limit ordinals. □

Theorem 3.5. *Let α, β, and γ be ordinals. Then $\beta < \gamma$ if and only if $\alpha+\beta < \alpha+\gamma$.*

Proof. Notice that

$$\alpha + \beta < (\alpha + \beta)^+ = \alpha + \beta^+$$

Now for any $\gamma > \beta$, we have $\beta^+ \leq \gamma$, and hence

$$\alpha + \beta < \alpha + \beta^+ \leq \alpha + \gamma$$

The rest follows from Theorem 3.4.

□

Theorem 3.6. *Let α and β be ordinals. If β is a limit ordinal, then $\alpha + \beta$ is a limit ordinal.*

Proof. Since β is a limit ordinal, we have

$$\alpha + \beta = \bigcup\{\alpha + \gamma : \gamma < \beta\}$$

Suppose now that $\delta < \alpha + \beta$, and fix an ordinal γ such that $\gamma < \beta$ and $\delta < \alpha + \gamma$. We then have $\gamma^+ < \beta$ because β is a limit ordinal, and hence

$$\delta^+ < (\alpha + \gamma)^+ = \alpha + \gamma^+ \leq \alpha + \beta$$

It follows that $\alpha + \beta$ is a limit ordinal, because this inequality shows that for any δ, $\alpha+\beta$ is not the successor of δ. This means that $\alpha+\beta$ has no immediate predecessor.
□

Theorem 3.7. *Let α, β and γ be ordinals. Then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*

Proof. Fix ordinals α and β. We will prove that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all ordinals γ by induction on γ. Suppose first that $\gamma = 0$. Then

$$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0)$$

Suppose now that γ is an ordinal and we know that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. Then by induction, we have

$$(\alpha + \beta) + \gamma^+ = ((\alpha + \beta) + \gamma)^+ = (\alpha + (\beta + \gamma))^+ = \alpha + (\beta + \gamma)^+ = \alpha + (\beta + \gamma^+)$$

Finally, let γ be a limit ordinal and suppose we know that $(\alpha+\beta)+\delta = \alpha+(\beta+\delta)$ for all $\delta < \gamma$. We then have

$$\begin{aligned}(\alpha + \beta) + \gamma &= \bigcup\{(\alpha + \beta) + \delta : \delta \leq \gamma\} \\ &= \bigcup\{\alpha + (\beta + \delta) : \delta \leq \gamma\} \\ &= \bigcup\{\alpha + \epsilon : \epsilon \leq \beta + \gamma\} \\ &= \alpha + (\beta + \gamma)\end{aligned}$$

The last line follows from the fact that $\beta + \gamma$ is a limit ordinal.

□

We can now begin our discussion of another type of number which can also be seen as an extension of the natural numbers. However, we will proceed in the construction of these new numbers, called *cardinal numbers*, quite differently than how we constructed the ordinals.

4. Cardinal Numbers

Definition 4.1. A *cardinal* is an ordinal α such that $\alpha \ncong \beta$ for any $\beta < \alpha$.

We can see just from this definition that we have already encountered many cardinal numbers

Theorem 4.2. *Every natural number is a cardinal, and ω is a cardinal.*

Proof. Clearly for any natural number n, $n \ncong m$ for $m < n$, since m and n are both finite sets with a different number of elements, and thus cannot have a bijection between them. In the case of ω, the ordinals less than ω are just the natural numbers, which are finite. Because a finite set cannot be put in bijection with an infinite set, we have $n \ncong \omega$ for all natural numbers n, which are precisely the ordinals less that ω.. □

If all the natural numbers and ω are cardinal numbers, then what about the successor of ω, namely $\omega + 1$? Well, is it the case that for every ordinal α less than $\omega + 1$ we have $\alpha \ncong \omega + 1$? If we consider ω, then we see that this is not the case, since there exists an isomorphism between ω and $\omega + 1$. To see this, recall the definition of ω and $\omega + 1$:

$$\omega = \{0, 1, 2, 3, 4, ...\}$$
$$\omega + 1 = \{0, 1, 2, 3, 4, ..., \omega\}$$

