AN INTRODUCTION TO SCHEMES

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Abstract. This paper serves as an introduction to the world of schemes used in algebraic geometry to the reader familiar with differentiable manifolds. After the basic definitions and constructions are motivated and laid out, an interesting result will be given that emphasizes the importance of such devices. The aim is to introduce some of the ideas, rather than work through theorems in detail.

Contents

1. A little motivation

The goal of this paper is to introduce the reader to the concept of schemes that is used widely in modern algebraic geometry. It will use the category of smooth manifolds as the primary motivation and analogy. Therefore it is assumed that the reader has some grasp of this subject. It is also assumed that the reader is familiar with the varieties of classical algebraic geometry as they serve as central objects of study in the subject and will appear from time to time in this paper. To prevent redundancy, all rings used in this paper are assumed to be commutative. The author would also like to thank his mentor for this project, Tom Church, for his help in choosing this topic and his guidance through it.

2. What are schemes?

There are really two parts to a scheme: a topological space, and a thing called the *structure sheaf* which we think of as functions on the space, all of which is subject to a few conditions regarding compatibility.

For the sake of analogy, let's consider manifolds. Topologically, a manifold M is a space that is "locally Euclidean," that is, there is an open cover $\{U_i\}$ of M such that each U_i is homeomorphic to some $\mathbb{R}^n$. In the smooth category though, this is not enough, the coordinate patches must satisfy compatibility conditions that allow us to define what we mean by a smooth function. In fact, we may have several essentially distinct smooth structures on a topological manifold, for example, exotic spheres. Another equivalent way to define a manifold is sort of by declaring which functions are to be smooth, and then demanding that locally this looks like the smooth functions on $\mathbb{R}^n$. We will come back to this second way later.

Thus, before we can define what a general scheme is, we need objects that will play the role that the $\mathbb{R}^n$'s do in the manifold categories. These are spaces associated to a ring R.

3. Affine Schemes

Definition 3.1. The *spectrum* of a commutative ring R, is the set of prime ideals in R, and is denoted by $Spec(R)$

Classically, we have a natural identification of the maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ and points in $\mathbb{C}^n$. These points are still here, but even for a polynomial ring over $\mathbb{C}$, we've just added in a bunch of extra points. Why? One reason is that if we have a homomorphism of rings $f : R \to S$, we want the *Spec* operation to give us a map f_* relating $Spec(R)$ and $Spec(S)$. A natural way to do this would be to define $f_*(p) = f^{-1}(p)$, which will be a prime ideal in R, but not necessarily a maximal one, even if p is maximal. Thus, we need all prime ideals to be included. The non-maximal points correspond to varieties.

Note that f_* goes the opposite direction of f: from $Spec(S)$ to $Spec(R)$.

We never really think of this as just a set, though. We equip it with a topology, making it a topological space, the first part of a scheme.

Definition 3.2. The closed sets of $Spec(R)$ are the sets of the form $V(I) = \{p \in Spec(R) | I \subseteq p\}$, where I is any ideal in R. This is called the Zariski topology on $Spec(R)$

It is fairly easy to verify that this is in fact a topology, at least given the facts that $V(I) \cap V(J) = V(I + J)$ and $V(IJ) = V(I) \cup V(J)$.

The map f_* we had from before is now a continuous map from $Spec(S)$ to $Spec(R)$.

However, this topology is often very far from being a nice one: it's not usually Hausdorff, or even T_1. In fact, the closed points will be exactly the maximal ideals in R, since $V(p)$ is the closure of $\{p\}$.

There are certain open sets of a spectrum that play a key role, we introduce them now.

Definition 3.3. The *distinguished open sets* of a ring R are the open sets of the form $Spec(R)_f = \{P \in Spec(R) | f \notin P\} = Spec(R) \setminus V(f)$.

These sets form a basis for the Zariski topology on $Spec(R)$.

Now we have a topological space to work with. We're not done yet, though. One of the central ideas of algebraic geometry is thinking of rings as sets of "functions" on certain spaces, namely their spectra. Also, in analogy with smooth manifolds, we really care about the "functions" more than the space, since they may not be completely determined by the space. So we need a way to think of R as functions on $Spec(R)$, and here's where things get complicated.

To really grasp what we mean by functions on a space, we need the notion of a sheaf.

