THE BANACH CONTRACTION PRINCIPLE

ALEX PONIECKI

ABSTRACT. This paper will study contractions of metric spaces. To do this, we will mainly use tools from topology. We will give some examples of contractions, and discuss what happens when we compose contraction functions. The paper will lead up to The Banach Contraction Principle, which states every contraction in a complete metric space has a unique fixed point.

CONTENTS

1. CONTRACTIONS: DEFINITION AND EXAMPLES

Definition 1 (Contraction)**.** Let (X, d) be a complete metric space. A function $f : X \to X$ is called a **contraction** if there exists $k < 1$ such that for any $x, y \in X$, $kd(x, y) \geq d(f(x), f(y))$.

Example 1. Consider the metric space $(\mathbb{R}, d)$ where d is the Euclidean distance metric, i.e. $d(x, y) = |x - y|$. The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{x}{a} + b$ is a contraction if $a > 1$. In this specific case we can find a fixed point. Since a fixed means that $f(x) = x$, we want $x = \frac{x}{a} + b$. Solving for x gives us $x = \frac{ab}{a-1}$.

Example 2. We can create a similar contraction in the metric space $(\mathbb{R}^2, d)$ with the Euclidean distance metric. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (\frac{x}{a} + b, \frac{y}{c} + b)$ is a contraction if $a, c > 1$. For a fixed point, we want $f(x, y) = (x, y)$. Solving just like we did above, we get $x = \frac{ab}{a-1}$ and $y = \frac{cb}{c-1}$.

2. COMPLETE METRIC SPACES

The Contraction Theorem will specify that the metric space must be complete. We will now define a complete metric space and explain its role in the Theorem.

Definition 2. A metric space X is complete if every cauchy sequence of points in X converges to a point in X.

This specification simply ensures that what would be our fixed point is actually in X. If we took the function $f(x) = \frac{x}{a} + b$, as in the above example, but only considered it on the space $\mathbb{R}^2 - \{ab/(a + 1\}$ (which is not complete), our function would not have a fixed point.

Date: July 31, 2008.

3. Applying Contractions Multiple Times

From these examples, we have reason to believe that contractions in general have fixed points. To show that any contraction has a fixed point we will find a point that should be fixed and prove that that point is indeed a fixed point. Let $f : X \to X$ be any contraction. If, for a moment, we believe that all contractions have fixed points, then $f^2(x)$ should have a fixed point since it's a contraction:

Claim 1. Suppose f is contraction. Then f^n is also a contraction. Furthermore, if k is the constant for f, k^n is the constant for f^n.

Proof. First we will show that the theorem is true for $n = 2$. Since f is a contraction, we know that for some $k < 1$, $kd(x, y) \geq d(f(x), f(y))$. We can apply f again to $f(x)$ and $f(y)$ to get $kd(f(x), (y)) \geq d(f^2(x), f^2(y))$. Since $kd(x, y) \geq d(f(x), f(y))$, we know that $k^2d(x, y) \geq kd(f(x), f(y))$. Thus we have

$$k^2d(x, y) \geq kd(f(x), f(y)) \geq d(f^2(x), f^2(y))$$

Thus $k^2d(x, y) \geq d(f^2(x), f^2(y))$. Note that since $k < 1$, $k^2 < 1$. Now assume f^n is a contraction. We can similarly show that f^{n+1} is a contraction. Our induction step is also proving that the constant for f^n is k^n, so we can assume that $k^nd(x, y) \geq d(f^n(x), f^n(y))$. Like in the above proof, we can apply f again to get

$$k^{n+1}d(x, y) \geq k^{n+1}d(f(x), f(y)) \geq d(f^{n+1}(x), f^{n+1}(y))$$

Thus $k^{n+1}d(x, y) \geq d(f^{n+1}(x), f^{n+1}(y))$. By induction, the theorem is true for all n.

□

Since each f^n is a contraction, each f^n has a fixed point. Moreover, we can show that each f^n has the same fixed point. If $f(x) = x$, then $f^2(x) = f(f(x)) = f(x) = x$. By induction, $f^n(x) = x$ for all n. Also, this proof shows that the distances between points get very small as we apply the contraction many times. This means that if the original space X was not infinitely large, the area covered by the image of f^n will get very small. That is, if the distance between any two points in X had a maximum, let's say m, then the distance between any two points in the image of f^m would have to be less than k^nm. In the next section, we will look at contractions on metric spaces where the all distances are bounded.