Then a suitable map $f : \omega \longrightarrow \omega + 1$ goes as follows: $f : 0 \longmapsto \omega$ and for every other element n of ω, $f : n \longmapsto n - 1$. Then there is an isomorphism between ω and $\omega + 1$, so $\omega + 1$ is not a cardinal. This type of analysis will allow us to develop isomorphisms between ω and many conceivable ordinals, like $\omega + \omega = \omega 2$, $\omega \cdot \omega = \omega^2$. We will next prove a theorem to show exactly what subclass of **ORD** are cardinals.

Theorem 4.3. *Every infinite cardinal is a limit ordinal.*

Proof. If this were not true, then there would exist an infinite cardinal, α ,which is not a limit ordinal. Because α is not a limit ordinal, it has an immediate predecessor. Then there exists β such that $\beta^+ = \alpha$. This is equivalent to $\beta + 1 = \alpha$ with α and β infinite, so $\alpha \cong \beta$, and $\beta < \alpha$. This contradicts the definition of a cardinal. □

We will now begin an inquiry into some examples of cardinals. However, we must first lay some seemingly unrelated groundwork.

Definition 4.4. We say that two sets A and B have the same *cardinality* if there exists a bijection between A and B. For a set A, the equivalence class of sets under bijection is called the *cardinality* of A, and is denoted by $|A|$.

The second part of this definition implies that two sets, A and B, are in bijection if and only if $|A| = |B|$. This consequence is consistent with the first part of the definition. Thus, we can discuss cardinality of a particular set, or compare cardinalities of several sets.

Definition 4.5. We write $|A| \leq |B|$ if there exists an injection from A into B.

Theorem 4.6. *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there is an injection from A into B and an injection from B into A, then there is a bijection from A into B.*

Proof. Let f be an injective mapping of X into Y and let g be an injective mapping of Y into X. Our goal is to find a one-to-one correspondence between X and Y. We can assume that $X \cap Y = \emptyset$, since if this is not true, we can easily match elements common to both sets to one another.

Hence, assume without loss of generality that $X \cap Y = \emptyset$. We will call an element x of X the *parent* of the element $f(x)$ in Y, and similarly, $y \in Y$ is the parent of $g(y) \in X$. Each element $x \in X$ has an infinite sequence of *descendants*, namely $f(x)$, $g(f(x))$, $f(g(f(x)))$ and so on. Similarly, every $y \in Y$ has descendants $g(y)$, $f(g(y))$, $g(f(g(y)))$ and so on. This definition implies that each term in the sequence is a descendant of all preceding terms. We will also say that each term in the sequence is an *ancestor* of all following terms.

For each element in either X or Y, one of three things must happen. If we keep tracing the ancestry of the element back as far as possible, then either we ultimately come to an element of X that has no parent and is consequently an element of $X \setminus g(Y)$, or we come to an element of Y with no parent and consequently an element of $Y \setminus f(X)$, or the regression continues ad infinitum.

Let X_X be the set of elements of X that originate in X, that is, the set X_X consists of the elements of $X \setminus g(Y)$ together with all their descendants in X. Let X_Y be the set of elements of X that originate in Y, that is X_Y consists of all the descendants in X of the elements of $Y \setminus f(X)$. Let X_∞ be the set of those elements of X which have no parentless ancestor, that is, elements of X which cannot be traced back to an ancestor without a parent. Partition Y similarly into Y_X, Y_Y, Y_∞.

