THE INDEX OF A VECTOR FIELD AS AN INVARIANT

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Introduction

The purpose of this paper is to demonstrate that the index of a vector field on a smooth manifold is an intrinsic invariant of the manifold. The paper is divided into three sections. The first section introduces the Inverse Function Theorem and provides the proof for the Implicit Function Theorem, laying the foundation for our discussion. In the second section we will introduce the concept of a vector field, degree, and the index of a vector field. We will also prove that the index of a vector field is invariant under diffeomorphism. In the last section we will prove and highlight the fact that the index of a vector field on a manifold is independent of the choice of vector field or the embedding of the manifold, demonstrating that the vector field index is an invariant of a smooth manifold.

The primary source for this paper was John W. Milnor's book, *Topology from the Differentiable Viewpoint.*

1. The Inverse Function Theorem and The Implicit Function Theorem

Definition 1.1. Given the open sets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$, we say that a mapping $f : U \to V$ is *smooth* if all the partial derivatives $\partial^n f/\partial x_{i_1} \ldots \partial x_{i_n}$ exist and are continuous.

Definition 1.2. A subset $M \subset \mathbb{R}^n$ is called a *smooth manifold* of dimension m if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset U of the euclidean space $\mathbb{R}^m$.

Any particular diffeomorphism $h : U \to W \cap M$ is called a *parametrization* of the region $W \cap M$.

We can think of h as a mapping from U to $\mathbb{R}^n$, so that we can define the derivative $dh_u : \mathbb{R}^m \to \mathbb{R}^n$. The *tangent space* of the manifold M at the point x, denoted TM_x, is the image $dh_u(\mathbb{R}^m)$ of dh_u. This construction is independent of the choice of parametrization.

Given another manifold, $N \subset \mathbb{R}^l$, a map $f : M \to N$ is called *smooth* if for each $x \in M$ there exists an open set $U \subset \mathbb{R}^n$ containing x and a smooth mapping $F : V \to \mathbb{R}^l$ which coincides with f throughout $V \cap M$.

The map dF_x maps TM_x into $TN_{f(x)}$. We call the restriction $dF_x|TM_x : TM_x \to TN_{f(x)}$ the derivative of f at the point x and denote it by df_x. This definition does not depend on the choice of F. Furthermore, $dF_x(v)$ belongs to TN_y. Therefore, $df_x(v) = dF_x(v)$ for all $v \in TM_x$.

Theorem 1.3 (Inverse Function Theorem). *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map. If the derivative $df_x : \mathbb{R}^k \to \mathbb{R}^k$ is nonsingular then f maps any sufficently small open set U about x diffeomorphically onto an open set $f(U)$.*

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Definition 1.4. For two arbitrary manifolds M and N, a regular point of the smooth map $f : M \to N$ is a point $x \in M$ such that df_x has the largest possible rank. a *regular value* will be a point $y \in N$ such that $f^{-1}(y)$ contains only regular points.

Theorem 1.5 (Implicit Function Theorem). *If the map $f : M \to N$ is smooth and maps between manifolds of dimension m and n respectively with $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

Proof. first let $x \in f^{-1}(y)$. Because y is a regular value, we know that the derivative df_x will map the the tangent space, TM_x, of M at the point x onto the tangent space, TN_y, of N at the point $f(x)$ or y. This means that the rank of df_x will be n and thus its null space will be an $m - n$ dimensional vector space.

Now if the manifold M is a subset of $\mathbb{R}^k$, then choose a a linear map $L : \mathbb{R}^k \to \mathbb{R}^{m-n}$ which is nonsingular on the null space of the derivative df_x. This subspace is a subset of the tanget space TM_x which in turn is a subspace of $\mathbb{R}^k$. Now define a new map

$$F : M \to N \times \mathbb{R}^{m-n} \tag{1.6}$$

by $F(z) = (f(z), L(z))$, for some z in M. The derivative of F is then given by the formula

$$dF_x(v) = (df_x(v), L(v)) \tag{1.7}$$

Clearly dF_x will be nonsingular. Thus by the inverse function theorem, F maps some neighborhood U around x diffeomorphically onto a neighborhood V of $(y, L(x))$. Under the map F, $f^{-1}(y)$ corresponds to $y \times \mathbb{R}^{m-n}$. More specifically, F maps $f^{-1}(y) \cap U$ diffeomorphically onto $(y \times \mathbb{R}^{m-n}) \cap V$. Therefore $f^{-1}(y)$ must be a smooth manifold of dimension $m - n$. □

