WORD THEORY AND THE MUSICAL SCALE

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ABSTRACT. For centuries scholars have wrestled to explain the ability of music to move and invoke our emotions. Music Theory is the attempt to explain the events that happen in a piece of music by characterizing its features and conceptualizing them into tangible ideas that can be used as a basis for comparison or understanding. However, once characterizations are complete, strenuous cognitive questions as to why certain features excite certain emotions or are better or worse to listen to are left unanswered. The romantic notion of the sublime, natural power inherent in music often dominates over the scientific characterization of sound. While the physical properties of pitch and the desire for modulation provide a strong argument for the development of a twelve-tone system, many questions as to why certain elements are preferred remain unanswered. In this paper, I bring forth some recent findings in the connection of music theory and word theory published by Clampitt, Domínguez, and Noll [2] that provide a surprisingly refined model for certain generated musical scales and suggest a natural mathematical relation giving preference for the Ionian Mode. Specifically I will focus on the notions of refined Christoffel duality, while highlighting other important connections along the way.

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1. INTRODUCTION

Since the mid-seventeenth century Western Art Music has been overwhelmingly centered around the major scale, or the Ionian Diatonic Mode. This collection of

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pitch-relations builds up the basic melodic and harmonic material of composition before the 20th century. While the minor scale plays a very strong role in composition before the 20th century, and the Dorian and Lydian scales have become increasingly used as the central material of a composition since the late 19th century, these scales remain in an underprivileged role, often positioned, theorized, and heard by their relations to the major scale. However, this preference is not universal. On a global scale, one finds that the pentatonic (five-note) scale holds a more dominant position than the diatonic (seven-note), as best seen in Indonesian Gamelan or West African string music.

In studying these scales mathematically, we adopt the equal-tempered system of tuning. This means we are only going to worry about scales in the 12-tone Western system with equal spacings between the notes as represented on a modern piano.

In this paper I am going to overview some key mathematical properties of these most commonly used scales, introduce some basic word theory, highlighting the relationship of Christoffel words and interval relations within scales, introducing the concept of a dual word and its music-theoretical importance, and a class of morphisms which generates these words.

2. Overview of the Scale and mathematical analogue

While the relation between word theory and the musical scale is a relatively new field of study, Algebra and Set theory have been used to study the properties of a strict mathematical representation of scales by the likes of David Lewin, John Clough, and Gerald Myerson for decades.

Definition 2.1. A *pitch* is a single sound at a distinguishable frequency. For example $A = 440mz$. The equivalence classes given by, $a \sim b$ determined by corresponding note names on the piano, or by octave frequency relation ($a \sim b$ iff $a/b = 2^j$ for some $j \in \mathbb{Z}$), are called *pitch classes.*

This equal spacing of pitch in the equal-tempered system allows for a natural bijection between the notes on a piano within an octave to the integers modulo 12. We give this bijection by sending the equivalence class of C to 0, $C\sharp$ to 1, $\dots B$ to 11 as demonstrated in figure 1. The *transposition*, *inversion*, and *the interval functions* on $\mathbb{Z}_{12}$ are the most relevant in understanding the musical notion of a scale.

Definition 2.2. *Transposition* by n, or translation by n, is the function $T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ given by $T_n(x) \equiv x + n \pmod{12}$.

Definition 2.3. For each $n \in \mathbb{Z}_{12}$, we have an *Inversion*, I_n, which is the bijective function $I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ given by $I_n(x) \equiv (-x + n) \pmod{12}$.

Definition 2.4. The interval function is the function $Int : \mathbb{Z}_{12} \times \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ such that $Int(x, y) \equiv x - y \pmod{12}$.

Definition 2.5. In the most general way, in music, a scale is just a collection of pitches. Therefore we consider a *scale* to be any subset S of $\mathbb{Z}_{12}$.

Remark 2.6. We consider two scales S_1 and S_2 to be of the same type if $S_1 = T_n(S_2)$ for some $n \in \mathbb{Z}_{12}$. For example, the F major scale and C major scales are both of the major type and are transpositions of each other. Often, the “type” is denoted by the a qualification of major, minor, or a mode name.

