TRANSLATION EQUIDECOMPOSABILITY

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ABSTRACT. Expository piece on M. Laczkovich's "Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem." A criterion for translation equidecomposability of two Jordan domains is given with an application.

CONTENTS

After the discovery of the Banach-Tarski paradox, Alfred Tarski asked, as an open problem in the 1925 issue of Fundamenta Mathematicae, whether a circle and a square are equidecomposable. [1] Although similar in spirit to the circle-squaring problem of the ancient geometers, Tarski disregarded issues concerning rulers and compasses and instead asked whether a disc in $\mathbb{R}^2$ could be broken into a finite number of pieces and then reassembled into a square of equal area.

Motivated by questions like Tarski's, this paper develops a criterion for translation equidecomposability of two Jordan domains in $\mathbb{R}^2$. In section one, we prove Poincare's formula and apply it to get a bound on the measure of a neighborhood of a Jordan curve. In section two, we use the notion of discrepancy to prove certain Jordan domains are "uniformly spread." Finally, we prove in section three that such evenly spread Jordan domains are translation equidecomposable. We conclude with a sketch of the solution to Tarski's circle-squaring problem.

This paper offers no original material, but aims to explicate the theorems on equidecomposability discovered by M. Laczkovich in 1990 in "Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem." However, developing a criterion for equidecomposability requires an interesting mix of mathematics, ranging from group theory, graph theory, integral geometry, and analysis.

[1]Tarski: "Un carré et un cercle dont les aires sont égales peuvent-ils être décomposés en un nombre fini de sous-ensembles disjoints respectivement congruents?"

1. Poincare's Formula

Definition 1.1. Let (O, x, y) be a fixed frame and let (O_1, X, Y) be a moving frame. Also, let a, b be the coordinates of O_1 and ϕ to be the angle between Ox and O_1X. The **kinematic density** is denoted by $dK_1 = da \wedge db \wedge d\phi$.

Definition 1.2. We call a continuous function $J : [p, q] \to \mathbb{R}^2$ a **closed simple Jordan curve** if J is injective on (p, q) and $J(p) = J(q)$. The domain enclosed by J is called a **Jordan domain**, denoted by $\tilde{J}$.

Lemma 1.3. *Let (O, x, y) be a fixed frame and let (O_1, X, Y) be a moving frame. Let J_0 and J_1 be simple Jordan curves such that*

(1) J_0, J_1 are twice differentiable and composed of a finite number of arcs.
(2) $x = x(s_0), y = y(s_0)$ are the equations of J_0 referred to the arc length s_0 and to the coordinate system (O, x, y)
(3) $X = X(s_1), Y = Y(s_1)$ are the equations of J_1 where s_1 denotes the arc length of J_1.

Letting θ be the angle between the tangent of J_0 and the tangent of J_1 at the point $P \in J_0 \cap J_1$, then $dK_1 = da \wedge db \wedge d\phi = |sin\theta| ds_0 \wedge ds_1 \wedge d\theta$

Proof. Letting a, b be the coordinates of O_1 and ϕ to be the angle between Ox and O_1X, with respect to the coordinate system (O, x, y), the equations of J_1 become

$$x = a + Xcos\phi - Ysin\phi, y = b + Xsin\phi + Ycos\phi$$

From this, we can see that the points of intersection between J_0 and J_1 are given by the system

$$x(s_0) = a + X(s_1)cos\phi - Y(s_1)sin\phi$$
$$y(s_0) = b + X(s_1)sin\phi + Y(s_1)cos\phi$$

with s_0, s_1 unknowns.

Deriving, we get

$$da = x'ds_0 - (X'cos\phi - Y'sin\phi)ds_1 + (Xsin\phi + Ycos\phi)d\phi$$
$$db = y'ds_0 - (X'sin\phi + Y'cos\phi)ds_1 - (Xcos\phi - Ysin\phi)d\phi$$

And further, by multiplying we see that

$$da \wedge db \wedge d\phi = [(X'y' - x'Y')cos\phi - (Y'y' + X'x')sin\phi]ds_0 \wedge ds_1 \wedge d\phi$$

If α_0 denotes the angle between the tangent to J_0 at the point $P \in J_0 \cap J_1$ and the x axis, and α_1 denotes the angle between the tangent to J_1 at the same point and the X axis, we have

$$x' = cos\alpha_0, y' = sin\alpha_0$$
$$X' = cos\alpha_1, Y' = sin\alpha_1$$

and the kinematic density, given by the equation $dK_1 = da \wedge db \wedge d\phi$ can be reformulated as follows:

$$dK_1 = da \wedge db \wedge d\phi = sin(\alpha_0 - \alpha_1 - \phi)ds_0 \wedge ds_1 \wedge d\phi$$

If θ is the angle between J_0 and J_1 at P, then $|\theta| = |\alpha_0 - \alpha_1 - \phi|$ and since α_0 and α_1 are functions only of s_0 and s_1, we conclude that

$$dK_1 = |sin\theta| ds_0 \wedge ds_1 \wedge d\theta$$

□

Lemma 1.4. *Given $(O, x, y), (O_1, X, Y), J_0, J_1$, let L_0 and L_1 be the lengths of J_0 and J_1 respectively. Then, letting n be the number of intersection points of J_0 and J_1, we have*

$$\int_{J_0 \cap J_1 \neq \emptyset} ndK_1 = 4L_0L_1$$

Proof. From Lemma 1.3, we have

$$dK_1 = |sin\theta| ds_0 \wedge ds_1 \wedge d\theta$$

And we know that

$$\int_0^{L_0} ds_0 = L_0$$

$$\int_0^{L_1} ds_1 = L_1$$

$$\int_{-\pi}^{\pi} |sin\theta| d\theta = 4.$$

Furthermore, because each position of J_1 gets counted for each of its intersection points with J_0, integrating both sides of the formula from Lemma 1.3 gives us,

$$\int_{J_0 \cap J_1 \neq \emptyset} ndK_1 = \int_0^{L_0} ds_0 \int_0^{L_1} ds_1 \int_{-\pi}^{\pi} |sin\theta| d\theta = 4L_0L_1$$

which is what we want. This formula is known within integral geometry as **Poincare's Formula**. □

Remark 1.5. We are most concerned here with a special case of Poincare's formula, where J_1 is a circle of radius r and midpoint $M = (a, b)$. Then $dK_1 = da \wedge db \wedge d\phi = dM \wedge d\phi$. Furthermore, as ϕ varies between 0 and 2π, n clearly does not change because rotating the circle would not alter the number of intersection points. This gives us

$$\int_{\mathbb{R}^2} ndM = 4rL_0 \tag{1.6}$$

Definition 1.7. Given a Jordan curve J and some constant δ, $U(J, \delta) = \{y : \exists x \ni |x - y| \leq \delta$. We say that $U(J, \delta)$ is the **closed δ-neighborhood** of J.