Definition 3.4. A *sheaf of rings* $\mathcal{O}_X$ on a topological space X is an assignment of a ring $\mathcal{O}_X(U)$ to each open set U in X, together with, for each inclusion $U \subseteq V$ a *restriction homomorphism* $res_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$, subject to the following conditions:

- $res_{U,U} = id_U$
- If $U \subseteq V \subseteq W$, then $res_{V,U} \circ res_{W,V} = res_{W,U}$
- For each open cover $\{U_\alpha\}$ of $U \subseteq X$ and for each collection of elements $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that for all α, β, if $res_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = res_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$, then there is a unique $f \in \mathcal{O}_X(U)$ such that for all α $f_\alpha = res_{U,U_\alpha}(f)$

We think of the elements of $\mathcal{O}_X(U)$ as "functions" defined on U. The restriction homomorphisms correspond to restricting a function on a big open set to a smaller one. Intuitively, the axioms say that these elements behave as functions should:

- Restricting a function to its original domain does nothing at all.
- Restricting, and then restricting again is the same as restricting all at once.
- If we have functions defined on some different open sets, and these functions agree on the overlaps, then we can glue them all together to get a unique function on the union of these open sets, and if we restrict this glueing to one of the open sets, we get the corresponding function back.

Definition 3.5. A topological space equipped with a sheaf of rings on it is called a *ringed space*.

For examples, consider the manifold category.

Example 3.6. Let M be a smooth manifold. Then for each open set U of M, we have $C(U)$, the set of real-valued continuous functions on U. Under point-wise addition and multiplication, this is a ring. If $V \subseteq U$ then we have the restriction homomorphism $C(U) \to C(V)$ given by actually restricting functions. It is easy to verify that this in fact a sheaf. Together with the next example, this is one of the prototypical examples of a sheaf, and serves as a basis for much of the intuition.

Example 3.7. With M still a smooth manifold, consider the sets $C^\infty(U)$ of C^∞ real-valued functions on U, where U is an open set in M. These are still closed under pointwise addition and multiplication, and the same restriction maps as above still work. It is easy to verify that this, too, is a sheaf. In fact, it is a "subsheaf" of the previous example.

One interesing thing about smooth manifolds is that they can also be defined the other way around; instead of defining them as a topological space with a certain open cover satisfying some conditions, and then deriving a sheaf from that, we can simply define them as a space together with a sheaf satisfying a similar property. This definition is equivalent to the coordinate charts definition, and will function as motivation for our later definition of general schemes.

Definition 3.8. A *smooth manifold* is a topological space, together with a sheaf of real-valued continous functions, subject to the condition that there exists an open covering $\{U_\alpha\}$, with the restriction sheaf to each U_α is isomorphic to some $\mathbb{R}^n$. Here $\mathbb{R}^n$ is equipped with the sheaf of standard differentiable functions.

In the sequel, we will abuse notation and often write just a space X, Y, etc. when we really mean that space together with a given sheaf of rings on it. Precisely which sheaf of rings will be clear from context.

There are other sorts of sheaves too, for instance sheaves of abelian groups, and more generally, of R-modules over some ring R, defined in exactly the same way, but we won't consider them here.

We should also define what a map between sheaves is.

Definition 3.9. Let X be a topological space, and $\mathcal{O}_X, \mathcal{O}'_X$ be two sheaves of rings on X. Then a *morphism* $\varphi : \mathcal{O}_X \to \mathcal{O}'_X$ is a collection of ring homomorphisms $\varphi(U) : \mathcal{O}_X(U) \to \mathcal{O}'_X(U)$, one for each open set $U \subseteq X$, which commute with the restriction maps. That is, if $V \subseteq U \subseteq X$ are open sets, then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\varphi(U)} & \mathcal{O}'_X(U) \\ {\scriptstyle res_{U,V}}\downarrow & & \downarrow{\scriptstyle res_{U,V}} \\ \mathcal{O}_X(V) & \xrightarrow[\varphi(V)]{} & \mathcal{O}'_X(V). \end{array}$$

There are also a few ways to create a sheaf from an existing one.

Given a sheaf $\mathcal{O}_X$ on a space X and an open subset $U \subseteq X$, we can naturally define what it means to restrict $\mathcal{O}_X$ to U.

Definition 3.10. If we have a sheaf $\mathcal{O}$ on a space X and U an open subset of X we can define a sheaf $\mathcal{O}|_U$ on U by taking $\mathcal{O}|_U(V) = \mathcal{O}(V)$, for any open subset V of U, and by keeping the same restriction maps. This will clearly be a sheaf if $\mathcal{O}$ is.