Definition 2.7. We further loosely define a scale *mode* by the "starting point" of a scale. We often denote the starting point simply with the letter it begins on, as in the "C" or "F" given in 2.6. Rigorously, the mode is a unique ordering of the scale-steps within a scale. For example, both the C-major scale and the A minor scale is the have the same collection of pitches, however the C-major scale has an interval pattern $2-2-1-2-2-2-1$ while the a-minor scale has an interval pattern $2-1-2-2-1-2-2$. We call the Major scale the Ionian mode and the minor scale the Aeolian mode after the church name precedent. An important mode to note for this paper is the Lydian mode, which has a scale step pattern of $2-2-2-1-2-2-1$.

Examples 2.8. Here our some clarifying examples:
The C-Major Scale

$$\{0,2,4,5,7,9,11\} \tag{2.9}$$

This has a scale step pattern $2-2-1-2-2-2-1$ as does any major scale. Note that we do include the distance from the last note to the first note of the scale as our last step-interval.
The F-Major Scale

$$\{5,7,9,10,0,2,3\} \tag{2.10}$$

This also has a scale step pattern $2-2-1-2-2-2-1$.
The a-minor Scale

$$\{9,11,0,2,4,5,7\} \tag{2.11}$$

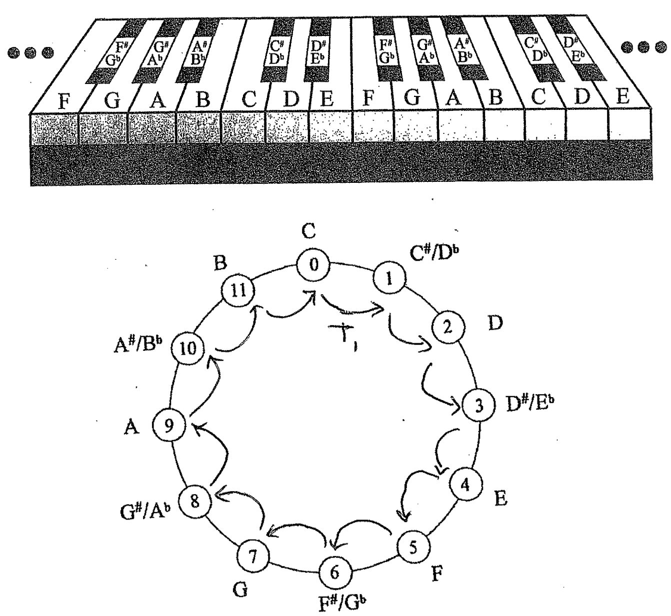

FIGURE 1. (Image from Fiore's REU talks [1]) $\mathbb{Z}_{12}$ in a clock-represenation with the natural bijection to the note names

This has a scale step pattern $2-1-2-2-1-2-2$, while having the same pitch material as the C major scale.
The Pentatonic Scale

$$\{0, 2, 4, 7, 9\} \tag{2.12}$$

The Pentatonic has scale step pattern $2-2-3-2-3$
The Tetractys

$$\{0, 2, 7\} \tag{2.13}$$

Scale Step Pattern: $2-5-5$
The Octatonic

$$\{0, 1, 3, 4, 6, 7, 9, 10\} \tag{2.14}$$

Scale Step Pattern: $1-2-1-2-1-2-1-2$
The Chromatic

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \tag{2.15}$$

Scale Step Pattern: $1-1-1-1-1-1-1-1-1-1-1-1$

Remark 2.16. Note in the classification of modes, interval classes are preserved, but the ordering of interval relations is not. For example $\{9, 11, 0, 2, 4, 5, 7\}$ is the aeolian diatonic mode or the "minor" scale and is different from the Major Scale.

Example 2.17. $\{6, 8, 10, 1, 3\} = T_6(\{0, 2, 4, 7, 9\})$ is the pentatonic scale. The two scales are of the same type but not the same *key*, as we can call the first the $F\sharp$-pentatonic and the latter the C-pentatonic. However, the ordered scale $\{2, 4, 7, 9, 0\}$ presents a different mode of the C-pentatonic scale. This transposition demonstrates an important property of the pentatonic scale: that the pentatonic scale is the complement of the major scale in $\mathbb{Z}_{12}$. In this specific case it is the complement of the C-Major scale.

Definition 2.18. The *scale interval* is the number of steps between two notes within a scale. Our older notion of interval in $\mathbb{Z}_{12}$ is called the *chromatic interval*. For example, in the C-Major scale the scale interval between 0 and 7 is five as there are five elements in the scale $0, 2, 4, 5, 7$ between them, while the chromatic interval is $7-0=7$.