Theorem 1.8. *Let A be a Jordan domain with boundary J. Then $\forall \delta$ such that $0 < \delta < \frac{1}{2} diam(J)$, we have $\lambda_2(U(J, \delta)) \leq 2\delta\lambda_1(J)$.* ***Note:*** *We use λ_1 to denote the one-dimensional Hausdorff measure λ_2 for the Lebesgue measure in $\mathbb{R}^2$.*

Proof. Let

$$n = n(x) = \big|\{y \in J : |y - x| = \delta\}\big|$$

.

Then, by Remark 1.5, we have

$$\int_{\mathbb{R}^2} ndx = 4\delta\lambda_1(J).$$

Furthermore, we can eliminate the possibility that $n(x) = 0$ because It is clear that if $n(x) = 0$, J must be contained within the circle with midpoint x and radius δ, so $diam(J) \leq 2\delta$ which contradicts our assumption. And J is a closed polygon, so for almost ever x either $n(x) = 0$ or $n(x) \geq 2$. Consequently, $n(x) \geq 2$ for almost every $x \in U(J, \delta)$ so $\lambda_2(U(J, \delta)) \leq 2\delta\lambda_1(J)$. □

2. Discrepancy and Uniform Spread

Definition 2.1. If $S \subset \mathbb{R}^2$ is discrete, i.e. a set where every bounded subset of S is finite, and $H \subset \mathbb{R}^2$ is bounded and measurable, then the **discrepancy$_\Delta$** of S with respect to H is given by

$$\Delta(S, H) = \big||S \cap H| - \lambda_2(H)\big|$$

Note: We add the Δ subscript to differentiate between a different kind of discrepancy to be introduced in section 4.

Definition 2.2. We call a square of the form $[a, a+1) \times [b, b+1)$ with $a, b \in \mathbb{Z}$ a **unit square**. Further, given $c, d \in \mathbb{Z}$ we say $Q(x) = [c, c+1) \times [d, d+1)$. If H is a union of unit squares, then we denote the boundary of H by ∂H. Likewise, we say $p(H) = \lambda_1(\partial H)$

Definition 2.3. A discrete set $S \subset \mathbb{R}^2$ is **uniformly spread** if there exist constants $C, a > 0$ so that for every Jordan domain A with $p(A) \geq a$, enclosed by Jordan curve J,

$$\Delta(S, A) \leq Cp(A),$$

with $p(A) = \lambda_1(J)$, i.e. the one-dimensional Hausdorff measure of J.

Definition 2.4. A point in $\mathbb{Z}^2$ is called a **lattice point**. Also, polygons with lattice point vertices and edges parallel to the coordinate axes are called **lattice polygons**. Accordingly, if a lattice polygon is a square, then we call it a lattice square. Given a lattice polygon P, we say $\tilde{P}$ is the domain bounded by P and $\hat{P}$ is the union of lattice squares in $\tilde{P}$. Given a lattice square Q, we denote the side length of Q by $s(Q)$.

Lemma 2.5. *Let $\mathcal{H}$ be the family of all non-empty sets which are unions of finitely many unit squares. For every $H \in \mathcal{H}$ one of more of the following are true:*

1. *There is a lattice polygon so that $H = \hat{P}$*
2. *There are sets $H_1, H_2 \in \mathcal{H}$ such that $H_1 \cap H_2 = \emptyset, H_1 \cup H_2 = H$, and $p(H) = p(H_1) + p(H_2)$*
3. *There are sets $H_1, H_2 \in \mathcal{H}$ such that $H_1 \subset H_2, H = H_2 \setminus H_1$ and $p(H) = p(H_1) + p(H_2)$*

Proof. Given $H \in \mathcal{H}$, let V denote the set of lattice points contained in ∂H. We can turn this into a graph theory problem by considering V as a set of vertices. We join two vertices $p, q \in V$ by an edge if $|p - q| = 1$ and if $[p, q]$ belongs to ∂H creating a set of edges E. This gives us a graph $G = (V, E)$ in which all vertices have degree 2 or 4 - visually, each vertex is either a corner of the lattice of squares or a common point between 4 squares. Consequently, every edge in G is contained in at least one circuit and, clearly, each circuit of G is lattice polygon.

Let P be a circuit in G and let $H_1 = H \cap \hat{P}, H_2 = H \setminus \hat{P}$, splitting H into two sets. This gives us three cases: either H_1 and H_2 are nonempty, H is contained in $\hat{P}$, or H and $\hat{P}$ have empty intersection.

In the first case, if $H_1 \neq \emptyset$, $H_2 \neq \emptyset$ then $H_1, H_2 \in \mathcal{H}, H_1 \cup H_2 = H$ and $p(H) = p(H_1) + p(H_2)$. In the first case, condition (2) is satisfied.