2. The Index of a Vector Field

Definition 2.1. Given an open set U of $\mathbb{R}^n$, a vector field v is a vector valued function $v : U \to \mathbb{R}^n$.

If $M \subset \mathbb{R}^n$ is a smooth m-dimensional manifold then a *smooth vector field on* M is a smooth function $v : M \to \mathbb{R}^n$ such that $v(x) \in TM_x$.

Before we can begin our discussion on the vector field index, we must first introduce such concepts as orientablity and the degree of a smooth mapping between manifolds.

Definition 2.2. Let $(b_1, \ldots b_n)$ and $(b'_1, \ldots, b'_n)$ be two ordered bases for $\mathbb{R}^n$. We say that $(b_1, \ldots b_n)$ and $(b'_1, \ldots, b'_n)$ are equivalent if $b'_i = \sum a_{ij} b_j$ and $\det(a_{ij}) > 0$. An *orientation* is an equivalance class of bases. Each equivalance class determines an orientation. If $\det(a_{ij}) < 0$ then the basis $(b_1, \ldots b_n)$ determines the opposite orientation. Thus, there will be exactly two equivalence classes, two orientations.

Definition 2.3. An *oriented smooth manifold* consists of a manifold M of dimension m together with a chosen orientation for each tangent space TM_x which piece together in the following way: For each point x of M there exists a neighborhood U of M and a diffeomorphism g mapping U onto an open subset of $\mathbb{R}^m$. For each $y \in U$, dg_y carries the chosen orientation for TM_y into the standard orientation, $(e_1, \ldots, e_m)$, for $\mathbb{R}^m$.

If M has a boundary then we can identify the following vectors in the tangent space TM_x at a specific boundary point:

(1) Vectors tangent to the boundary forming the $m-1$ dimensional subspace $T(\partial M)_x$.
(2) Vectors pointing outward forming an open half space bounded by $T(\partial M)_x$
(3) Vectors pointing inward forming a complementary half space

The orientation for ∂M will be determined as follows: For each $x \in \partial M$ we will choose an oriented basis $(v_1, v_2, \ldots, v_m)$ for TM_x in such a way that $v_2, \ldots, v_m$ are tangent to the boundary and that v_1 is an outward pointing vector. Then $(v_2, \ldots, v_m)$ determines the required orientation for ∂M at x.

Definition 2.4. Let M and N be oriented n-dimensional manifolds without boundary. Consider a smooth map $f : M \to N$. If M is compact and N is connected then the degree of f is given by the following:

Choose a regular point x of f in M. Then, the linear isomorphism $df_x : TM_x \to TN_{f(x)}$, between two oriented vector spaces, is defined. The 'sign' of df_x is

$$\text{sign } df_x = \begin{cases} +1 & \text{if } df_x \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

And for any regular value, y, in N, the *degree* of f is

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \text{sign } df_x. \tag{2.5}$$

Note that this integer is a locally constant function on the set of regular values.

Definition 2.6. Given an open set U in $\mathbb{R}^m$ and a smooth vector field $v : U \to \mathbb{R}^m$ with an isolated zero z, then the function

$$\bar{v}(x) := v(x)/||v(x)|| \tag{2.7}$$

maps a small sphere centered at z into the unit sphere. The degree of this mapping is called the *index* of the vector field v at the zero z, denoted $\mathbf{i}(z)$.