We can also push sheaves forward along continous functions.

Definition 3.11. Let X and Y be topological spaces, $\mathcal{O}_X$ a sheaf on X, and $f : X \to Y$ be a continuous function. We define the pushforward sheaf $f_*\mathcal{O}_X$ on Y by declaring $f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$ for any open set U in Y, with the obvious restriction maps. It is eay to check that this will be a sheaf.

One more way of constructing sheaves deserves a comment. If we have a sheaf on X and a basis of open sets for X, then the sheaf is completely determined by its values on the basis. That is, to define a sheaf on X, it is enough to declare what it does on a basis and check that it is a sheaf. Once we have done so, we are assured of this sheaf being entirely defined. This is not entirely obvious, but we will not prove it here.

We use this fact to define a certain natural sheaf of rings on the space $Spec(R)$ via defining it on the distinguished open sets.

Definition 3.12. The *structure sheaf* of $Spec(R)$ is the scheme $\mathcal{O}_{Spec(R)}$ defined by $\mathcal{O}_{Spec(R)}(Spec(R)_f) = R_f$, the localization of R at the element f.

Again, we will abuse notation and from now on just say $Spec(R)$ when we mean the set of prime ideals in R together with the Zariski topology and this sheaf of rings on it.

Spectra of rings are our first example of a scheme, and will play the same part in defining general schemes as $\mathbb{R}^n$ do in defining smooth manifolds. Thus, we give them a name.

Definition 3.13. A spectrum of a ring (with the sheaf of rings defined above) is also called an *affine scheme*.

Now, we offer a few examples.

Example 3.14. If k is a field, then $Spec(k)$ is the one point space with $\mathcal{O}_{Spec(k)}(*) = k$

Example 3.15. $Spec(\mathbb{Z})$ is one point for each prime number (corresponding to the maximal ideal (p)), as well as one non-closed point, (0).

When the ring has nilpotents, an interesting and entirely non-classical phenomenon occurs.

Example 3.16. Let k be a field, and $R = k[x]/(x^2)$. Then R has only one prime ideal, namely, (x), so $Spec(R)$ is one point, with $k[x]/(x^2)$ at that point. The key fact here is that *functions are no longer determined by their values.* In particular, the function x is everywhere zero, but is not the zero function.

A question that arises is: what exactly do we mean when we think of elements of a ring R as functions on $Spec(R)$? There is a way in which we can sort of make this rigorous.

Definition 3.17. For a point $p \in Spec(R)$, we have the following canonical map:

$$R \to R/(p) \to \kappa(p),$$

where $\kappa(p)$ is the fraction field of $R/(p)$. For an element $f \in R$, we define $f(p)$ to be the image of f under this map.

This definition does not always yield actual functions though, as in the following example.

Example 3.18. Let $X = Spec(\mathbb{Z})$, and consider the element $f = 7 \in \mathbb{Z}$. Then $f((2)) = 1$ in the ring $\mathbb{Z}/2\mathbb{Z}$, $f((5)) = 2$ in the ring $\mathbb{Z}/5\mathbb{Z}$, and $f((7)) = 1$ in the ring $\mathbb{Z}/7\mathbb{Z}$. In particular, note that the values of f *lie in defferent fields.*

The set $\{p \in Spec(R) | f(p) = 0\}$ still does make sense though. Also, if k is an algebraically closed field, and $R = k[x_1, \dots, x_n]$, then for all maximal ideals $\mathfrak{m}$, $\kappa(\mathfrak{m}) = k$, since it is a finite extension of an algebraically closed field. Therefore, they really are functions in the classical case.

4. General Schemes

With the notion of affine scheme and isomorphism of sheaves, we can define a general scheme.

Definition 4.1. A *scheme* is topological space X, together with a sheaf of rings which is locally affine in the following sense: the is an open covering $\{U_\alpha\}$ of X so that the restriction of $\mathcal{O}_X$ to each U_α is isomorphic to an affine scheme.

This is just like the smooth manifold category, where we can define a smooth manifold to be a topological space with a sheaf of differentiable functions on it (the sheaf of rings), that is locally isomorphic (in the same sense as above) to some $\mathbb{R}^n$ with its standard sheaf of differentiable functions.

