ELECTRICAL NETWORK THEORY AND RECURRENCE IN DISCRETE RANDOM WALKS

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Abstract. This paper investigates recurrence in one dimensional random walks. After proving recurrence, an alternate approach using electrical network theory is analyzed. Using harmonic functions, the function governing voltage and the function describing the probability in a random walk are proven to be the same. Then, electrical theory is used to prove that the one dimensional random walk is recurrent.

Contents

1. Random Walks

Definition 1.1 (Simple Random Walk)**.** A simple random walk is a random walk on the regular d-dimensional lattice with vertices on the integers $\mathbb{Z}^d$ and edges between adjacent points, i.e. points where only one coordinate differs by ± 1. The probability of moving in each possible direction of travel is equal, so this probability is $1/(2d)$. Formally, consider the space of all infinite paths beginning at the origin. Fix a path with length n; the probability that an infinite path begins in this manner is $[1/(2d)]^n$.

Random walks can take place on non-regular graphs, but for simplicity, this paper is only concerned with simple random walks.

Figure 1. A one dimensional lattice.

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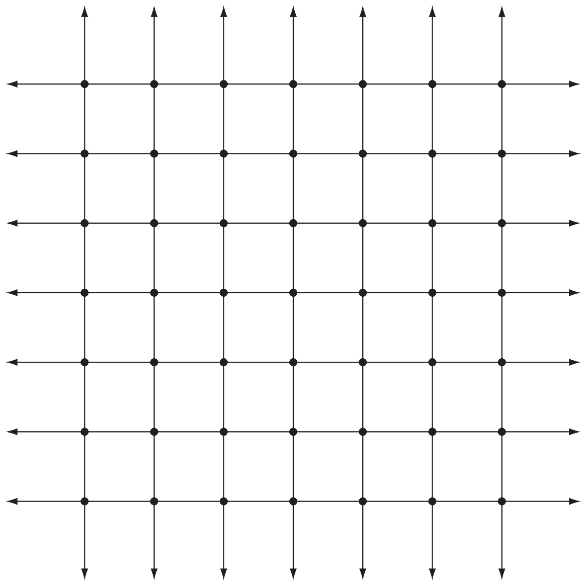

FIGURE 2. A two dimensional lattice.

In one dimension, an easy way to visualize the probability of arriving at a point in k steps is Pascal's triangle. Importantly, the probabilities at any step all sum to 1.

	Location								
Step	-4	-3	-2	-1	0	1	2	3	4
0					1				
1				1/2	0	1/2			
2			1/4	0	2/4	0	1/4		
3		1/8	0	3/8	0	3/8	0	1/8	
4	1/16	0	4/16	0	6/16	0	4/16	0	1/16

TABLE 1. Pascal's Triangle.

2. Recurrence

We begin by introducing the notion of escape probability.

Definition 2.1 (Escape probability)**.** Consider the set B_k of points on an n-dimensional lattice that have at least one coordinate equal to k or $-k$. Note that on a three dimensional lattice, this set is a cube of sidelength $2k$ with its center at the origin. The escape probability $p_e^{(k)}$ is the probability that starting at the origin, the walker reaches B_k before returning to the origin. Since any path meeting B_{k+1} necessarily meets B_k, $p_e^{(k+1)} \leq p_e^{(k)}$. The escape probability p_e is defined as

$$p_e = \lim_{k\to\infty} p_e^{(k)}.$$

As $p_e^{(k)}$ is a nonincreasing sequence bounded below by 0, this limit exists.

Lemma 2.2. *If a walker visits a vertex v infinitely many times with probability 1, then for any other vertex w, it visits w with probability 1.*

Proof. Starting at v, there is a non-zero probability $0 < p_1 \leq 1$ that the walker will pass through w before returning to v. By our assumption, the walker will return to v infinitely many times. Accordingly, the probability that the walker will pass through w before returning to v on the second attempt and not the first is $p_2 = p_1 \cdot (1 - p_1)$ since $1 - p_1$ is the probability that the walker will not pass through w on the first try, and p_1 is the probability that on the second try, the walker will pass through w. Generally, the probability of passing through w before returning to v on exactly the nth try is

$$p_n = p_1 \cdot (1 - p_1)^n.$$

Thus the probability of passing through w is

$$p = \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} p_1 (1 - p_1)^i = p_1 \sum_{i=1}^{\infty} (1 - p_1)^i.$$

Since $0 < p_1 < 1$, we have a geometric series multiplied by p_1, which converges to

$$p_1 \left(\frac{1}{1 - (1 - p_1)} \right) = \frac{p_1}{p_1} = 1.$$

Therefore, the walker has probability 1 of visiting w, as desired. □

Lemma 2.3. *If the escape probability is 0, then the walker has a probability 1 of returning to the origin.*

Proof. Since the escape probability is zero, there exists some bounded set of points inside which the walker remains with probability 1. By the pigeonhole principle, there is some point v which is visited infinitely many times with probability 1. Therefore, by Lemma 2.2, the walker has probability 1 of returning to the origin. □

Definition-Proposition 2.4 (Recurrence)**.** *A simple random walk is recurrent if any of the following equivalent statements holds:*

(1) a walker has a probability 1 of returning to the origin.
(2) a walker has a probability 1 of returning to the origin infinitely many times.
(3) a walker has a probability 1 of passing through every point of the lattice at least once.