To consider the second and third cases, we may assume that whenever P is a circuit in G then either $H \subset \hat{P}$ or $H \cap \hat{P} = \emptyset$. Let p be a vertex of G with minimal y-coordinate, and let P_0 be a circuit containing p. It is easy to see that in this case

$H \cap \hat{P}_0 = \emptyset$ is impossible and hence $H \subset \hat{P}_0$. If $H = \hat{P}_0$ then (1) is satisfied. If $H \neq \hat{P}_0$ then $\hat{P}_0 \setminus H \neq \emptyset$. Let $q = (a, b)$ be a lattice point in $\hat{P}_0 \setminus H$ with minimal y-coordinate. Let

$$Q = [a, a+1) \times [b, b+1), Q' = [a, a+1) \times [b-1, b),$$

then $Q \subset \hat{P}_0 \setminus H$ and $Q' \cap (\hat{P}_0 \setminus H) = \emptyset$ by the minimality of b. If $Q' \cap H = \emptyset$ then $Q' \cap \hat{P}_0 = \emptyset$ and, as $Q \subset \hat{P}_0$, the common side of Q and Q' belongs to P_0. However, this is impossible, since $P_0 \subset \partial H$ and $(Q \cup Q') \cap H = \emptyset$. Therefore $Q' \subset H$ and consequently, the common side of Q and Q', the segment $l = [a, a+1] \times \{b\}$, belongs to ∂H.

Since $Q' \subset H \subset \hat{P}_0$ and $Q \subset \hat{P}_0$, l cannot belong to P_0. Let P_1 be a circuit of G containing l; then $P_1 \neq P_0$. If $H \subset \hat{P}_1$ then $P_0 \subset \partial H \subset \hat{P}_1 \cup P_1$ and $P_1 \subset \partial H \subset \hat{P}_0 \cup P_0$ and hence $P_0 = P_1$ which is impossible. Therefore we have $H \cap \hat{P}_1 = \emptyset$.

Now we take $H_1 = \hat{P}_1$ and $H_2 = H \cup \hat{P}_1$. Then $H_1, H_2 \in \mathcal{H}$, $H_1 \subset H_2$, and $H = H_2 \setminus H_1$. Since $P_1 \subset \partial H$ and $H \cap \hat{P}_1 = \emptyset$, it is easy to check that $p(H) = p(H_1) + p(H_2)$. Hence (3) holds, and this completes the proof.

□

Lemma 2.6. *Let $C = \sum_{n=0}^{\infty} \frac{\Psi(2^n)}{2^n} < \infty$. For every Jordan domain A with boundary J there are non-overlapping lattice squares $Q_1, Q_2, \dots, Q_m$ so that*

$$A \setminus U(J, \sqrt{2}) \subset \bigcup_{j=1}^{m} Q_j \subset A \tag{2.7}$$

and

$$\sum_{j=1}^{m} \Psi(s(Q_j)) < 7Cp(A) \tag{2.8}$$

Proof. Let $\mathcal{L}$ be the set of lattice squares which are in A and are of the form $[a2^k, (a+1)2^k] \times [b2^k, (b+1)2^k]$, for some $a, b \in \mathbb{Z}$, and $k = 0, 1, 2, \dots$. However, we want a set of non-overlapping squares. Fortunately, if two squares in $\mathcal{L}$ overlap, i.e. have nonempty intersection, it is clear that one of them must be contained in the other, because for given a and b, a lattice square $[a2^k, (a+1)2^k] \times [b2^k, (b+1)2^k]$ is in $[a2^{k'}, (a+1)2^{k'}] \times [b2^{k'}, (b+1)2^{k'}]$ for all k, k' such that $0 \leq k \leq k'$. This tells us that for every $Q \in \mathcal{L}$ we can choose a $Q' \in \mathcal{L}$ so that $Q \subset Q'$. In this way, Q' is maximal, with respect to containment. Now, let $\mathcal{L}' = \{Q' | Q \in \mathcal{L}\}$, so then the elements of $\mathcal{L}'$ are non-overlapping with $\bigcup \mathcal{L}' = \bigcup \mathcal{L}$.

Let $\mathcal{L}' = \{Q_1, Q_2, \dots, Q_m\}$. If $x \in A \setminus U(J, \sqrt{2})$ then there is a unit square Q such that $x \in Q \subset A$. Thus $Q \in \mathcal{L}$ and hence $x \in \bigcup \mathcal{L} = \bigcup \mathcal{L}' = \bigcup_{j=1}^{m} Q_j$.

Let $\mathcal{L}_k = \{Q \in \mathcal{L}' | s(Q) = 2^k\}$ and $n_k = |\mathcal{L}_k|$, with $k = 0, 1, \dots$ if $Q \in \mathcal{L}_k$ then, as Q is a maximal element of $\mathcal{L}$, there is a lattice square Q^* such that $Q \subset Q^*, s(Q^*) = 2^{k+1}$ and $Q^* \not\subset A$. This implies $dist(Q, J) \leq \sqrt{2} \cdot 2^k$ and $Q \subset A \cap U(J, 2\sqrt{2} \cdot 2^k)$. Therefore,

$$\bigcup \mathcal{L}_k \subset A \cap U(J, 2\sqrt{2} \cdot 2^k). \tag{2.9}$$

Here we must deal with two cases: either $2\sqrt{2}\cdot 2^k < \frac{1}{2}diam(J)$ or $2\sqrt{2}\cdot 2^k \geq \frac{1}{2}diam(J)$.

In the first case, if $2\sqrt{2}\cdot 2^k < \frac{1}{2}diam(J)$ then by Theorem 1.8,

$$\lambda_2(U(J, 2\sqrt{2}\cdot 2^k)) \leq 4\sqrt{2}\cdot 2^k\lambda_1(J) = 4\sqrt{2}\cdot 2^k p(A)$$

For the second case, we know that measuring the left hand side of (2.9) gives us $\lambda_2(\bigcup\mathcal{L}_k) = n_k\cdot 2^{2k}$. Putting this together with the above equation, (2.9) gives us $n_k \leq 4\sqrt{2}\cdot p(A)\cdot 2^{-k}$. If $2\sqrt{2}\cdot 2^k \geq \frac{1}{2}diam(J)$ then, because we know that circles are area maximizing,

$$n_k 2^{2k} = \lambda(\bigcup\mathcal{L}_k) \leq \lambda_2(A) \leq \frac{\pi}{4}(diam(J))^2 \leq \frac{\pi}{4}(4\sqrt{4}\cdot 2^k)^2 = 8\pi\cdot 2^{2k}$$

and so $n_k \leq 8\pi$. But n_k is an integer, so we can do even better and say $n_k \leq 25$.