Now that have objects to study we want to define maps between them. For the smooth manifold case, a continuous map $\psi : M \to N$ is C^∞ if for all differentiable functions f defined on an open set U of N, the pullback $f \circ \psi$ is differentiable on $\psi^{-1}U \subseteq M$. We would somehow like to translate this into a definition of a morphism of schemes. To do this notice that a continous function $\psi : M \to N$ between differentiable manifolds gives a map of sheaves on N

$$\psi^\sharp : C(N) \to \psi_* C(M)$$

by sending $f \in C(N)(U)$ to its pullback $f \circ \psi \in \psi_* C(M)(U)$. With this then, a continous map ψ is differentiable if $\psi^\sharp$ takes $C^\infty(N)$ into $\psi_* C^\infty(M)$. That is, the diagram

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{\psi^\sharp} & \psi_* C^\infty(M) \\ \downarrow & & \downarrow \\ C(N) & \xrightarrow{\psi^\sharp} & \psi_* C(M) \end{array}$$

commutes, where the vertical arrows are the inclusion maps. The problem with adapting this definition directly to schemes is that the structure sheaf on a scheme is not a subscheme of a sheaf of functions that already exists. Therefore, we need to specify a continous map $\psi : X \to Y$ *and* a pullback map $\psi^\sharp : \mathcal{O}_Y \to \psi_* \mathcal{O}_X$. We also need a compatibility condition like the above diagram. The only thing that makes sense involves zeros of functions. Thus, we have the following defintion.

Definition 4.2. A *morphism* between two schemes X and Y is a continuous map $\psi : X \to Y$ along with a map of sheaves on Y $\psi^\sharp : \mathcal{O}_Y \to \psi_* \mathcal{O}_X$ subject to the condition that if for any point $p \in X$, any neighborhood U of $q = \psi(p)$ in Y, and any $f \in \mathcal{O}_Y$, f vanishes at q if and only if $\psi^\sharp f$ vanishes at p.

This may all seem like a lot of unnecessary complications. Why don't we just consider affine schemes? The first answer is that there are interesting schemes that are not affine, such as projective schemes. The second answer is we need more general schemes to get a "nice" category, and the third answer is that we don't really gain anything at all considering affine schemes over general schemes. That is, anything that we could do with affine schemes, we could do equally well with just commutative rings. The following theorem is a rigorous statement of this sentiment, but will be stated without proof.

Theorem 4.3. *Let X be an arbitrary scheme and R a ring. Then there is a bijection*

$$Hom(X, Spec(R)) \cong Hom(R, \mathcal{O}_X(X))$$

That is, the set of scheme morphisms from X to $Spec(R)$ can be identified with ring homomorphisms from R to the ring of global sections of X.

In particular, if $X = Spec(S)$ is also an affine scheme, then the maps $Spec(S) \to Spec(R)$ are basically the same thing as maps $R \to S$, except going the other direction. This is a statement of the fact the category of affine schemes is equivalent to the opposite category of the cateory of commutative rings. This fact will be used in the upcoming constructions.

5. Constructions

We will now define a few useful constructions on schemes.

One of the basic ways to construct new topological spaces out of old ones is to glue them together. We can also do this to schemes.

Construction 5.1. Consider a collection of schemes $\{X_\alpha\}$ and an open set $X_{\alpha\beta}$ in X_α for each $\beta \neq \alpha$. If we also have isomorphisms of schemes

$$\psi_{\alpha\beta} : X_{\alpha\beta} \to X_{\beta\alpha}$$

with the conditions that $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$,

$$\psi_{\alpha\beta}(X_{\alpha\beta} \cap X_{\alpha\gamma}) = X_{\beta\alpha} \cap X_{\beta\gamma}$$

and

$$\psi_{\beta\gamma} \circ \psi_{\alpha\beta}|_{(X_{\alpha\beta} \cap X_{\alpha\gamma})} = \psi_{\alpha\gamma}|_{(X_{\alpha\beta} \cap X_{\alpha\gamma})}$$

then we can define a new scheme X by identifying the X_α along the maps $\psi_{\alpha\beta}$.

Definition 5.2. Given morphisms of schemes $f : X \to S$ and $g : Y \to S$, the fiber product of X and Y over S is a scheme $X \times_S Y$ together with maps $X \times_S Y \to X$ and $X \times_S Y \to Y$ that makes the following diagram a pullback:

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & S. \end{array}$$

By virtue of the fact that fiber products are pullbacks, they are unique if they exist. Also note that the fiber product really does depend on the maps f and g, despite the terminology and notation.

We will now go about actually constructing these things. We will start as always by considering affine schemes.