If $x \in X_X$, then $f(x) \in Y_X$, so the restriction of f to X_X is a one-to-one correspondence between X_X and Y_X. If $x \in X_Y$, then x belongs to the domain of the inverse function g^{-1} and $g^{-1}(x) \in Y_Y$, so the restriction of $g^{-1}(x)$ to X_Y is a one-to-one correspondence between X_Y and Y_Y. If $\in X_\infty$, then $f(x) \in Y_\infty$, so the restriction of f to X_∞ is a one-to-one correspondence between X_∞ and Y_∞. By combining these three one-to-one correspondences, we obtain our desired one-to-one correspondence between X and Y. □

This important result is called the *Schroeder-Bernstein Theorem.* In addition to being a very strong and profound result in itself, it will be useful in our later characterizations of cardinal numbers.

Recall that an ordinal α is defined as a set containing all predecessors of α. With this understanding of ordinals in mind, we can now return to cardinal numbers.

Theorem 4.7. *Let A be a set. Then there is an ordinal α such that $|\alpha| \not\leq |A|$.*

Proof. Let $F = \{(B, R) \in P(A) \times P(A \times A) : \text{R is a well-ordering on } B\}$. By our Axioms of Separation and Collection discussed in the remarks after Theorem 2.13, $A = \{\text{order-type } (B, R) : (B, R) \in F\}$ is a set of ordinals. Let α be an ordinal such that $\alpha > \bigcup A$. Such an ordinal exists because **ORD** is a proper class. Notice that $|\alpha| \not\leq |A|$ because if $f : \alpha \to A$ were an injection, we could let $B = ran(f)$ and let R

be the well-ordering on B obtained by transferring the order of α. We would then have $\alpha \in A$, since $(B, R) \in F$ and (B, R) has order-type α. This is a contradiction, hence $|\alpha| \not\leq |A|$. □

Definition 4.8. Let A be a set. The least ordinal α such that $|\alpha| \not\leq |A|$ is called the *Hartogs Number* of A, and is denoted $H(A)$.

Theorem 4.9. *For every set A, $H(A)$ is a cardinal.*

Proof. Let A be a set and let $\alpha = H(A)$. Suppose that $\beta < \alpha$ and $\alpha \cong \beta$. Let $f : \alpha \to \beta$ be a bijection guaranteed by the assumption that $\alpha \cong \beta$. Since $\beta < \alpha = H(A)$, there exists an injection $g : \beta \to A$. We then have that $g \circ f : \alpha \to A$ is an injection. This contradicts $|\alpha| \not\leq |A|$. It follows that $\alpha \not\cong \beta$ for any $\beta < \alpha$, so $\alpha = H(A)$ is a cardinal. □

Definition 4.10. If κ is a cardinal, we define κ^+ to be $H(\kappa)$. In other words, the successor of κ is $H(\kappa)$.

Definition 4.11. We define $\aleph_\alpha$ for $\alpha \in \mathbf{ORD}$ as:
1.) $\aleph_0 = \omega$
2.) $\aleph_{\alpha+1} = \aleph_\alpha^+$
3.) $\aleph_\alpha = \bigcup\{\aleph_\beta : \beta < \alpha\}$ if α is a limit ordinal

Theorem 4.12. *If α is an ordinal, then $\alpha \leq \aleph_\alpha$*

Proof. We will prove this by induction on **ORD**. If $\alpha = 0$, then $\aleph_\alpha = \aleph_0 = \omega$, and clearly $0 \leq \omega$. Assume the statement holds for α. Then we want to show that $\alpha + 1 \leq \aleph_{\alpha+1}$. Well, $\aleph_{\alpha+1} = \aleph_\alpha^+$. Then since we have $\alpha \leq \aleph_\alpha$, it follows that:

$$\alpha^+ \leq \aleph_\alpha^+ \implies \alpha + 1 \leq \aleph_{\alpha+1}$$

If α is a limit ordinal, then assume the theorem holds for all $\beta < \alpha$. Since $\alpha = \bigcup \beta$, we have $\bigcup \beta \leq \sum_{\beta<\alpha} \aleph_\beta$. From Theorem 5.3, we have $\bigcup \beta \leq \sum_{\beta<\alpha} \aleph_\beta \leq \aleph_\alpha$, which gives $\alpha \leq \aleph_\alpha$. By Theorem 2.16, our proof by induction is complete. □

This proof relies heavily on ideas not presented until section 5. While this might seem frustrating to the reader, sections 4 and 5 present two different, though equally compelling ways to understand cardinals. Hence, it is useful to draw on these differing approaches when proving theorems such as this one. We assure the reader that the proof of Theorem 5.3 does not use Theorem 4.12 in any way. Also, the statement $\bigcup \beta \leq \sum_{\beta<\alpha} \aleph_\beta$ will be justified in the first lines of section 5.