2.1. **Properties of Scales and Generation of The Major Scale.**

Definition 2.19. A scale is said to be *generated* if it can be obtained by an iterated application of T_n to some $x \in \mathbb{Z}_{12}$ for a fixed $n \in \mathbb{Z}_{12}$.

Example 2.20. The C-Major Scale in Example 1.7 is generated by applying T_7 to 5 seven times. Similarly, the important pentatonic and tetractys scales are generated by the transposition five times and three times respectively, while the chromatic scale, or all of $\mathbb{Z}_{12}$ is generated by T_7 as 7 is relatively prime to 12, therefore it is a generator of the cyclic group.

Note that we do not require T_n of the final note to be the initial note in the definition of generated.

Remark 2.21. The Octatonic scale cannot be generated.

Proof. The Octatonic scale has eight elements, so we can immediately eliminate the possibility of generation by any T_n such that n is not relatively prime to 12. This is because if n is not relatively prime to 12, then $< n > \leq \frac{12}{2} = 6$, as well as any translation $< n > +i$, which is precisely the same as continuously applying $T_n(i)$. So our only other options of generators in the twelve tone system are T_1, T_5, T_7, and T_{11}. Now we know that T_1 and T_{11} generate the chromatic by adding half-steps, so any 8-note generation using either of them will not contain any whole steps. Since the Octatonic has 4 whole steps, then it cannot be generated in this fashion. Similarly, we know T_5 or T_7 applied 7 times generates the major scale. If we add any note to this major scale, we will get a string of at least two consecutive half-steps, as there are no steps in this scale of length 3 or greater. As the Octatonic has no consecutive half-steps, then T_5 or T_7 cannot generate the Octatonic. Therefore there are no possible generators in the 12-tone system for the Octatonic. □

Definition 2.22. A scale is *well-formed* if each generating interval spans the same number of scale steps, including the return to origin interval.

Example 2.23. The Major Scale is well-formed. Consider the C-Major Scale $\{0,2,4,5,7,9,11\}$. Between n and $T_7(n) = 7 + n \pmod{12}$ there are 5 scale steps. Here the return to origin is $B = 11$ to $F = 5$, which also contains 5 scale steps.

Definition 2.24. A scale satisfies the *Myhill Property* if each scale interval comes in two chromatic step sizes.

Examples 2.25.

- The Major Scale is Myhill. For the scale interval of the second, we can find both major and minor varieties, $2 - 0 = 2$ and $0 - 11 = 1$, for the third we get major and minor third, $4 - 0 = 4$ and $5 - 2 = 3$ and so forth for each scale interval.
- The Octatonic is not Myhill because any scale interval of a third (two scale steps) only spans 3 chromatic steps.

Myhill's property lends itself to many interesting geometric results and seems to single out a collection of important scales which include the diatonic collection and pentatonic scales. One such property is Cardinality equals variety.

Definition 2.26. [6] *Cardinality equals variety* In the traditional diatonic scale, each numerical interval (second, third, and so forth) appears in two sizes; the scale includes three kinds of triads (a three-note collection); and the diatonic tetrachord (four-note collection) has exactly four species, etc. It holds that all k-note chords come in k species for all diatonic chords of $1 - 6$ notes.

Theorem 2.27. *Myhill Property implies Cardinality equals Variety.*

Proof in [6].

While these mathematical properties provide a possible expression of the importance and preference of these scales, progress in word theory and the remarkable analogue it provides for the scale opens up many more possibilities to answer the questions of why certain scales are used and desired over others.

3. Christoffel words and their conjugates

Following the work of Clampitt-Domínguez-Noll [2], there have been startling connections between the notions of Christoffel dual words and the modes of scales and their generations.

We begin with some basic definitions of word theory.

Definition 3.1. Consider the 2-letter alphabet $\{a, b\}$. A *word* in this alphabet is a sequence of a's and b's.

We denote the free monoid on the set $\{a, b\}$ by $\{a, b\}^*$. Elements of $\{a, b\}^*$ are the words in the alphabet $\{a, b\}$. Here, multiplication is concatenation of words, and the unit element is the empty word.

Examples 3.2. Some examples of words include $\emptyset, a, b, ab, aab, baaaba$.