If $Q \in \mathcal{L}_k$, then $4\cdot 2^k = p(Q) \leq p(A)$ and we get

$$n_k \leq 25 \leq \frac{25}{4}p(A)2^{-k} < 7p(A)2^{-k}.$$

Consequently, from both cases we can conclude that whenever $\mathcal{L}_k \neq \emptyset$ then $n_k < 7p(A)\cdot 2^{-k}$. This gives us

$$\sum_{j=1}^{m}\Psi(s(Q_j)) = \sum_{k=0}^{\infty} n_k\Psi(2^k) < 7p(A)\sum_{k=0}^{\infty}\frac{\Psi(2^k)}{2^k} = 7Cp(A)$$

$\square$

Lemma 2.10. *Let H be a set such that either $H \in \mathcal{H}$ or $\mathbb{R}^2 \setminus H \in \mathcal{H}$. If, for every $C > 0$, we say $N(C)$ is the smallest $N \in \mathbb{N}$ such that*

$$\left(1 + \frac{1}{4C}\right)^N > 16(N+1)^2, \tag{2.11}$$

then for every $C > 0$, there exists an $n \in \mathbb{Z}$ such that $1 \leq n \leq N(C)$ and

$$\lambda_2(H_n \setminus H) \geq Cp(H_n)$$

with H_n defined as in Lemma 2.5.

Proof. We put $s_n = p(H_n)$ with $n = 1, 2, \ldots$. Our assumption on H implies that s_n is finite for every n. Obviously, ∂H_n is the union of s_n segments of unit length. Each of these segments belongs to the boundary of one of the unit squares contained in $H_{n+1}\setminus H_n$. On the other hand, each unit square in $H_{n+1}\setminus H_n$ contains at most four of these segments, and hence

$$\lambda_2(H_{n+1}\setminus H_n) \geq \frac{s_n}{4}. \tag{2.12}$$

We'll prove the lemma from here by contradiction. Suppose the lemma is false, then

$$\lambda_2(H_n\setminus H) < Cp(H_n) = Cs_n \leq 4C\lambda_2(H_{n+1}\setminus H_n) = 4C\lambda_2(H_{n+1}\setminus H) - 4C\lambda_2(H_n\setminus H)$$

for all n such that $1 \leq n \leq N(C)$. Consequently, for such n,

$$\lambda_2(H_{n+1}\setminus H) > \left(1 + \frac{1}{4C}\right)^{N(C)}\lambda_2(H_1\setminus H).$$

and so

$$\lambda_2(H_{N(C)+1} \setminus H) > \Big(1 + \frac{1}{4C}\Big)^{N(C)} \lambda_2(H_1 \setminus H). \tag{2.13}$$

We want to show

$$\lambda_2(H_{N(C)+1} \setminus H) \leq 4(N(C)+1)^2 s_0 \tag{2.14}$$

because $\lambda_2(H_1 \setminus H) \geq \frac{s_0}{4}$ by (2.12), (2.14) contradicts (2.13) and (2.11), and so the contradiction would prove the Lemma.

To this end, let Q be a unit square with $Q \subset H_{N(C)+1} \setminus H$. Then there is a sequence of unit squares $Q_0, Q_1, \ldots, Q_n$. such that, for $n \leq N(C)+1$, $Q_0 \subset H$, Q_i and Q_{i-1} are adjacent for every $i = 1, \ldots, n$, and $Q_n = Q$. Since $Q \not\subset H$, there is an $i \geq 1$ such that $\partial Q_i \cap \partial H \neq \emptyset$. If p is a lattice point in $\partial Q_i \cap \partial H$ and T is a lattice square with center p and with $s(T) = 2(N(C)+1)$, then $Q \subset T$. As ∂H contains at most s_0 lattice points, this argument shows that $H_{N(C)+1} \setminus H$ can be covered by s_0 squares of area $4(N(C)+1)^2$. This proves (2.14) which completes the proof. □

Lemma 2.15. *For every $H \in \mathcal{H}$ and $C > 0$, there exists a $K \in \mathcal{H}$ such that $H \subset K \subset H_{N(C)}$ and*

$$\lambda_2(H_{N(C)}) \geq \lambda_2(K) + Cp(K). \tag{2.16}$$

Proof. Applying Lemma 2.10 to the set $A = \mathbb{R}^2 \setminus H_{N(C)}$ we get

$$\lambda_2(A_n \setminus A) \geq Cp(A_n) \tag{2.17}$$

for $1 \leq n \leq N(C)$

We put $K = \mathbb{R}^2 \setminus A_n$. Then $K \subset \mathbb{R}^2 \setminus A = H_{N(C)}$. We show $H \subset K$. Let Q be a unit square in H and suppose that $Q \not\subset K$. Then $Q \subset A_n$ and hence there are unit squares $Q_0, \ldots, Q_n$ such that $Q_0 \subset A$, Q_i and Q_{i-1} are adjacent for every $i = 1, \ldots, n$ and $Q_n = Q$. Since $Q \subset H$, this implies $Q_0 \subset H_n \subset H_{N(C)} = \mathbb{R}^2 \setminus A$ which is impossible. Hence we have $H \subset K \subset H_{N(C)}$. Since $A_n \setminus A = H_{N(C)} \setminus K$ and $p(K) = p(A_n)$, (2.16) follows from (2.17).

□

Lemma 2.18. *Let S be a discrete subset of $\mathbb{R}^2$ and suppose that*

$$\Delta(S, \hat{P}) \leq C\lambda_1(P) \tag{2.19}$$

holds for every lattice polygon with a constant $C > 0$. Then there is a bijection $\phi : S \to \mathbb{Z}^2$ such that

$$|\phi(x) - x| \leq M$$

holds for every $x \in S$, where $M = N(C) + \sqrt{2}$.

Proof. We do this proof in two parts. First, we do an induction proof, preparing us for the second part where we repeat the previous trick of turning the problem into a graph theory problem. For this to work, however, we'll need to use a theorem of graph theory, which we'll dub the Rado theorem since it was proven by R. Rado:

Given any system of k vertices in S (or $\mathbb{Z}^2$) that is adjacent to at least k vertices in $\mathbb{Z}^2$ (or S, respectively), Γ contains a one-factor.