Theorem 4.13. *For ordinals α and β, if $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.*

Proof. We will prove this using induction on $\alpha \in \mathbf{ORD}$. For $\alpha = 0$, we must consider $\aleph_0$. Well, for every $\beta > 0$, we have $\aleph_\beta \geq H(\aleph_0) > \aleph_0$. Thus, consider the case where α is a successor ordinal. If we have $\gamma^+ = \alpha$ and $\gamma < \beta \Rightarrow \aleph_\gamma < \aleph_\beta$, then we clearly have $\alpha < \beta + 1 \Rightarrow H(\aleph_\gamma) < H(\aleph_\beta)$. This gives that

$$\alpha < \beta + 1 \Rightarrow \aleph_\alpha < \aleph_{\beta+1}$$

which proves the theorem for successor ordinals.

To see the case where α is a limit ordinal, assume the theorem holds for all $\kappa < \alpha$. Then we have $\alpha = \bigcup_{\kappa_i < \alpha} \kappa_i$, and for each κ_i, $\kappa_i < \beta_i \Rightarrow \aleph_{\kappa_i} < \aleph_{\beta_i}$. Then we have

$$\bigcup_{\kappa_i < \alpha} \kappa_i < \bigcup_{\kappa_i < \beta_i} \beta_i \Rightarrow \sum_{\kappa_i < \alpha} \aleph_{\kappa_i} < \sum_{\kappa_i < \beta_i} \aleph_{\beta_i}.$$

This gives us that

$$\alpha < \bigcup_{\kappa_i < \beta_i} \beta_i \Rightarrow \sum_{\kappa_i < \alpha} \aleph_{\kappa_i} < \sum_{\kappa_i < \beta_i} \aleph_{\beta_i}$$

Let $\beta = sup_{\kappa_i < \beta_i}\{\beta_i\} = \bigcup_{\kappa_i < \beta_i} \beta_i$ (the second equality follows from Theorem 2.12). Since $\beta_i > \kappa_i$ for every $\kappa_i < \alpha$, Theorem 5.3 gives us:

$$\alpha < \beta \Rightarrow \aleph_\alpha < \aleph_\beta$$

This proves the theorem for limit ordinals, which concludes our proof by induction. □

Theorem 4.14. *Let κ be an ordinal. Then κ is an infinite cardinal if and only if there exists $\alpha \in$ **ORD** with $\kappa = \aleph_\alpha$.*

Proof. ($\Rightarrow$)We will prove that $\aleph_\alpha$ is an infinite cardinal for all $\alpha \in$ **ORD** by induction. Notice that $\aleph_0 = \omega$ is an infinite cardinal by Theorem 4.3. Also, if $\aleph_\alpha$ is a cardinal, then $\aleph_{\alpha+1} = \aleph_\alpha^+ = H(\aleph_\alpha)$ is a cardinal by Theorem 4.10. Suppose then that α is a limit ordinal and that $\aleph_\beta$ is a cardinal for all $\beta < \alpha$. Notice that $\aleph_\alpha$ is an ordinal by Theorem 2.12. Suppose that $\gamma < \aleph_\beta$. Then $\beta + 1 < \alpha$, since $\beta < \alpha$ and α is a limit ordinal. Since $\aleph_{\beta+1} \not\precsim \aleph_\beta$, it follows that $\aleph_{\beta+1} \not\precsim \gamma$, so $\aleph_\alpha \not\precsim \gamma$. Therefore, $\aleph_\alpha \not\cong \gamma$ for any $\gamma < \aleph_\alpha$, thus $\aleph_\alpha$ is a cardinal.