Definition 3.3. Two elements w and w' of $\{a, b\}^*$ are *conjugate* if there exist words u and v such that $w = uv$ and $w' = vu$.

Example 3.4. The words $aabab, baaba, abaab, babaa$, and $ababa$ are all conjugate.

Note that these words are just rotations of each other. This is the case for all conjugates in the free monoid$\{a, b\}^*$.

Lemma 3.5. *Two elements w and w' in the free monoid $\{a, b\}^*$ are conjugate if and only if they are conjugate in the free group on the set $\{a, b\}$.*

Note that in the free group $< a, b >$ is the set of all reduced words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$ and the inverse of any word is constructed by taking reverse spelling and inverting each element. For example, $(aab)^{-1} = b^{-1}a^{-1}a^{-1}$.

Proof.

(1) (All conjugates in the free monoid are conjugates in the free group.) Let w and w' be words in $\{a, b\}$. First, suppose they are conjugate in the free monoid. Then w is some rotation of w', which is equivalent to saying that $w = uv$ and $w' = vu$. Since $v \in \{a, b\}^*$ then $v \in< a, b >$. Consider $vwv^{-1} = vuvv^{-1} = w'$, so w and w' are conjugate in the free group.

(2) (There are no other conjugates in the free group that are also elements of the free monoid.) Now if we are to act on a word $w = w_1 \dots w_n$ by conjugation, we will show that in order for the resulting word to be an element of the free monoid $\{a, b\}^*$ the element $g \in< a, b >$ of the free group must be of the form $v = w_i \dots w_n$ or $u^{-1} = (w_1 \dots w_i)^{-1}$ (neglecting any complete repetitions of the word w or of w^{-1}). It is clear from the first part that any g such v or u^{-1} will result in a conjugate if we then factor $w = uv$. Suppose now there is some $h \in< a, b >$ that is not of the form $h = w_i \dots w_n$, but hwh^{-1} is an element of the free monoid. Then there exists some $h_i \neq w_{n-i}$ or $h_i{}^{-1} \neq w_i$. In the first case, if $h_i \in \{a^{-1}, b^{-1}\}$, then the resultant word $hwh^{-1} = w'$ will have $w'_i = h_i$ and therefore $w' \notin \{a, b\}^*$. So then $h_i \in \{a, b\}$ and therefore $h_i^{-1} \in \{a^{-1}, b^{-1}\}$ and since $h \neq v$ for some v a suffix of w, then h_i^{-1} will not cancel with w_{n-i} and therefore $h_i^{-1} = w'_{i+n+1}$ (Assuming h_i is the first element which varies from a possible v) and again it follows that $w' \notin \{a, b\}^*$. A similar argument holds for $h \notin u^{-1}$.

$\square$

Conjugation on Lydian	Result	Mode Name
$b \circ aaabaab \circ b^{-1}$	$baaabaa$	Phrygian
$ab \circ aaabaab \circ b^{-1}a^{-1}$	$abaaaba$	Dorian
$aab \circ aaabaab \circ b^{-1}a^{-1}a^{-1}$	$aabaaab$	Ionian
$baab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}b^{-1}$	$baabaaa$	Locrian
$abaab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}b^{-1}a^{-1}$	$abaabaa$	Aeolian
$aabaab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}a^{-1}b^{-1}a^{-1}$	$aabaaba$	Mixolydian
$aaabaab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}a^{-1}b^{-1}a^{-1}a^{-1}$	$aaabaab$	Lydian

Definition 3.6. Let p and q be relatively prime positive integers, then the *Christoffel Word* of slope p/q and length $n = p + q$ is the lower discretization of the line $y = \frac{p}{q} \cdot x$ and can be obtained through the equation

$$w_i = \begin{cases} a & \text{if } p \cdot i \pmod{n} > p \cdot (i-1) \pmod{n} \\ b & \text{if } p \cdot i \pmod{n} < p \cdot (i-1) \pmod{n}. \end{cases}$$

We will look closely at three specific Christoffel words. The Lydian word of slope 2/5, the Pentatonic word of slope 2/3 and the Tetractys word of length 2/1.

Examples 3.7.

- (Lydian) The Christoffel word of slope 2/5 is precisely *aaabaab*.
- (Pentatonic) The Christoffel word of slope 2/3 is precisely *aabab*.
- (Tetractys) The Christoffel word of slope 2/1 is precisely *abb*.