Let v be a vector field on a manifold $M \subset \mathbb{R}^n$ and $z \in M$ an isolated zero of v. If h is a parametrization, $h : U \to M$ of a neighborhood of $z \in M$ then the *index of v at z* is the index of the corresponding vector field $dh^{-1} \circ v \circ h$ on U at the zero $h(z)^{-1}$.

However, in order to define the index of a vector field on a manifold in this way, we first need to show that the index is independent of the choice of parametrization. Corollary 2.21 will prove this fact. We first require the following lemmas and theorems:

Lemma 2.8. *Suppose that M is the boundary of a compact oriented manifold X and that M is oriented as the boundary of X. If $f : M \to N$ extends to a smooth map $F : X \to N$ then $deg(f; y) = 0$ for every regular value y.*

Proof. Suppose that y is a regular value for F, and also for $f = F|M$. Then $F^{-1}(y)$ will be a 1-dimensional manifold by the implicit function theorem. Furthermore, we know that since $F^{-1}(y)$ is a 1-dimensional manifold then it will be made up of a union of arcs and circles. Note that the boundary points of these arcs lie on $M = \partial X$. Let us denote one of these arcs as A and let the boundary of A be $\partial A = a \cup b$.

Now let us consider the orientation of A. Given an $x \in A$, let $(v_1, \ldots, v_{n+1})$ denote the basis defining a positive orientation for TX_x where v_1 is tangent to A. Note that v_1 determines the required orientation for TA_x if and only if dF_x carries $(v_1, \ldots, v_{n+1})$ into a positively oriented basis for TN_y.

Let $v_1(x)$ denote the unit vector tangent to A at x which determines a positive orientation for TA_x. Now, note that this v_1 points outward at one boundary point of A, say a, and inwards at the point b. This follows from the three types of vectors that were defined to be in the tangent space of a manifold with boundary at a boundary point. Therefore this means that

$$\text{sign } df_a = -1, \quad \text{sign } df_b = +1 \tag{2.9}$$

and

$$\text{sign } df_a + \text{sign } df_b = 0. \tag{2.10}$$

Thus if we add up over all arcs like A then we prove that $\deg(f; y) = 0$.

Now to generalize this result, we must consider the case in which y is a regular value for f but not for F. Since the function $\deg(f; y)$ is locally constant on the set of regular values of f this means that it will be constant within some neighborhood U of y. Therefore, all we have to do is choose a regular value for F in U, say y_0, and observe that $\deg(f; y) = \deg(f; y_0) = 0$. □

Lemma 2.11. *Given a smooth homotopy $F : [0,1] \times M \to N$ between two mappings $f(x) = F(0, x), g(x) = F(1, x)$, the degree $deg(g; y)$ is equal to $deg(f; y)$ for any common regular value y*

Proof. If we consider the manifold $[0,1] \times M$ then its boundary will consist of $1 \times M$ and $0 \times M$. Note that, as in the previous theorem, these two boundary components will have opposite orientations. Thus, the degree of $F|\partial([0,1] \times M)$ is equal to

$$\deg(g; y) - \deg(f; y) \tag{2.12}$$

which by the previous theorem must be equal to zero. □

Theorem 2.13. *Any orientation preserving diffeomorphism f of $\mathbb{R}^n$ is smoothly isotopic to the identity.*

Proof. Let us first assume that $f(0) = 0$. Now consider the derivative of f. recall that the derivative of f at a point x, $df_x : \mathbb{R}^n \to \mathbb{R}^n$ is defined by the formula

$$df_x(h) = \lim_{t \to 0} (f(x + th)) - f(x))/t$$

Thus, at $x = 0$ the derivative will be

$$df_0(h) = \lim_{t \to 0} f(th)/t \tag{2.14}$$

We will define an isotopy $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ in the following way:

$$F(h, t) = f(th)/t \text{ for } 0 < t \leq 1, \tag{2.15}$$

$$F(h, 0) = df_0(h) \tag{2.16}$$

Now we must show that F is smooth. Therefore, in order to demonstrate this, let us write f in the following manner;

$$f(h) = h_1 g_1(h) + \ldots + h_n g_n(h), \tag{2.17}$$

where $g_1, \ldots, g_n$ are smooth functions. When we substitute the equation above for f into our definiton of $F(h,t)$ we obtain

$$F(h,t) = h_1 g_1(th) + \ldots + h_n g_n(th) \tag{2.18}$$

for all values of t. This construction demonstrates that F is smooth everywhere, even as $t \to 0$. Thus, we have shown that f is smoothly isotopic to the linear mapping df_0 and by the connectedness of $GL^+(n, \mathbb{R})$, df_0 is isotopic to the identity. □