($\Leftarrow$)Suppose that κ is an infinite cardinal. By Theorem 4.13, we have $\kappa \leq \aleph_\kappa$. If $\kappa = \aleph_\kappa$, we are done. Suppose then that $\kappa < \aleph_\kappa$ and let α be the least ordinal such that $\kappa < \aleph_\alpha$. Notice that $\alpha \neq 0$ because κ is infinite and α cannot be a limit ordinal. For if α is a limit ordinal, $\kappa < \aleph_\beta$ for some $\beta < \alpha$. Thus, there exists β such that $\alpha = \beta^+$. By our choice of α, we have $\aleph_\beta \leq \kappa$. If $\aleph_\beta < \kappa$, then $\aleph_\beta < \kappa < \aleph_{\beta^+} = H(\aleph_\beta)$, contradicting the definition of $H(\aleph_\beta)$. It follows that $\kappa = \aleph_\beta$. □

At this point, we can now make highly precise and rigorous the ideas presented in context of the Schroeder-Bernstein Theroem. Namely, for a given set A, we can give a numerical definition to $|A|$.

Theorem 4.15. *Let A be a set. There exists an ordinal α such that $A \cong \alpha$ if and only if A can be well-ordered.*

Proof. ($\Rightarrow$) Suppose there exists an ordinal α such that $A \cong \alpha$. We use a structure-preserving bijection between A and α to transfer the ordering on the ordinals to an ordering on A. Let $f : A \to \alpha$ be such a bijection. Define a relation $<$ on A by letting $a < b$ if and only if $f(a) < f(b)$. Then because $(\alpha, \in_\alpha)$ is a well-ordering, we clearly have $(A, <)$ is a well-ordering.

($\Leftarrow$) Suppose that A is well-ordered. Fix a relation $<$ on A so that $(A, <)$ is a well-ordering. By Theorem 2.19, there is an ordinal α such that $A \cong \alpha$. □

Definition 4.16. Let A be a set that can be well-ordered. We define $|A|$ to be the least ordinal α such that $A \cong \alpha$.

Lemma 4.17. *If A can be well-ordered, then $|A|$ is a cardinal.*

Proof. A can be well-ordered, so by Theorem 4.16, there is an ordinal α such that $\alpha \cong A$. Many ordinals α could have the property that $\alpha \cong A$. Then take the least element of this set and call it β. This number exists because **ORD** is well-ordered, and is also the definition of $|A|$. Clearly $\beta \ncong \alpha$ for any $\alpha < \beta$, or else β would not be the least α such that $\alpha \cong A$. Thus, $|A| = \beta \ncong \alpha$ for any $\alpha < |A|$. Thus, $|A|$ is a cardinal. □

5. Cardinal Arithmetic

In this section, we will lay some framework for arithmetic between cardinals to introduce a particularly strange type of cardinal, called an *inaccessible cardinal*. We will discuss cardinal arithmetic in terms of cardinal numbers and the sets which are described by that cardinal as shown in Definition 4.17

Definition 5.1. Let I be a set and κ_i for $i \in I$ a collection of cardinals. Let $\{A_i : i \in I\}$ be a family of disjoint sets such that, for each i, $|A_i| = \kappa_i$. Recall that that the *Cartesian product* of the family $\{A_i : i \in I\}$ is

$$\prod_{i \in I} A_i = \{f : I \to \bigcup_{i \in I} A_i : \forall i, f(i) \in A_i\}$$

Define *cardinal addition* by

$$\sum_{i \in I} \kappa_i = |\bigcup_{i \in I} A_i|$$

Define *cardinal multiplication* by

$$\prod_{i \in I} \kappa_i = |\prod_{i \in I} A_i|$$

Definition 5.2. For a set I and cardinals κ and γ such that $\gamma = |I|$, we define *cardinal exponentiation* by

$$\kappa^\gamma = \prod_{i \in I} \kappa$$

It is clear from these definitions that for cardinals κ and γ that $\kappa + \gamma = \gamma + \kappa$ and $\kappa \cdot \gamma = \gamma \cdot \kappa$. The following theorem give us a strong characterization of cardinal addition and multiplication for infinite cardinals.