Proof of equivalence. To prove that $1 \Rightarrow 2$, we prove that a walker has a probability 0 of returning to the origin exactly i times. We know that a walk returns to the origin with a probability 1. The walk is infinite, so once the walker returns to the origin, it has a probability 1 of returning to the origin again. Accordingly, all walks have a probability 0 of returning to the origin exactly i times. The origin is thus returned to infinitely many times as all walks must return to the origin.

To prove that $2 \Rightarrow 3$, note that the assumption is that the walker returns to the origin infinitely many times. For any point w, the walker has a probability 1 of passing through w at least once by Lemma 2.2.

$3 \Rightarrow 1$ is trivial as if the walker must pass through every point of the lattice, it must return to and pass through the origin. □

Definition 2.5 (Transience)**.** A random walk is transient if it is not recurrent.

Theorem 2.6 (Recurrence in one dimension)**.** *A one-dimensional simple random walk is recurrent.*

Proof. We know that $p_e^{(k)}$ is a nonincreasing sequence bounded below by 0. Thus to show that $p_e = \lim_{k\to\infty} p_e^{(k)}$ is 0, it suffices to show that the sequence $p_e^{(k)}$ becomes arbitrarily small.

In a one dimensional random walk, if the walker is at the point x, by symmetry he is equally likely to reach the point $2x$ before returning to the origin as it is to reach the origin before reaching $2x$. The probability that a walker reaches x from the origin is equal to the probability of reaching the origin from x as the probability of moving left or right at any point is equal. It thus follows that the probability

$$p_e^{(2n)} = \frac{p_e^{(n)}}{2}$$

as the walker is half as likely to reach a point $2n$ away from the origin before returning to the origin as it is to reach a point n away. Since the walker always moves left or right before returning to the origin, $p_e^{(1)} = 1$, and it follows that $p_e^{(2)} = 1/2$, $p_e^{(4)} = p_e^{(2)}/2 = 1/4$, and generally,

$$p_e^{(2^k)} = \frac{1}{2^k}.$$

Therefore, $p_e^{(k)}$ becomes arbitrarily small, so

$$p_e = \lim_{k\to\infty} p_e^{(k)} = 0.$$

□

3. Harmonic Functions

In the remaining sections, we shift focus to analyzing random walks through the framework of electric networks.

Definition 3.1 (Harmonic function)**.** A harmonic function is a function that satisfies the property that the value at a non-boundary point is equal to the average of the adjacent points.

Example 3.2 (One-dimensional harmonic function)**.** Let $S = \{0, 1, \ldots, n\}$. Let $I = \{1, 2, \ldots, n-1\}$, which is the set of interior points of S. Let $B = \{0, n\}$, which is the set of boundary points of S. A function $h : S \to \mathbb{R}$ is harmonic if for all $x \in I$,

$$h(x) = \frac{h(x+1) + h(x-1)}{2}.$$

A few important properties of harmonic functions must now be proven.

Lemma 3.3 (Maximum Principle)**.** *A one dimensional harmonic function h achieves its maximum value M and its minimum value m on B.*

Proof. Let $M = \max\{h(x) \mid x \in S\}$. To avoid triviality, assume $x \notin B$. Since h is harmonic, $h(x+1) = h(x-1) = M$. If $x - 1 \in I$, then the same argument proves that that $h(x-2) = h(x-1) = M$. This process continues until a boundary point is reached, when $h(0) = M$. The proof is similar to show that the minimum is achieved on the boundary. □

Theorem 3.4 (Uniqueness Principle)**.** *If g and h are one dimensional harmonic functions such that for all $x \in B$, $g(x) = h(x)$, then $g = h$.*

Proof. Let $f = g - h$. If $x \in I$,

$$\frac{f(x+1)+f(x-1)}{2} = \frac{g(x+1)+g(x-1)}{2} - \frac{h(x+1)+h(x-1)}{2}.$$

Accordingly, f is harmonic. If $x \in B$, then $f(x) = 0$, which by the Maximum Principle means that the maximum and minimum values of f are 0. Thus, $f = 0$ for all $x \in S$, so $g = h$. □

Theorem 3.5. *The function modeling a simple random walk in one dimension is harmonic.*

Proof. Define $S = \{0, 1, \ldots, n\}$, $I = \{1, 2, \ldots, n-1\}$, and $B = \{0, n\}$. In a simple, one dimensional random walk on the integers, let $p : S \to \mathbb{R}$ be defined by letting $p(x)$ be the probability that that the walker, starting at a given point x, gets to n before 0. Note that $p(n) = 1$ and $p(0) = 0$ by definition. There is a 1/2 chance of moving to the left and a 1/2 chance of moving to the right by 1 at any point $x \in I$. If the walker moves to the left, then the new probability of reaching n before 0 is $p(x-1)$, and if the walker moves to the right, this probability is $p(x+1)$. We thus have the relation that

$$p(x) = (1/2)p(x-1) + (1/2)p(x+1) = \frac{p(x-1)+p(x+1)}{2}.$$

Therefore, $p(x)$ is harmonic. □

4. Electrical Networks

4.1. **Basic Facts.** The following facts of physics will not be proven in this paper [1]. If you are already familiar with basic circuits, feel free to skip this section.