4. Contractions on Balls

Definition 3. For $c \in X$ and $0 < r \in \mathbb{R}$ let

$$B_r(c) = \{x \in X \mid d(c, x) < r\}$$

be the ball of radius r centered at c. Also, we define the closure of $B_r(c)$, denotated by $\overline{B_r(c)}$, to be the set containing $B_r(c)$ and all of its limit points.

Remark. If a subset of a complete metric space contains all of its limit points, then all cauchy sequences contained in the set converge to a point in the set. Thus by using closed balls, we ensure that they are complete. Also note that $\overline{B_r(c)} = \{x \in X \mid d(c, x) \leq r\}$.

The above definition implies the distance between any two points in a ball $\overline{B_r(x)}$ has to be less than or equal to $2r$. Thus, if we restrict f to some $\overline{B_r(x)}$ we will be able to specify which points are contained in the image. But we have to make sure we pick our ball so that the hypothetical fixed point of X is in it. If we pick our ball so that the image of the ball is a subset of the original ball, then $f : \overline{B_r(x)} \to \overline{B_r(x)}$ would itself be a contraction, so it would have a fixed point. This would also be the same point that is fixed in X! So we just have to make sure that such a ball exists:

Claim 2. Suppose that f is a contraction. Let $x \in X$. For large enough values of r, $f(B_r(x)) \subset B_r(x)$.

Proof. We want to find r such that for any $y \in B_r(x)$, $f(y) \in B_r(x)$. Thus we want to find r such that $d(x, f(y)) < r$. By the triangle inequality, we know that

$$d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y))$$

Since f is a contraction, we know that $d(f(x), f(y)) \leq kd(x, y)$. If $B_r(x)$ is any ball and $y \in B_r(x)$, we know that $d(x, y) \leq r$. Thus $d(f(x), f(y)) \leq kr$. Thus

$$d(x, f(y)) \leq d(x, f(x)) + kr$$

.

So if we choose r so that $d(x, f(x)) + kr < r$, we would know that $d(x, f(y)) < r$ for any $y \in B_r(x)$. Thus $f(y) \in B_r(x)$. Thus we can solve for r to find which $r's$ will work for a specific k and x:

$$\begin{aligned} d(x, f(x)) + kr &< r \\ d(x, f(x)) &< r - kr \\ d(x, f(x)) &< r(1 - k) \\ \frac{d(x, f(x))}{(1 - k)} &< r \end{aligned}$$

Thus we know that if $\frac{d(x,f(x))}{(1-k)} < r$, then $f(y) \in B_r(x)$ for any $y \in B_r(x)$.

□

Remark. In general, we know that if $A \subset B$ and p is a limit point of A, then p is also limit point of B. Thus $\overline{A} \subset \overline{B}$. Since $f(B_r(x)) \subset B_r(x)$, we know that $\overline{f(B_r(x))} \subset \overline{B_r(x)}$.

Now, since we know how large the ball is to begin with, we can specify how small it gets when we apply f many times:

Claim 3. For any n, $f^n(B_r(x)) \subset B_{r'}(f^n(x))$, where $r' = k^n r$.

Proof. The proof of this Claim is very similar to the proof of Claim 1. First we will show that it is true for $n = 1$. We know that $r > d(x, y)$ for any $y \in B_r(x)$. This implies that $r' = kr > kd(x, y)$. Since f is a contraction, we also know that $kd(x, y) \geq d(f(x), f(y))$, which means $kr \geq d(f(x), f(y))$. Since every point in $f(B_r(x))$ equals $f(y)$ for some $y \in B_r(x)$, we know that every point in $f(B_r(x))$ is less than kr away from $f(x)$. Thus all points in $f(B_r(x))$ are in a ball around $f(x)$

or radius kr.