Theorem 5.3. *Let κ and γ be cardinals, at least one of which is infinite. Then $\kappa + \gamma = \kappa \cdot \gamma = max\{\kappa, \gamma\}$.*

Proof. Let $a = \max\{\kappa, \gamma\}$ and suppose that γ is infinite. Let A and B be disjoint sets such that $|A| = \kappa$ and $|B| = \gamma$. Since $\kappa \leq a$, and $\gamma \leq a$, it follows that $\kappa + \gamma \leq a + a = a$. Also, since $a \leq |A \cup B|$, we have $a \leq \kappa + \gamma$. Thus, since all cardinals, including κ and γ, are ordinals and thus antisymmetric, we have $\kappa + \gamma = a = \max\{\kappa, \gamma\}$. To see the multiplicative case, notice that $\kappa \cdot \gamma \leq a \cdot a = a$, using the same reasoning as before. Also, $a \leq \prod A \times B = \kappa \cdot \gamma$. Hence, we have $\kappa \cdot \gamma = a = \max\{\kappa, \gamma\}$. □

The reader should have noticed that this theorem used the fact that for an infinite cardinal a, $a + a = a \cdot a = a$. This has not been proven, and its proof requires either a further investigation of well-ordering on ordinal tuples, or the use of Zorn's Lemma. We direct the reader to pages 96 and 97 of *Naive Set Theory* by Paul Halmos, or pages 164 and 165 of Mileti's notes for further reading.

We now know how to get from one infinite cardinal to a larger one, or at least how it cannot be done: we cannot pass from an infinite cardinal to a larger one through finite addition or finite multiplication by cardinals of the same or smaller size. However, the following important theorem, often called *Cantor's Theorem*, shows how we *can* move to a larger infinite cardinal.

Theorem 5.4. *Recall that the power set of a set A is the collection of all subsets of A and is denoted 2^A. For every set A, A has a smaller cardinality than its power set. That is, for all sets A, $|A| < |2^A|$.*

Proof. There is an injection of A into 2^A, namely the mapping that associates with every $x \in A$ the singleton $\{x\} \in 2^A$. So $|A| \leq |2^A|$. We must now show that $|A| \neq |2^A|$.

Assume that $f : A \to 2^A$ is a bijection. Consider $X = \{x \in X : x \notin f(x)\}$. In other words, X is the set of those elements of A that are not contained in $f(x)$, which is a subset of the power set of A. Since $X \in 2^A$ and since f maps A into 2^A, there exists an an element $a \in A$ such that $f(a) = X$. The element a is either an element of X or it is not an element of X. If $a \in X$, the by the definition of X, we must have $a \notin f(a)$, and since $f(a) = X$, this is impossible. If $a \notin X$, then again, by the definition of X, we must have $a \in f(a)$, and this too is impossible. We thus have a contradiction, since we proved that $a \in X$ and $a \notin X$ and both impossible. Thus, f cannot be a bijection, so $|A| \neq |2^A|$. Hence, $|A| < |2^A|$. □

To conclude this section, we will enter a brief investigation into a theoretical curiosity. Do there exist cardinal numbers that cannot be reached even by considering the power set of a set? What about cardinals that cannot be reached through infinite addition or multiplication of smaller cardinals? If these cardinals do exist, what properties do they have? These possibilities have been considered, and these hypothetical cardinals are called *inaccessible cardinals*. We say that these cardinals are hypothetical because their existence is independent of the traditionally accepted axioms of set theory. This proof is quite difficult and out of the scope of this paper.