Since affine schemes are dual to commutative rings, a pushout of commutative rings, dualized, would make a perfectly good fiber product of affine schemes. Now we notice that the following diagram is, in fact a pushout in the category of commutative rings:

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow \\ B & \longrightarrow & A \otimes_R B, \end{array}$$

where the maps f and g give A and B R-algebra structures, and the tensor product is taken with this structure in mind. This diagram is a pushout by the universal property of the tensor product.

Dualizing this we get:

Definition 5.3. Given maps $\phi : Spec(A) \to Spec(R)$ and $\psi : Spec(B) \to Spec(R)$, we define the fiber product to be

$$Spec(A) \times_{Spec(R)} Spec(B) := Spec(A \otimes_R B).$$

For arbitrary schemes, we simply decompose them into affine schemes, apply this definition, and glue them back together using the gluing construction.

6. Some Results

We will now demonstrate the interaction of schemes and algebraic geometry with another area of mathematics, namely field theory. The proofs of the results, however, are a little too far afield.

Let k_0 be any field at all, k the algebraic closure of k_0 and X a variety over k defined by polynomials $f_1, \dots f_n$. If all the coefficients of the f_i are in k_0, then we can consider the variety X_0, which is defined in the same way as X, but with k_0 as the base field. Let $R = k[X_1, \dots X_m]/(f_1, \dots f_n)$ be the coordinate ring of X and $R_0 = k_0[X_1, \dots X_m]/(f_1, \dots f_n)$ be that of X_0. Then $R = R_0 \otimes_{k_0} k$, so by duality,

$$X = X_0 \times_{Spec(k_0)} Spec(k)$$

In fact, the other way also holds. If we have affine varieties X over k and X_0 over k_0 satisfying a minor finiteness condition, and so that the last equation is true, then they are essentially "defined by" the same set of polynomials. But this equation can hold for any schemes at all; it does not depend on them being affine. Thus, we can use it as a jumping off point to translate between the algebraically closed case and the non-algebraically closed case.

To this end, we now define an action of the Galois group $Gal(k/k_0)$ on the topological space X.

Definition 6.1. Let $\sigma \in Gal(k/k_0)$, and let $\varphi : Spec(k) \to Spec(k)$ be the map of schemes induced by $\sigma^{-}1$. Then we have a map of schemes $id_{X_0} \times \varphi : X = X_0 \times_{Spec(k_0)} Spec(k) \to X_0 \times_{Spec(k_0)} Spec(k) = X$. We define σ_X to be the map of topological spaces this morphism defines.

One checks that in fact $(\sigma \cdot \tau)_X = \sigma_X \circ \tau_X$ and so we get the Galois group acting on the space X. We now can state the following theorem:

Theorem 6.2. *Let X_0 be a scheme over k_0, k be the algebraic closure of k_0, $X = X_0 \times_{Spec(k_0)} Spec(k)$, and p be the canonical projection map $p : X \to X_0$. Then p is surjective, open, and closed. Also, if $x, y \in X$, then $p(x) = p(y)$ if and only if x and y are in the same orbit of the action by the Galois group. In particular, X_0 is, as a topological space, the quotient of X by the action of the Galois group.*

Note that for the nontrivial action of the Galois group to exist, and thus this result, we needed to venture into the realm of sheaves.

Another related result, concerns rational points.

Definition 6.3. A closed point $x \in X_0$ is called rational over k_0 if $\kappa(x) = k_0$.

Such points are of great interest. The following result helps to find them.

Theorem 6.4. *If k_0 is perfect (in particular, finite or characteristic 0), and $x \in X$, then under a mild finiteness condition, $p(x)$ is rational over k_0 if and only if x is a fixed point of the action of the Galois group.*

And finally, one example using these theorems.

Example 6.5. Take $k = \mathbb{C}$, $k_0 = \mathbb{R}$, and $X_0 = Spec(\mathbb{R}[X, Y]/(X^2 + Y^2 - 1))$. Then X, without its nonclosed point, looks like a sphere with north and south poles at infinity. The real points lie along the equator, and the action of the Galois group (complex conjugation) exchanges the two hemispheres. Thus, X_0 looks like a disc, and the rational points are the ones along the boundary, corresponding to the maximal ideals $(X-\alpha, Y-\beta)$, with (α, β) on the unit circle. The rest of the closed points correspond to the maximal ideals $(X^2+Y^2-1, \alpha X+\beta Y-1)$, where (α, β) is in the interior of the unit disc.

References

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