Recall from 2.8 that the Lydian Diatonic Mode has a scale step pattern of $2-2-2-1-2-2-1$. Notice that this directly corresponds with the Christoffel word we call Lydian, *aaabaab*, if we allow a to represent a whole step and b represents a half step. A Similar relation holds for the Pentatonic word *aabab* as the pentatonic has scale step pattern $2-2-3-2-3$ and the Tetrachtys word *abb* with scale step pattern $2-5-5$. This relation is the main connection between word theory and music theory.

The seven Diatonic modes are often called the Church modes from their development and use in pre-medieval music history, though the names initially derive from Ancient Greek scale names. The modes as we know them developed in the medieval times and throughout pre-baroque history arguments can be made for the preference of the Dorian and other modes. However beginning in the Baroque era and stretching through today the Ionian has been the mode of choice.

Using the Lemma, we reach an important conclusion:

Proposition 3.8. *All diatonic mode words are conjugate to the Lydian word, and moreover any conjugate of the Lydian word in the free monoid $\{a, b\}^*$ is a diatonic mode word.*

3.1. **Christoffel Dual Words.**

We see that musically, Christoffel words that are *dual* to each other present an important relation.

Definition 3.9. Given a Christoffel word w of slope $\frac{p}{q}$, we define the *dual Christoffel word* w^* of slope $\frac{p^*}{q^*}$ where $p \cdot p^* = 1 \pmod{n}$ and $q \cdot q^* = 1 \pmod{n}$ and $n = p + q$.

We know that these inverses exist because p and q are relatively prime and therefore p and q are relatively prime to $n = p + q$. Therefore, p^* and q^* are relatively prime.

Examples 3.10.

- Recall the Lydian word, $aaabaab$, is the Christoffel word of slope $\frac{2}{5}$. As $2 \cdot 4 = 1 \pmod{7}$ and $5 \cdot 3 = 1 \pmod{7}$. Its dual word, w^* is the Christoffel word of slope $\frac{4}{3}$. This gives $w^* = xyxyxyy$.
- The Pentatonic Christoffel word, $aabab$ is dual to the Christoffel word of slope $\frac{3}{2}$, $xyxyy$.
- The Tetrachtys Christoffel word of slope $\frac{2}{1}$, abb is self-dual, as when $n = 3$, 2 and 1 are both inverses of themselves. So $w^* = xyy$.

Note that we use the alphabet $\{x, y\}$ to denote a dual word to one in the alphabet $\{a, b\}$, however, from a word theory point of view, the alphabets are isomorphic.

The musical relationship between dual words will be illustrated in Section 3.3.

3.2. **Palindromization.** The relationship between Christoffel words and their duals is further strengthened by the conception of an underlying palindrome within these words.

Definition 3.11. A palindrome is a word $w = w_1 \dots w_n$ in which $w_i = w_{n-i+1}$ for $1 \leq i \leq n$.

All Christoffel words have an important composition, as will be presented in 3.18: If w is Christoffel of slope $\frac{p}{q}$, then $w = aub$ where u is a palindrome. The palindrome of this type is called the *central palindrome*.

Proposition 3.12 (Prop. 4.3 from [7])**.** *Let w be a word. Write $w = uv$, where v is the longest suffix of w that is a palindrome. Then $w^+ = w\tilde{u}$, with $\tilde{u} = u_n \dots u_1$ when $u = u_1 \dots u_n$, is the unique shortest palindrome having w as a prefix.*

Proof. Suppose there is a shorter palindrome p such that w is a prefix than the constructed w^+ with $|w^+| = n + |u|$ where $w = uv$ with v being the longest suffix of w that is a palindrome. Let $k = |u|$ So $p = p_1 \dots p_m$ with $n < m < n + k$, and $p_1 \dots p_n = w_1 \dots w_n$. Therefore $m - n < k$. Now since p is a palindrome, we know that $p_m = w_1 = u_1$, $p_{m-1} = w_2 = u_2$, $\dots$, $p_{m-k-1} = w_{k-1} = u_{k-1}$. But then we have $p_{m-k} = w_{n-(k-(m-n))} = w_k = u_k$. However, this result contradicts v being the longest suffix that is a palindrome, as we now arrive at one that has length at least $|v| + 1$. $\square$

Definition 3.13. This word w^+ is called the *right palindromic closure* of w.