Definition 5.5. For a cardinal ξ, ξ is *strongly inacessible* if

1.) ξ is not the sum of fewer, smaller cardinals
2.) $\forall \kappa,\ \kappa < \xi \Longrightarrow 2^\kappa < \xi$

To see how strange these cardinals are, we give one theorem that characterizes inaccessible cardinals.

Theorem 5.6. *If ξ is a strongly inaccessible cardinal and $\xi = \aleph_\alpha$, then $|\alpha| = \xi$.*

Proof. First assume α is not a limit ordinal. Let β be the immediate predecessor of α, that is, $\beta^+ = \alpha$. Then if $|\alpha| < \xi$, $\aleph_\beta < \xi$. Well, if we take the power set of the set that has cardinality $\aleph_\beta$, call it B, then because ξ is inaccessible, we have $|2^B| < \xi$. But $|2^B| \geq \aleph_{\beta+1} = \aleph_\alpha$. This gives us $\aleph_\alpha < \xi$, which is a contradiction.

Then consider if α is a limit ordinal. Then if $|\alpha| < \xi$, consider some ordinal $\beta < \alpha$, but with $\beta > \gamma$, where γ is the largest limit ordinal less than α. In other words, β is in between α and the largest limit ordinal less than α. Since $\beta < \alpha$, $\aleph_\beta < \xi$. Consider a sequence of ordinals $I = \{\beta, \beta + i_1, \beta + i_2, ...\}$ such that $\beta + i_n < \beta + i_{n+1}$ and such that this sequence converges to α. Then consider the sequence of cardinals $\{\aleph_\beta, \aleph_{\beta+i_1}, \aleph_{\beta+i_2}, ...\}$ indexed over I such that this sequence converges to $\aleph_\alpha$. We also define a family of sets $\{B_{\beta+i_n}\}$ such that for all $i_n \in I$, $B_{\beta+i_n} \subseteq B_{\beta+i_{n+1}}$ and for each i_n, $|B_{\beta+i_n}| = \aleph_{\beta+i_n}$. Because ξ is inaccessible and for each i, $\aleph_{\beta+i_n} < \xi$, we have

$$\sum_{i_n \in I} \aleph_{\beta+i_n} < \xi.$$

But by the definition of cardinal addition, this implies

$$|\bigcup_{i_n \in I} B_{i_n+\beta}| < \xi$$

Since α is a limit ordinal, $\alpha = \bigcup\{\beta : \beta < \alpha\}$. We know that the sequence $\{|B_{\beta+i_n}|\}$ increases monotonically over $I \subset \mathbf{ORD}$, and that all of our i_n's are ordinals, so $\bigcup_{i_n \in I} B_{i_n+\beta} = B_\alpha$. But $|B_\alpha| = \aleph_\alpha$, which implies that $\aleph_\alpha < \xi$, which is a contradiction. Thus, we must have $|\alpha| = \xi$. □

6. Cardinality of Sets

We conclude this paper with an investigation into a topic alluded to in Definitions 4.5 and 4.6. Cardinal numbers are closely linked with the "sizes" of sets. In the finite case, a cardinal number is associated with the number of elements in a set. However, we can use cardinals to extend the notion of set size to infinite sets, as we already discussed. We can now prove the cardinalities of some familiar sets.

Recall that $\aleph_0$ was the smallest infinite cardinal, which was equal to ω. If a set has cardinality $\aleph_0$, we say the set is *countable*. If a set has cardinality greater than $\aleph_0$, we say it is *uncountable*. This terminology comes from the fact that if a set has cardinality $\aleph_0$, it can be put in bijection with $\mathbb{N}$, the set of natural numbers. Hence, we can match each element of the set with a natural number and "count" the elements of the set.

Theorem 6.1. *$\mathbb{Z}$ has cardinality $\aleph_0$.*

Proof. Define a bijection $f : \mathbb{Z} \to \mathbb{N}$ by

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ 2|n| + 1 & \text{if } n < 0 \end{cases}$$

Then $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$.