Fact 4.1.1 (Kirchoff's Loop Laws)**.**

(1) *Let x be a point on a circuit. The current flowing into x is equal to the current flowing out of x.*
(2) *In any closed circuit, the sum of the potential differences is zero.*

Fact 4.1.2 (Ohm's Law)**.** *The potential difference v across a resistor is equal to the product of the current i and the resistance R. In other words, $v = iR$.*

Additionally, one fact about resistors is necessary.

Fact 4.1.3. *When adding resistors in series (one after another),*

$$R_{total} = \sum_{i=1}^{n} R_i.$$

4.2. **Electrical Functions.** First, we must describe the circuit. Take a simple, one-dimensional graph with n vertices $\{0, 1, \ldots, n\}$. Put a resistor between every pair of adjacent vertices $(i, i+1)$ with all resistors having equal resistance. Apply a 1 volt potential difference to the circuit. Note that the current travels such that it hits the vertex n first and the vertex 0 last. Ground the vertex $x = 0$ such that $v(0) = 0$. Therefore, $v(n) = 1$.

Lemma 4.2.1. *The voltage across a simple one-dimensional circuit (resistors connected in series) is a harmonic function.*

Proof. By Ohm's Law, between points a and b on a circuit that are connected by a resistance R, the current is

$$i_{a,b} = \frac{v(a) - v(b)}{R}.$$

By Kirchoff's Loop Law, if $x \neq 0, n$,

$$\frac{v(x+1) - v(x)}{R} - \frac{v(x) - v(x-1)}{R} = 0.$$

Multiplying by R, we have

$$v(x+1) - v(x) - v(x) - v(x-1) = 0$$

as well. Solving for $v(x)$ yields

$$v(x) = \frac{v(x+1) + v(x-1)}{2}.$$

Therefore, v is harmonic. □

Theorem 4.2.2. *The simple random walk and the voltage across this circuit are the same; that is, the function p from Section 2 is equal to the function v.*

Proof. Both v and p are harmonic. Since $p(0) = v(0) = 0$ and $p(n) = v(n) = 1$, their boundary values are equal. Therefore, by the Uniqueness Principle, $v = p$. □

4.3. **Effective Resistance and Recurrence.** If there is a voltage between points x and y such that the voltage $v_x = v$ and $v_y = 0$, then by Ohm's Law, a current will flow into x. Let

$$i_x = \sum_n i_n,$$

which is the sum of the current flowing from x to the points connected to x.

Definition 4.3.1 (Effective Resistance)**.** By Ohm's Law, the effective resistance R_e between x and y in the circuit is

$$R_e = \frac{v_x}{i_x}.$$

An important fact to note about effective resistance is that the actual voltage applied does not affect the effective resistance.

We now consider a one dimensional random walk and sketch a proof of recurrence using the language of electrical networks.

Since this is an electrical network, positive charges flow from the positive terminal of the voltage supply to the negative terminal.

First, we explain how current can be interpreted in a probabilistic manner. Consider a circuit where the point a is at the positive terminal of a 1 volt voltage source and b is at the negative terminal. Between x and y where x and y are points on the circuit between a and b, the current between x and y is equal to the the expected number of times a walker who starts at a will pass along the part of the circuit between x and y before reaching b.

We can now get to the proof. Consider a finite one dimensional lattice consisting of the integer vertices from $-k$ to k such that between adjacent vertices, there is a one Ohm resistor. Connect both sides of the lattice to the negative terminal of the voltage source, which applies a one volt potential difference. Since these resistors are all in series, the effective resistance from 0 to k is

$$\sum_{i=1}^{k} R_i = k$$

where $R_i = 1$ for all i. Also, note that for the circuit as a whole, $v = iR_e \Rightarrow i = v/R_e$. Thus, as $k \to \infty$, $R_e \to \infty$. Accordingly,

$$\lim_{k\to\infty} \frac{v}{R_e} = 0,$$

indicating that the current in the system goes to zero. Accordingly, there is no current flowing through the system. As discussed above, the lack of current indicates that the probability a walker can go infinitely far away from the origin before returning is zero, and thus the random walk is recurrent.

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References

[1] J. D. Cutnell and K. W. Johnson. Physics. John Wiley & Sons. 2003

[2] P. G. Doyle and J. L. Snell. Random Walks and Electric Networks. Mathematical Association of America. 1984.