For the first part, we show that

$$\Delta(S, H) \leq Cp(H), \forall H \in \mathcal{H} \tag{2.20}$$

by induction over $p(H)$. Let $H \in \mathcal{H}$ be given and suppose that the statement is true for every $H' \in \mathcal{H}$ with $p(H') < p(H)$. By Lemma 2.5, atleast one of (1), (2), and (3) holds. In case of (1), (2.20) follows from (2.19). If (2), then $p(H_i) \leq p(H)$ for $i = 1, 2$, and this implies that (2.20) holds for H_1 and H_2, by the induction hypothesis. This gives us

$$\Delta(S, H) \leq \sum_{i=1}^{2} \Delta(S, H_i) \leq C(p(H_1) + p(H_2)) = Cp(H)$$

If (3) holds, then we have $p(H_i) < p(H)$ for $i = 1, 2$ once more and so

$$\Delta(S, H) = |(|S \cap H_2| - \lambda_2(H_2)) - (|S \cap H_1| - \lambda_2(H_1))| \leq \sum_{i=1}^{2} \Delta(S, H_i) \leq C(p(H_1) + p(H_2)) = Cp(H)$$

So (2.20) regardless of which case we have.

Now for the second part, we restate the lemma in terms of graph theory. The lemma is equivalent to the claim that the bipartite graph

$$\Gamma = \{(x, y) : x \in S, y \in \mathbb{Z}^2, |x - y| \leq M\}$$

contains a one-factor, i.e. there is a set of edges such that each vertex in Γ is incident to exactly one edge in the set. The degree of each vertex of Γ is finite since both S and $\mathbb{Z}^2$ are discrete. Therefore, by the Rado theorem the existence of a one-factor in Γ follows from the following condition:

Any system of k vertices in S is adjacent to at least k vertices in S.

Let $A \subset \mathbb{Z}^2$ be given with $|A| = k$. Let H be the union of all unit squares meeting A, then $H \in \mathcal{H}$ and $\lambda_2(H) = k$. By Lemma 2.10, there is an integer $1 \leq n \leq N(C)$ such that

$$\lambda_2(H_n) - Cp(H_n) \geq \lambda_2(H) = k$$

Then, by (2.20),

$$|S \cap H_n| \geq \lambda_2(H_n) - Cp(H_n) \geq k$$

Obviously,

$$H_n \subset U(H, n) \subset U(A, n + \sqrt{2}) \subset U(A, N(C) + \sqrt{2}) = U(A, M)$$

and hence $|S \cap U(A, M)| \geq k$. This shows that A is adjacent with at least k vertices in S.

Next, let $B \subset S$ be given with $|B| = k$. Let H be the union of all unit squares meeting B. Then $H \in \mathcal{H}$ and hence, by Lemma 2.15, there exists a $K \in \mathcal{H}$ such that $H \subset K \subset H_{N(C)}$ and

$$\lambda_2(H_{N(C)}) \geq \lambda_2(K) + Cp(K) \tag{2.21}$$

From (2.20) and (2.21), we have

$$k \leq |S \cap H| \leq |S \cap K| \leq \lambda_2(K) + Cp(K) \leq \lambda_2(H_{N(C)})$$

and so

$$|\mathbb{Z}^2 \cap H_{N(C)}| = \lambda_2(H_{N(C)}) \geq k.$$

Since

$$H_{N(C)} \subset U(H, N(C)) \subset U(B, N(C) + \sqrt{2}) = U(B, M),$$

we have $|\mathbb{Z}^2 \cap U(B, M)| \geq k$. Therefore, B is adjacent with at least k points in $\mathbb{Z}^2$. Thus the condition of the Rado theorem is fulfilled, proving the lemma.

□

Theorem 2.22. *Let $\Psi : [0, \infty) \to \mathbb{R}$ be a nonnegative, increasing and continuous function so that*

$$C = \sum_{n=0}^{\infty} \frac{\Psi(2^n)}{2^n} < \infty$$

If $S \subset \mathbb{R}^2$ is discrete and

$$\Delta(S, \tilde{Q}) \leq \Psi(s(Q))$$

for every square Q with $s(Q) \geq 1$, then S is uniformly spread.

Proof. Let Q be a lattice square. It is then clear that

$$|S \cap \hat{Q}| \leq |S \cap \tilde{Q}| \leq \lambda_2(\tilde{Q}) + \Psi(s(Q)).$$

However, if $s(Q) \geq 2$, then we take a square I_ϵ so that $I_\epsilon \subset intQ$ (?) and $s(I_\varepsilon) = s(Q) - \varepsilon, 0 < \varepsilon < 1$. This gives us

$$|S \cap \hat{Q}| \geq |S \cap I_\varepsilon| \geq \lambda_2(I_\varepsilon) - \Psi(s(I_\varepsilon))$$

Then, letting ε tend to 0, we get $|S \cap \hat{Q}| \geq \lambda_2(\tilde{Q}) - \Psi(s(Q))$ and

$$\left| |S \cap \hat{Q}| - \lambda_2(\tilde{Q}) \right| \leq \Psi(s(Q)). \tag{2.23}$$

Also, if $s(Q) = 1$, then

$$|S \cap \hat{Q}| \geq 0 = \lambda_2(\tilde{Q}) - 1.$$

Then, if we replace Ψ by $max\{1, \Psi\}$, (2.23) will be satisfied by every lattice square Q. Let P be a lattice polygon. Furthermore, Lemma 2.6 makes clear that if J is a lattice polygon, then it is equal to the union of non-overlapping lattice squares. By Lemma 2.6, then there are lattice squares $Q_1, Q_2, \ldots, Q_m$ such that (2.8) holds. Then $\hat{P}$ is the disjoint union of the sets $\hat{Q}_j, j = 1, \ldots, m$ and hence

$$\Delta(S, \hat{P}) \leq \sum_{j=1}^{m} \left| |S \cap \hat{Q}_j| - \lambda_2(\tilde{Q}_j) \right| \leq \sum_{j=1}^{m} \Psi(s(Q_j)) \leq 7C \cdot \lambda_1(P).$$

So the theorem follows from Lemma 2.18. □

3. Translation Equidecomposability

As alluded to briefly in the introduction, we are primarily concerned with the development of a criterion for determining when two Jordan domains are equidecomposable. More generally, we say two sets $A, B \subseteq \mathbb{R}^2$ are **equidecomposable** when, letting $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{i=1}^n B_i$ with $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $i \neq j$, there is a group of bijections G such that for each i there is a $\gamma_i \in G$ with $\gamma_i(A_i) = B_i$. Put simply, A and B are equidecomposable if they can each be decomposed into the same number of pieces congruent by G. If A and B can be shown to be equidecomposable with only translations in G, then we say that A and B are **translation equidecomposable** which we denote by $A \sim^{tr} B$.