□

Theorem 6.2. *$\mathbb{Q}$ has cardinality $\aleph_0$.*

Proof. We clearly have an injection $f : \mathbb{N} \to \mathbb{Q}$ given by inclusion, so it suffices to show that $|\mathbb{Q}| \leq \aleph_0$. We define an injection $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by mapping a rational number $\dfrac{n}{m} \to (n, m)$, where we make sure the map is well-defined by regarding all rational numbers only in reduced form, that is, n and m have no common factors, and by regarding only rational numbers with a positive denominator. Under these conditions, we obtain that $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}|$. But $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$ by Theorem 6.1, so $|\mathbb{Z} \times \mathbb{N}| = \aleph_0$ by Corollary 5.4. Thus, we have $\aleph_0 \leq |\mathbb{Q}| \leq \aleph_0$, so $|\mathbb{Q}| = \aleph_0$. □

We have shown that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$. However, it is not hard to find a set with cardinality greater than $\aleph_0$.

Theorem 6.3. $[0, 1] \subset \mathbb{R}$, *has cardinality greater than* $\aleph_0$.

Proof. If $x \in [0, 1] \subset \mathbb{R}$, then x can be written as a binary decimal expansion $0.a_1a_2a_3...a_n...$, where for every i, a_i is either 0 or 1. Then since this expansion has a term for each natural number, the cardinality of the set of all of these decimal expansions is $2^\omega = 2^{\aleph_0}$. Well, we know from Theorem 5.5 that for every set, $|A| < |2^A|$. This gives that $|\mathbb{N}| < |2^\mathbb{N}|$, which implies $\aleph_0 < 2^{\aleph_0}$. We showed that $|[0, 1]| = 2^{\aleph_0}$, so we have $\aleph_0 < |[0, 1]|$. □

Corollary 6.4. $|\mathbb{R}| > \aleph_0$

It is interesting to consider what exactly the cardinality of $|[0, 1]|$ or $|\mathbb{R}|$ is. For although we showed these sets have cardinality $2^\aleph_0$, (It is easy to prove this for $\mathbb{R}$ using Theorem 6.3) this is not a cardinal of the form we have investigated. Is it the case that $2^{\aleph_0} = \aleph_1$? More generally, is it the case that for any ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$? The statement of equality in this second statement is called *The General Continuum Hypothesis*, and its validity has been proven to be independent of the accepted axioms of set theory. That is, it is not known to be true or false, because both possibilities do not contradict the normally accepted foundations of set theory. This proof, like the similar statement regarding inaccessible cardinals, is out of the scope of this paper.

As a final interesting bit, which illustrates another paradox in working with cardinal and ordinal numbers, consider the collection C of all cardinal numbers. We have shown that there is no largest ordinal number in Theorem 2.13, and we will now prove a similar result for cardinal numbers.

Theorem 6.5. *There is no largest cardinal number.*

Proof. Assume this is not true. Then there is a largest cardinal, $\aleph_\tau$. Let A be a set such that $|A| = \aleph_\tau$. Then by Theorem 4.6, $|A| < |2^A|$. Thus, we cannot have $\aleph_\tau$ be the largest cardinal, since $|2^A|$ has a cardinality larger than $\aleph_\tau$. Hence, there is no largest cardinal. □

This final theorem, often referred to as *Cantor's Paradox*, emphasizes once more just how large these transfinite sets **ORD** and C are. Since there is no largest cardinal and the cardinals are indexed by the ordinals, we have a bijection between **ORD** and C. Thus, the cardinals also form a proper class.

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References

[1] Naive Set Theory. Paul R. Halmos. D. Van Nostrand Company. 1961.
[2] Transfinite Combinatorics: Lecture 1. Lázsló Babai. 2007.
[3] Mathematical Logic for Mathematicians. Joseph R. Mileti. 2007.