Definition 2.19. Consider the diffeomorphism $f : M \to N$ with a vector field v on M and v' on N. These vector fields are said to *correspond* under f if the derivative df_x carries $v(x)$ into $v'(f(x))$ for each $x \in M$. Thus, the following relation will be defined:

$$v' = df \circ v \circ f^{-1}. \tag{2.20}$$

Corollary 2.21. *Assuming that the vector field v on U corresponds to the vector field $v' = df \circ v \circ f^{-1}$ on U' under a diffeomorphism $f : U \to U'$, then the index of v at an isolated zero z is equal to the index of v' at $f(z)$.*

Proof. If we consider the case where f preserves orientation, then the corollary is a quick consequence of Theorem 2.20. Now let us consider the case in which f does not preserve orientation. Let ρ denote a reflection which reverses orientation. We will define

$$v' = \rho \circ v \circ \rho^{-1} \tag{2.22}$$

such that the function $\bar{v}'(x) = v'(x)/||v'(x)||$ satisfies

$$\bar{v}' = \rho \circ \bar{v} \circ \rho^{-1} \tag{2.23}$$

on the ϵ sphere. Thus, by combining the two equations above, it is clear to see that the degree of $\bar{v}'$ equals the degree of $\bar{v}$. This completes the proof. □

This corollary demonstrates that the index of a vector field is invariant under a diffeomorphism. This allows us to define the index of a vector field on a manifold in the desired way.

3. The Index of a Vector Field as an Invariant

Now let us consider the index of a vector field on a manifold with boundary and with isolated zeros which points outward along the boundary.

Definition 3.1. Let $X \subset \mathbb{R}^m$ be a compact m-manifold with boundary. The *Gauss mapping* is:

$$g : \partial X \to S^{m-1} \tag{3.2}$$

which assigns to each $x \in \partial X$ the unit normal outward vector at the point x.

Theorem 3.3. *For $X \subset \mathbb{R}^m$, a compact m-manifold with boundary, If $v : X \to \mathbb{R}^m$ is a smooth vector field with isolated zeros, and if v points out of X along the boundary, then the index sum, denoted $\sum \boldsymbol{i}(z_i)$, is equal to the degree of the gauss mapping from ∂X to S^{m-1}.*

Proof. To prove this statement let us consider removing an ϵ ball around each isolated zero. We therefore obtain a new manifold with boundary which includes the original boundary of X and also the boundaries of the ϵ-balls which were removed. Now, once the zeros have been removed, the function $\bar{v}(x) = v(x)/||v(x)||$ is well defined on what is left of the manifold X. Therefore, if we consider $\bar{v} : X - \bigcup_{z_i} \{B_\epsilon(z_i)\} \to S^{m-1}$ as an extension of a map $\partial X \coprod \bigcup_{z_i} \{\partial B_\epsilon(z_i)\} \to S^{m-1}$, where $\{B_\epsilon(z_i)\}$ denote the ϵ balls removed around each zero, then by Lemma 2.8 we note that the sum of the degrees of $\bar{v}$ restricted to this new boundary will be zero. Furthermore, because $\bar{v}$ points outward, we know that $\bar{v}|\partial X$ will be homotopic to the Gauss mapping g. Therefore, the degree of $\bar{v}|\partial X$ will be the same as the degree of g. Since the sum of the degrees of $\bar{v}$ must add up to zero, $\deg(g)$ must cancel out with the sum of the degrees on all the other boundary components. This sum is equal to the negative index sum of $\bar{v}$ at all the various isolated zeroes. This is because we are computing the degree of the same map only we are changing the orientation of the spheres. Thus, $\deg(g) - \sum \mathbf{i}(z_i) = 0$ and $\deg(g) = \sum \mathbf{i}(z_i)$. □

One important consequence of this theorem is that the index of a vector field does not depend on the choice of the vector field. As long as the vector field points outward along the boundary the result will be the same. This brings us closer to our goal of showing that the index of a vector field is an intrinsic invariant of a smooth manifold.