As mentioned before, we'll now define a different kind of discrepancy.

Definition 3.1. Let $I = [0,1)$, and suppose $S \subset I^n$, $|S| = N$, and $H \subset I^n$ is measurable, then the **discrepancy**$_D$ of S with respect to H is

$$D_n(S,H) = \left|\frac{1}{N}|S \cap H| - \lambda_n(H)\right|.$$

Furthermore, the discrepancy of the finite set $S \subset I^n$ is

$$D_n(S) = sup D_n(S,J),$$

where the sup is taken over all subintervals $J = \times_{i=1}^n [a_i, b_i) \subset I^n$

Theorem 3.2. *Let Ψ be a nonnegative function on $\mathbb{N}$ so that*

$$C = \sum_{k=0}^{\infty} \frac{\Psi(2^k)}{2^k} < \infty.$$

Now let $H_1, H_2 \subset I^2$ be measurable with $\lambda_2(H_1) = \lambda_2(H_2) > 0$. Suppose that there are $x, y \in \mathbb{R}^2$ so that

1. *the vectors $x, y, i = (0,1), j = (0,1)$ are linearly independent over $\mathbb{Q}$, and*
2. *$N^2 \cdot D_2(S_N(u,x,y), H_r) \leq \Psi(N), \forall u \in \mathbb{R}^2, N \in \mathbb{N}, r \in \{1,2\}$*

Then H_1 and H_2 are translation-equidecomposable.

Proof. Let $\lambda_2(H_1) = \lambda_2(H_2) = \alpha^2, \alpha > 0$. Also let

$$S_r(u) = \{(n,k) | (u + nx + ky) \in H_r\}$$

with $u \in \mathbb{R}^2, r = 1,2$. First, we want to show that there is a bijection $\phi_u : S_1(u) \to S_2(u)$ so that $|\phi_u(z) - z|$ is uniformly bounded. This first requires proving that the sets $\alpha S_r(u)$ are uniformly spread for all u and for all $r \in \{1,2\}$.

If Q is a lattice square then, for $r = 1,2$

$$\left| |S_r(u) \cap \hat{Q}| - \alpha^2 \lambda_2(Q) \right| \leq \Psi(s(Q)) \tag{3.3}$$

Now, let $Q = [a, a+N] \times [b, b+N]$, which gives us

$$\begin{aligned} |S_r(u) \cap \hat{Q}| &= |\{(n,k) : a \leq n < a+N, b \leq k < b+N, (u + nx + ky) \in H_r\}| \\ &= |\{(n',k'); 0 \leq n', k' < N, (u + ax + by + n'x + k'y) \in H_r\}| \\ &= |s_N(u + ax + by, x, y) \cap H_r| \end{aligned}$$

Which gives us

$$\left| \frac{1}{N^2} |S_r(u) \cap \hat{Q}| - \alpha^2 \right| = D_2(s_N(u + ax + by, x, y); H_r)$$

So the equation (3.3) follows from the second condition above.

Let P be a lattice polygon, and let $J = \alpha^{-1}P$ and $A = \alpha^{-1}\tilde{P}$. By Lemma 2.6, there are lattice squares $Q_1, Q_2, \ldots, Q_m$ such that (2.7) and (2.8)) hold. It is easy to see that (2.7) implies $\bigcup_{j=1}^m \hat{Q}_j \supset A \setminus U(J, \sqrt{2})$ and, as the sides of P are parallel to the coordinate axes, $\bigcup_{j=1}^m \hat{Q}_j \subset \alpha^{-1}\hat{P}$. Therefore,

$$\bigcup_{j=1}^m \hat{Q}_j \subset \alpha^{-1}\hat{P} \subset U(J, \sqrt{2}) \cup \bigcup_{j=1}^m \hat{Q}_j. \tag{3.4}$$

Let $U = \alpha^2 \sum_{j=1}^{m} \lambda_2(Q_j)$, $V = \sum_{j=1}^{m} \Psi(s(Q_j))$ and $W = |S_r(u) \cap U(J, \sqrt{2})|$. Then (3.3) and (3.4) imply

$$U - V \leq |S_r(u) \cap \alpha^{-1}\hat{P}| \leq U + V + W. \tag{3.5}$$

Suppose that $diam(J) > 4\sqrt{2}$. Then (3.4) and Theorem 1.8 imply

$$|U - \lambda_2(\hat{P}) \leq \lambda_2(\alpha U(J, \sqrt{2}) \leq \lambda_2(U(J, \sqrt{2})) \leq 2\sqrt{2} \cdot \lambda_1(J) = 2\sqrt{2}\alpha^{-1}\lambda_1(P).$$

Since $S_r(u) \subset \mathbb{Z}^2$, we have

$$W \leq \left|\mathbb{Z}^2 \cap U(J, \sqrt{2}\right| \leq \lambda_2(U(J, 2\sqrt{2})) \leq 4\sqrt{2}\alpha^{-1}\lambda_1(P).$$

Furthermore, (2.8) gives us $V \leq 7C\alpha^{-1}\lambda_1(P)$. Substituting these estimates of U, V, and W into (3.5), we get

$$\left||S_r(u) \cap \alpha^{-1}\hat{P}| - \lambda_2(\hat{P})\right| \leq (6\sqrt{2} + 7C)\alpha^{-1}\lambda_1(P) \tag{3.6}$$

supposing that $diam(J) > 4\sqrt{2}$. If $diam(J) \leq 4\sqrt{2}$ then $\alpha^{-1}\hat{P}$ can be covered by a square Q with $s(Q) \leq 7$ and hence

(3.7)

$$\left||S_r(u) \cap \alpha^{-1}\hat{P}| - \lambda_2(\hat{P})\right| \leq \left|S_r(u) \cap \hat{Q}\right| + 49 \leq 49\alpha^2 + \Psi(7) + 49 \leq (25 + \frac{\Psi(7)}{4})\lambda_1(P).$$

as $\lambda_1(P) \geq 4$.