Examples 3.14.

- $(aba)^+ = aba$
- $(ab)^+ = aba$
- $(aab)^+ = aabaa$
- $(aabab)^+ = aababaa$.

Definition 3.15 (Defn. 4.5 from [3]). . Define a function $Pal : \{a,b\}^* \to \{a,b\}^*$ recursively as follows. For the empty word, $\emptyset$, define $Pal(\emptyset) = \emptyset$. If $w = vz \in \{a,b\}^*$ for some $z \in \{a,b\}$, then let

$$Pal(w) = Pal(vz) = (Pal(v)z)^+. \tag{3.16}$$

The resultant word $Pal(w)$ is called the *iterated palindromic closure* of w.

Examples 3.17.

- We want to calculate $Pal(aab)$: First, we need to know $Pal(a) = (a)^+ = a$. Then, we'll need to calculate $Pal(aa) = (Pal(a)a)^+ = (aa)^+ = aa$. Lastly, we can then put together $Pal(aab) = (Pal(aa)b)^+ = (aab)^+ = aabaa$.
- Through the same process we find $Pal(yxx) = yxyxy$.

At this point, it is important to take a break and notice a musically historical connection. The central palindrome of the Lydian word, and therefore a fundamental center to the creation of all the Diatonic mode words, is precisely the Guidonian Hexachord, a six note scale characterized by its interval relations of T-T-S-T-T; or tone, tone, semi-tone, tone, tone; or in modern terms, whole, whole, half, whole, and whole steps. In order to learn and memorize a long and complicated piece of music without ever having a written copy, monks assigned each step in the Guidonian hexachord a syllable, a predecessor of today's solfege. As it only allowed for a range of six notes, in order to accommodate songs with larger spans, singers would shift among three varieties of the hexachord: the soft hexachord which began on the note F, the hard hexachord which began on a G, and the natural hexachord which began on C. For example, if a singer starts on F and wanted to span 8 steps up to F', then he would sing the first five steps of the soft hexachord, then switch to the first step of the natural hexachord, where he would then be able to reach the desired pitch. [8] This hexachordal system slowly evolved into the diatonic system we are more familiar with and the ties of it as a historical 'center' for the diatonic scales is strong. The mathematical analogue demonstrates a similar importance to this hexachord in its position as the central palindrome and characterizing element of the Christoffel words which generate the diatonic mode words. Further, the choice of starting points for the three main hexachords results in the Tetractys, or the first three notes in a generation of $T_7(5)$. What is remarkable about this connection is that without any mathematical conception of these systems, the Guidonian Hexachord was in prominent use by the early 11th century.

Theorem 3.18 (Thm 4.6 and Prop. 4.14 in [7]). *Let $v \in \{a,b\}^*$. Then $w = xPal(v)y$ is a Christoffel word, and if w is a Christoffel word, then there exists some $v \in \{a,b\}^*$ such that $w = xPal(v)y$.*

Proof. Proof in [7] □

We call the *directive word of w* the word $dir(w) = u$ such that $w = Pal(u)$. One can notice in our example that the directive words for the central palindromes in the Lydian word and its dual are reverse spellings on an equivalent two-letter alphabet. This is not a mere coincidence, as it holds for all Christoffel words w and their duals w^* that if $dir(w) = u_1u_2 \dots u_n$, then $dir(w^*) = u_n \dots u_1$. [5] This relationship of the palindromic closures of the central palindromes of Christoffel words and their duals provides another view into the interaction between these two groups. However, the relationship between these words and their musical representations as step-interval patterns and the foldings of generated scales strengthens the connection while providing another point of reference for the preference of certain scales.

3.3. **Musical Folding.**

Recall that the major scale, the pentatonic, and the tetractys are all generated scales by the transposition T_7. For clarity, we will consider the C-major scale and its similarly generated counterparts in the pentatonic and tetractys, so we will be observing the three such scales generated beginning on $F = 5$, $\{5, 0, 7\}$, $\{5, 0, 7, 2, 9\}$, $\{5, 0, 7, 2, 9, 4, 11\}$ or in their ordered sense, $\{5, 7, 0\}, \{5, 7, 9, 0, 2\}, \{0, 2, 4, 5, 7, 9, 11\}$.