Now assume the Claim is true for n. Thus we know that $f^n(B_r(x)) \subset B_{r'}(f^n(x))$, which implies $d(f^n(x), f^n(y)) \leq k^n r$ for any $y \in B_r(x)$. This implies that $kd(f^n(x), f^n(y)) \leq k^{n+1}r$. Also, since f is a contraction, we know that $d(f^{n+1}(x), f^{n+1}(y)) \leq kd(f^n(x), f^n(y))$. So $d(f^{n+1}(x), f^{n+1}(y)) \leq k^{n+1}r$. This implies that $f^{n+1}(B_r(x)) \subset B_{r'}(f^{n+1}(x))$, as desired. By induction, the Claim is true for all n.

□

Remark. Again, since $f^n(B_r(x)) \subset B_{r'}(f^n(x))$, we know that $\overline{f^n(B_r(x))} \subset \overline{B_{r'}(f^n(x))}$.

5. The Intersection of Nested Closed Sets

We know that when we apply f many times the image gets very small and we have a precise way of saying how small the image is. Also, our fixed point should be in the image of each f^n. Since the fixed point should be in all $\overline{f^n(B_r(x))}$, it should be in the intersection of all $\overline{f^n(B_r(x))}$. We will complete the proof by studying the intersection. First, we will make sure there is at least one point in the intersection:

Claim 4. The intersection of all $\overline{f^n(B_r(x))}$ is nonempty.

Proof. First we will find a sequence of points that is cauchy, so it must converge to a limit. We will then show that this limit must be in the intersection.

Take the sequence of points a_n such that $a_n \in \overline{f^n(B_r(x))}$. We will show that this sequence is cauchy. We will show that for any ϵ there exists N such that if $n, m > N$, $d(a_n, a_m) \leq \epsilon$. Let N be such that $\epsilon > k^N r$. We know that $\overline{f^N(B_r(x))} \subset \overline{B_{r'}(f^N(x))}$, where $r' = k^N r$, as described in Claim 3. Also, note that if $p > N$, then $\overline{f^p(B_r(x))} \subset \overline{f^N(B_r(x))}$. Thus if $n, m > N$, we know that $a_n, a_m \in \overline{f^N(B_r(x))}$. Thus $a_n, a_m \in \overline{B_{r'}(f^N(x))}$. Thus $d(f^N(x), a_n) \leq r'$ and $d(f^N(x), a_m) \leq r'$. Adding these together get $d(f^N(x), a_n) + d(f^N(x), a_m) \leq 2r'$. Since this is a metric space we know that $d(a_m, a_n) \leq d(f^N(x), a_n) + d(f^N(x), a_m)$. Thus $d(a_m, a_n) \leq 2r < \epsilon$, as desired.

Since the sequence is cauchy, it must converge to a limit. Now we will show that this limit is in the intersection. Say that the limit p was not in the intersection. Then p is not in some $\overline{f^n(B_r(x))}$. Let $B_j(p)$ be a ball around p such that $j = d(f(x), p) - r'$, where $r' = k^n r$. Thus no point in $\overline{f^n(B_r(x))}$ is in $B_j(p)$. But there are an infinite number of points from the sequence a_n in $\overline{f^n(B_r(x))}$ since we have $a_m \in \overline{f^m(B_r(x))} \subset \overline{f^n(B_r(x))}$ if $m > n$. Since every ball around p must only have a finite number of points of a_n outside of it, this contradicts the fact that a_n converges to p. Thus p is in the intersection.

□

Remark It is true that in a complete metric space, for any sequence of closed sets $A_1 \supset A_2 \supset A_3 \supset \ldots$, the intersection of the sets A_i is nonempty.

Furthermore, we can show that p is the only point in the intersection.

Claim 5. The intersection of all $\overline{f^n(B_r(x))}$ has only one point.

Proof. Say the intersection had more then one point. Let a, b be any two points in the intersection. Let $\epsilon = d(a, b)$. Take n such that $\epsilon/2 > k^n r$. We know that $\overline{f^n(B_r(x))}$ is contained $\overline{B_{r'}(f(x))}$, where $r' = k^n r$. Also, no two points in this ball can have a distance of $2r'$ or more. Since $d(a, b) = \epsilon = 2r'$, both a and b cannot be in $\overline{f^n(B_r(x))}$. Thus they are not in the intersection of all $\overline{f^n(B_r(x))}$. □

6. Two Proofs of the Contraction Theorem

Now we know that the intersection has exactly one point. Since we have guessed that our fixed point is in all $\overline{f^n(B_r(x))}$, this must be our fixed point. We will now prove that this is so.