It is also clear that $|S_r(u) \cap \alpha^{-1}\hat{P}| = |\alpha S_r(u) \cap \hat{P}|$, so (3.6) and (3.7) give us

$$D_2(\alpha S_r(u), \hat{P}) \leq C_1\lambda_1(P) \tag{3.8}$$

with $C_1 = max\{(6\sqrt{2} + 7C)\alpha^{-1}, 25 + \frac{\Psi(7)}{4}\}$. Since (3.8) holds for every lattice polygon $P, u \in \mathbb{R}^2$ and $r = 1, 2$, we may apply Lemma 2.18 and obtain the bijection $\phi_{u,r} : \alpha S_r(u) \to \mathbb{Z}^2$ such that $|\phi_{u,r}(z) - z| \leq N(C_1) + \sqrt{2}, \forall z \in \alpha S_r(u)$. We put $\phi_u = \alpha^{-1}\phi_{u,2}^{-1}(\phi_{u,1}(\alpha z))$ so $\phi_u : S_1(u) \to S_1(u)$ is a bijection such that

$$|\phi_u(z) - z| \leq 2\alpha^{-1}(N(C_1) + \sqrt{2}) = C_2, \forall z \in S_1(u) \tag{3.9}$$

This theorem will be completed by dealing with some group theory. Let G denote the group generated by the operator + and x, y, i, j with $i = (0, 1), j = (1, 0)$. We define an equivalence relation, denoted by $\sim$, where for $z_1, z_2 \in \mathbb{R}^2$,

$$z_1 \sim z_2 \iff (z_1 - z_2) \in G$$

Let E be an equivalence class and pick some $u \in E$. Then we can pick $n, k, l, m \in \mathbb{Z}$ so that every $z \in E$ can be described uniquely by

$$z = u + nx + ky + li + mj.$$

If $z \in H_1$, then $(u + nx + ky) \in H_1$ and so $(n, k) \in S_1(u)$. Let the function $\phi_u((n, k)) = (n', k')$. As $(n', k') \in S_2(u)$, we have $(u + n'x + k'y) \in H_2$ and so there exist $l', m' \in \mathbb{Z}$ so that

$$u + n'x + k'y + l'i + m'j \in H_2.$$

Now, let $\chi_u(z) = u + n'x + k'y + l'i + m'j$. Then χ_u is a well-definted map from $H_1 \cap E$ to $H_2 \cap E$. At this point, it is worth noting that n' and k' uniquely determine the integers l', m'. Furthermore, because ϕ_u is a bijection from $S_1(u)$ to $S_2(u)$, we know χ_u is a bijection from $H_1 \cap E$ onto $H_2 \cap E$.

By (3.9), $|n' - n| \leq C_2$ and $|k' - k| \leq C_2$. Since z, $\chi_u(z) \in I^2$, we have $|\chi_u(z) - z| \leq \sqrt{2}$ and so

$$|(l' - l)i + (m' - m)j| \leq \sqrt{2} + |n' - n| \cdot |x| + |k' - k| \cdot |y|).$$

Hence, $\forall z \in H_1 \cap E, \exists a, b, c, d$ so that

$$\chi_u(z) = z + ax + by + ci + dj$$

and

$$|a|, |b| \leq C_2,\ |c|, |d| \leq C_3 \tag{3.10}$$

Now, let $\{d_t\}_{t=1}^K$ be an enumeration of the vectors $ax + by + ci + dj$, where a, b, c, d satisfy (3.10). Then $K \leq (2C_2+1)^2(2C_3+1)^2$ and C_2, C_3 only depend on α and Ψ. We have now proved that $\forall z \in H_1 \cap E$, there is $1 \leq t \leq K$ so that $\chi_u(z) = z + d_t$. Since the equivalence class E was selected arbitrarily and $\forall t, d_t \in G$, this implies that there is a bijection $\chi : H_1 \to H_2$ so that for all z there is a t such that $\chi(z) = z + d_t$. Let

$$A_t = \{z \in H_1 : \chi(z) = z + d_t\}$$

with $t = 1, \ldots, K$.

Then $\bigcup_{t=1}^K A_t$ and $\bigcup_{t=1}^K (A_t + d_t)$ are disjoint decompositions of H_1 and H_2 respectively, which is what we want.

□

4. Circle-Squaring

Now that we have developed a criterion for translation equidecomposability, we can put it to use in an interesting application to Tarski's circle squaring problem. The problem was posed in 1925 and was not solved until 1990. Accordingly, the complete solution, although not conceptually difficult, is quite complicated and lengthy. Reproducing it completely would have easily added 20 pages to the paper, so instead we will merely sketch a solution by assuming the following three theorems.

Theorem 4.1. *If P_1 and P_2 are polygons of the same area, then $P_1 \sim^{tr} P_2$.*

Theorem 4.2. *For almost every pair of vectors $x, y \in \mathbb{R}^2$ and for every $\varepsilon > 0$ there is a constant C such that*

$$D_2(s_N(u, x, y)) \leq C \frac{l^{6+\varepsilon}(N)}{N^2} \tag{4.3}$$

for every $u \in \mathbb{R}^2$ and $N \in \mathbb{N}$.

Theorem 4.4. *Let f be twice differentiable on $[0, 1]$, let $f(0) = 0, f(1) = 1$, and suppose that there are positive constants a, b, c, d such that*

$$a \leq f'(x) \leq b, c \leq |f''(x)| \leq d, \forall x \in [0, 1] \tag{4.5}$$

Then for almost every pair of vectors $x, y \in \mathbb{R}^2$ and for every $\varepsilon > 0$ there is a constant C such that

$$D_2(s_N(u,x,y); H_f) \leq CN^{-4/3} l^{6+\varepsilon}(N) \tag{4.6}$$

for every $u \in \mathbb{R}^2$ and $N \in \mathbb{N}$.