We now want to show that the results of this theorem are also true for the case of a vector field with non-degenerate zeros on a compact manifold without boundary and embedded into any finite dimensional euclidean space.

Definition 3.4. Given an open set, U of $\mathbb{R}^m$ and a vector field $w : U \to \mathbb{R}^m$ such that $dw_z : \mathbb{R}^m \to \mathbb{R}^m$, then the vector field w is said to be *nondegenerate* at the the zero z if dw_z is nonsingular.

Lemma 3.5. *The index of w at a nondegenerate zero is equal to the sign of the determinant of dv_z.*

Proof. Let us first think of w as a diffeomorphism from some convex neighborhood U of z which maps into $\mathbb{R}^n$. If w preserves orientation then by Theorem 2.13, $w|U$ can be smoothly deformed into the identity, without introducing new isolated zeros. Thus the index will simply be equal to +1. If w reverses, then in a similar demonstration as that of Lemma 3.6, we can show that w can be smoothly deformed into a reflection. Thus, the index of w will be -1 in this case. □

We must now generalize this result to a manifold of a dimension that does not agree with the dimension of the space into which the manifold is embbeded. Therefore, let us consider the case of an n-dimensional manifold $M \subset \mathbb{R}^k$ with a vector field $v : M \to \mathbb{R}^k$. Note that at a zero z we also have the map $dv_z : TM_z \to \mathbb{R}^k$. In order to ensure that we can calculate the index of the vector field v the same way we did in the previous lemma, we must show that dv_z is in fact defined by the mapping $dv_z : TM_z \to TM_z$ which will allow us to calculate its determinant. The following lemma will prove this fact.

Lemma 3.6. *Given the vector field $v : M \to \mathbb{R}^k$, The map dv_z can be considered as a linear transformation from TM_z to itself. If this linear transformation has determinant $\neq 0$ then z is a nondegenerate zero of v with index equal to +1 or -1 according to the sign of the determinant of dv_z.*

Proof. See Milnor [1], page 37 □

We now have the necessary tools to demonstrate that the index of a vector field is an intrinsic invariant of the manifold. Let's consider the following: given a compact manifold $M \subset \mathbb{R}^k$ without boundary and of dimension less than k, we will let N_ϵ denote the closed ϵ-neighborhood of M which can be defined more specifically as the set of all x in $\mathbb{R}^k$ such that $||x - y|| \leq \epsilon$ for some y in M. It follows from Sard's Theorem and the Implicit Function Theorem that, for a sufficiently small ϵ, N_ϵ is a smooth manifold with boundary.

Theorem 3.7. *For any vector field v on M with only nondegenerate zeros, the index sum is equal to the degree of the Gauss mapping $g : \partial N_\epsilon \to S^{k-1}$.*

Proof. Let us consider a point x on the ϵ-neighborhood of M, N_ϵ. Now let $r(x)$ be the closest point to x in M. For ϵ sufficiently small, r will be a continuous function and $r(x)$ will be unique. This implies that $x - r(x)$ is perpendicular to the tangent space $TM_{r(x)}$. We will consider the square distance function

$$\phi(x) = ||x - r(x)||^2$$

computing the gradient of this function gives,

$$\text{grad } \phi = 2(x - r(x))$$

Thus for any point x on the boundary, ∂N_ϵ, of N_ϵ, the unit normal vector pointing outward is given by the function

$$g(x) = \text{grad } \phi / ||\text{grad } \phi|| = (x - r(x))/\epsilon. \tag{3.8}$$

Now we extend the vector field v on M to a vector on N_ϵ

$$w(x) := (x - r(x)) + v(r(x)) \tag{3.9}$$

Note that the dot product between $g(x)$ and $w(x)$ yields ϵ. This means that the projection of w onto the unit normal vector g will be positive, implying that $w(x)$ will also point outwards along the boundary. Since the two summands that make up w, $(x-r(x))$ and $v(r(x))$, are mutually orthogonal, this means that w will vanish only at the zeros of v. Thus, w and v have the same zeros.