Theorem 1 (The Banach Contraction Principle)**.** Let (X, d) be a complete metric space, $f : X \to X$ a contraction. Then f has a unique fixed point in X.

Proof. By the previous two claims, there exists exactly one point y in the intersection of all $\overline{f^n(B_r(x))}$. Say that it was not fixed. Let $\epsilon = d(f(y), y)$. We know that ϵ is not 0 since $f(y) \neq y$. Take n such that $\epsilon/2 = k^n r$. Thus we know that $\overline{f^n(B_r(x)} \subset \overline{B_{\epsilon_2}(f(x))}$. Thus no two points in $\overline{f^n(B_r(x)}$ can have a distance greater than ϵ. y is in $\overline{f^n(B_r(x)}$ because it's in all $\overline{f^n(B_r(x)}$ and $f(y)$ is in $\overline{f^n(B_r(x)}$ because each contraction is a subset of the last. But the distance between y and $f(y)$ is ϵ. Thus we must have $f(y) = y$. Thus y is a fixed point.

Note that we cannot have more than one fixed point. If had two, say x, y, we would have $d(f(x), f(y)) = d(x, y)$. But we know that $d(f(x), f(y)) \leq kd(x, y)$ where $k < 1$ since f is a contraction.

□

Alternate Proof. We can prove the Theorem an alternate way using similar concepts and methods developed in this paper. However, this proof is framed around the idea of creating a cauchy sequence that will 'lead' us to our fixed point.

Proof. Take any $x \in X$. Let a_n be the sequence such that $a_k = f^k(x)$. Choose r so that $\overline{f(B_r(x))} \subset \overline{B_r(x)}$, as described in Claim 2. First we will show that this sequence is cauchy. For any ϵ, we will find j such that if $m, n > j$ then $d(a_m, a_n) < \epsilon$. Take any ϵ. Take j such that $\epsilon/2 > k^j r$. So we know that the distance between any two points in $\overline{f^j(B_r(x))}$ is less than ϵ (see Claim 3). We also know that $a_k \in \overline{f^k(B_r(x))}$ and that if $m, n > j$, then $\overline{f^m(B_r(x))}, \overline{f^n(B_r(x))} \subset \overline{f^j(B_r(x))}$. Thus $a_m, a_n \in \overline{f^j(B_r(x))}$ and so $d(a_m, a_n) < \epsilon$, as desired.

Since the sequence is cauchy, it converges to a limit y. Say that y was not fixed by f. Take $\epsilon = d(f(y), y)/3$. Since a_n converges to y, we can find N such that for all $k > N$, $a_k \in B_\epsilon(y)$. Find that N and take any such k. By the triangle inequality, we know that $d(f(y), y) \leq d(f(y), a_k) + d(a_k, y)$. This implies that $d(f(y), y) - d(a_k, y) \leq d(f(y), a_k)$. Since $d(a_k, y) < d(f(y), y)/3$,

$$d(f(y), y) - d(f(y), y)/3 < d(f(y), y) - d(a_k, y) \leq d(f(y), a_k)$$

Thus $\frac{2d(f(y),y)}{3} < d(f(y), a_k)$. Since $a_{k+1} = f(a_k)$, a_{k+1} is in the same ball as a_k. Thus the exact same statement can be made for a_{k+1}, i.e. $\frac{2d(f(y),y)}{3} < d(f(y), a_{k+1})$. Thus we have

$$d(y, a_k) < \frac{d(f(y), y)}{3} < \frac{2d(f(y), y)}{3} < d(f(y), a_{k+1})$$

This contradicts the fact that f is a contraction, since a contraction requires that $d(f(y), f(a_k)) < d(y, a_k)$. Thus y must be fixed.

□

Acknowledgments. I would like to thank my mentors Tom Church and Katie Mann. Tom was very helpful in getting me started on Topology topics and answering some preliminary questions. Katie provided much guidance during our meetings, and was particularly good at helping me understand the concepts and ideas behind the math we were working with.

References

[1] Charles C. Pugh. Real Mathematical Analysis. Springer. 2003.