Theorem 4.7. *Let J be a simple closed Jordan curve and let $O, A, B \in J$ ad suppose that the subarcs OA, AB, BO have the following properties:*

(1) OA and AB are line segments
(2) BO is a twice differentiable curve
(3) If P denotes the parallelogram having O, A and B as vertices then the arc BO is contained in P and neither of the sides of P is a tangent of BO.
(4) There are positive constants δ and K such that the curvature of BO lies between δ and K at each point of BO.

If Q is a square with $\lambda_2(Q) = \lambda_2(\tilde{J})$ then $\tilde{J} \sim^{tr} Q$.

Proof. Let U be a linear transformation of $\mathbb{R}^2$ such that $U(O) = (0,0), U(A) = (1,0)$ and $U(B) = (1,1)$. Then the image $F = U(BO)$ of the arc BO is contained in $[0,1]^2$. The conditions (2) and (4) easily imply that the curve BO is convex or concave and hence F is the graph of an increasing function $f : [0,1] \to [0,1]$. Then it follows from (2) and (3) that there are positive constants a, b such that $a \leq f'(x) \leq b$, $\forall x \in [0,1]$.

Since the curvature of F at the point $(x, f(x))$ equals $\frac{f''(x)}{(1+[f'(x)]^2)^{\frac{3}{2}}}$. it follows from (4) that there are positive constants c, d such that $c \leq |f''(x)| \leq d, \forall x \in [0,1]$, so f satisfies (4.5).

Therefore, by Theorem 4.4, for almost every pair of vectors $x, y \in \mathbb{R}^2$ there is a constant C such that

$$D_2(s_N(u,x,y); H_f) \leq CN^{\frac{-4}{3}} l^7(N) \tag{4.8}$$

for every $u \in \mathbb{R}$ and $N \in \mathbb{N}$. Let $Q_1 \subset [0,1)^2$ be a square with $\lambda_2(Q_1) = \lambda_2(H_f)$. Then, by Theorem 4.2, for almost every pair of vectors $x, y \in \mathbb{R}^2$ there is a constant C' such that

$$D_2(s_N(u,x,y); Q_1) \leq C' \frac{l^7(N)}{N^2} \tag{4.9}$$

for every $u \in \mathbb{R}$ and $N \in \mathbb{N}$.

Therefore, we can fix a pair of vectors $x, y \in \mathbb{R}^2$ and constants C, C' such that (4.8) and (4.9) hold for every $u \in \mathbb{R}$ and $N \in \mathbb{N}$. Let $\Psi(N) = max(C, C')N^{2/3} l^7(N)$, giving us

$$\sum_{k=0}^{\infty} \frac{\Psi(2^k)}{2^k} < \infty$$

Also, $N^2 \cdot D_2(s_N(u,x,y); H_f) \leq \Psi(N)$ and $N^2 \cdot D_2(s_N(u,x,y); Q_1) \leq \Psi(N)$ hold for every $u \in \mathbb{R}$ and $N \in \mathbb{N}$. Thus, by Theorem 3.1, $H_f \sim^{tr} Q_1$. This easily implies that

$$U^{-1}(H_f) \sim^{tr} U^{-1}(Q_1) =^{def} P_1.$$

Now we have $U(\tilde{J}) = H_f \cup (\{1\} \times [0,1])$ and hence $\tilde{J}$ differs from $U^{-1}(H_f)$ in the line segment AB. Since P_1 is a parallelogram, this implies that $\tilde{J} \sim^{tr} P_1$. Also, $\lambda_2(P_1) = \lambda_2(\tilde{J}) = \lambda_2(Q)$ and hence, by Theorem 4.1, $P_1 \sim^{tr} Q$. Therefore, we have $\tilde{J} \sim^{tr} Q$, completing the proof. □

Theorem 4.10. *Let J be a simple closed Jordan curve such that J can be decomposed into finitely many subarcs $J_1, J_2, \ldots, J_n$ with the following properties:*

(1) J_i is a twice differentiable curve for every $i = 1, 2, \ldots, n$.
(2) For every $i = 1, 2, \ldots, n$, either J_i is a line segment, or there are positive constants δ, k such that the curvature of J_i lies between δ and K at each point of J_i.
(3) J has no cusps, i.e. at the common end-point of J_i and J_j with $i \neq j$ the half tangents of J_i, J_j do not coincide.

If Q is a square with $\lambda_2(Q) = \lambda_2(\tilde{J})$, then $\tilde{J} \sim^{tr} Q$.

Proof. Let $\Phi = \{A_0, \ldots, A_{m-1}, A_m = A_0$ be a subdivision of J containing the end-points of the arcs J_i. It is easy to see that if Φ is fine enough, then we can find points $P_0, \ldots, P_{m-1}, P_m = P_0 \in \tilde{J}$ such that

(1) The line segments $p_i = P_iA_{i-1}$ and $q_i = P_iA_i$ are in $\tilde{J}$ and hence p_i, q_i, and the subarc $A_{i-1}A_i$ of J constitute a simple closed Jordan curve T_i for every $i = 1, 2, \ldots, m$.
(2) $\tilde{T}_1, \ldots, \tilde{T}_m$ are non-overlapping.
(3) Either T_i is a triangle or it satisfies the conditions of theorem 4.7, $\forall i$.

Consequently, there are non-overlapping squares $Q_1, \ldots, Q_m$ such that $T_i \sim Q_i, \forall i$. Since $S = \tilde{J} \setminus \bigcup_{i=1}^m \tilde{T}_i$ is a polygon, there is a square Q_0 such that $S \sim Q_0$ by Theorem 4.1. We may assume that $Q_0 \cap Q_i$ for every $i = 1, \ldots, m$. Then $\tilde{J} \sim \bigcup_{i=0}^m Q_i$ and hence applying Theorem 4.1 again, we obtain $\tilde{J} \sim Q$, which is what we want. □

Remark 4.11. Clearly, if J is a circle, then it fulfills the conditions of Theorem 4.10, so the enclosed disc is translation equidecomposable with a square of equal volume. This provides an affirmative answer to Tarski's circle-squaring problem.

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