Now we use Lemmas 3.5 and 3.6 to compute the indices of v and w at a zero $z \in M$. Using Lemma 3.5, we can compute the index of w by calculating the determinant of dw_z. Likewise, Lemma 3.6 allows us to compute the index of v on M using the same technique of calculating the determinant of the linear transformation dv_z. Therefore, note that when we compute the derivative of w we obtain:

$$dw_z(h) = dv_z(h) \; \forall h \in TM_z$$
$$dw_z(h) = h \; \forall h \in TM_z^{\perp}.$$

This means that the determinant of dw_z is equal to the determinant dv_z. Thus, the index of w at the zero z is equal to the index of v at the z.

Since w points outward along the boundary of N_e, we know by Theorem 3.3 that the index of w and v, since they are the same, is equal to the degree of the Gauss mapping g. This completes the proof. □

The proof works by thickening the manifold M into a new k-dimensional manifold with boundary, N_ϵ. We then extend the vector field v on M to a new vector field w on N_ϵ. Note that in the proof, w was explicitly constructed in order to demonstrate three crucial steps that together would prove the desired result. The first step was to show that the vector field w points outward along the boundary of N_ϵ. This was done by calculating the projection of w onto the outward normal vector g. The second step was to demonstrate that the vector field w on N_ϵ had the same zeros as the vector field v on the manifold M. The third and last step was to demonstrate that the indices of v and w agreed at some zero. By Lemmas 3.5 and 3.6 we knew that the indices of v and w could be explicitly computed and the construction of w demonstrated that these indices were in fact equal.

The most important result of this theorem is that the index of a vector field does not depend on the choice of the vector field on M. Furthermore, the index of a vector field does not depend on the way in which M is embedded.

4. Concluding Remarks

Interestingly, we can also conclude that two very different Gauss mappings can have the same degree. If, for instance, we have two manifolds, M embedded in $\mathbb{R}^{k_1}$ and N embedded in $\mathbb{R}^{k_2}$, and there exists a diffeomorphism, $f : M \to N$, then the degree of the Gauss mapping $\partial N_{1_\epsilon} \to S^{k_1-1}$ is equal to the degree of $\partial N_{2_\epsilon} \to S^{k_2-1}$. We know by Corollary 2.21 that the index of a vector field is invariant under diffeomorphism and by Theorem 3.11 that the index of the vector fields on M and N are equal to the degrees of their respective Gauss mappings. Let's consider a simple example:

Example 4.1. Take $M = S^1$ embedded in $\mathbb{R}^2$. The boundary of the ϵ neighborhood of S^1 is just the disjoint union of two circles. Therefore, the Gauss mapping is given by $\partial N_\epsilon = S^1 \coprod S^1 \to S^1$. Now consider the same circle yet this time embedded in $\mathbb{R}^3$. The boundary of N_ϵ is now a torus. The Gauss mapping is $\partial N_\epsilon = S^1 \times S^1 \to S^2$. Even though these are two completely different mappings, they have the same degree, zero.

Together, the results in this paper show that index number is really an intrinsic invariant of a smooth manifold: We showed in Theorem 3.11 that the index does not depend on the choice of the vector field or on the embedding of the manifold and in Corollary 2.21 we showed that it is a diffeomorphism invariant. By the Poincare-Hopf Theorem, we can show that this invariant is equal to the Euler characteristic.

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References

[1] John W. Milnor. Topology from the Differentiable Viewpoint. Princeton University Press. 1997.