Definition 3.19. The *span* of a scale is the chromatic space between its highest and lowest notes.

As the span isn't necessarily restrained to $\mathbb{Z}_{12}$ we need to re-establish a bijection between keys on the piano that maintains uniqueness among notes of the same pitch class, but of a different octave. For this paper, it is sufficient to maintain that the normal ordering refers to the lowest spoken of octave, and each higher octave will be designated with a " $*$ ". For example, the distance between two notes 5^* and 4 is $(5 + 12) - 4 = 13$.

Example 3.20. The span of the F-Lydian Scale (the Lydian scale which begins on F) is the space from 5 to 5^*

Definition 3.21. We call the *musical folding* the unique way the ordered generation falls into the span of a scale S. That is, begin with the starting note in the genaration, k with T_n being the generating transposition. If $k + n \leq U$, where U is the highest note in the scale, then the first step is up and we add $k + n$, and we denote this by x. If $k + n > U$, then we subtract $U - n$ from k, and we denote this step by y. We do this until we have covered all the notes in our generated scale.

This notion of a folding may seem unnecessary and peculiar in a mathematical sense, but in a music-theoretic application it is entirely appropriate. Despite the relation in harmonic frequency of pitches at an octave relation allowing for an almost "unified" sound, the human ear is highly-sensitive to musical range. As we generate pitches in a scale (take for instance with the generation of T_7) the resulting notes on a piano would not fit into an octave or even a close range. The notes comprising a major scale if we keep translating up 7 steps would span over 3 octaves! In order to adjust this into a more compositionally functional collection the span is contracted by using this process of folding, so we get a collection of pitches comprising the scale, but within a reasonable range to work with. Further, composers naturally encorporate this concept of folding a generated scale by sequences of fifths and fourths that occur throughout the canon of classical music.

In the both Figure 3.3 and Figure 3.3 we find an actualization of the generation of the scale by a fifth through an ascending fifth diatonic sequence and the compositional decision to 'fold' the root notes of the chords. This is apparent even in more modern musics. For example, the bridge to The Beatles "Here Comes the Sun" is an ascending fifth progression and elements of a folding can be heard in the instrumentation.

Examples 3.22. Consider the Lydian scale. We know that it is generated by T_7 and, specifically, the Lydian mode beginning on F spans the octave from 5 to 5^*. So

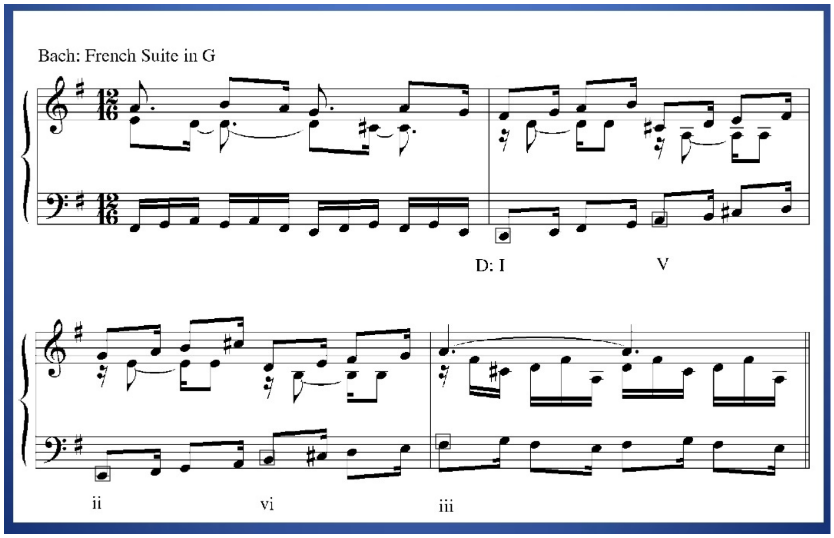

FIGURE 2. In Bach's French Suite in G, we find a precise ascending fifth musical folding in the bass clef. The boxed notes represent the structural notes of the harmonic progression and we see in a musical example how after the initial leap of a fifth, the motion from A back down to E in the second and third measures is a descent of a fourth. The folding ends as bach leaves the octave boundaries of D when it reaches $F\sharp$ in the fourth measure. However, this coincides with the end of a sequence and the motion out of the folding coincides with the beginning of a new and different musical section.

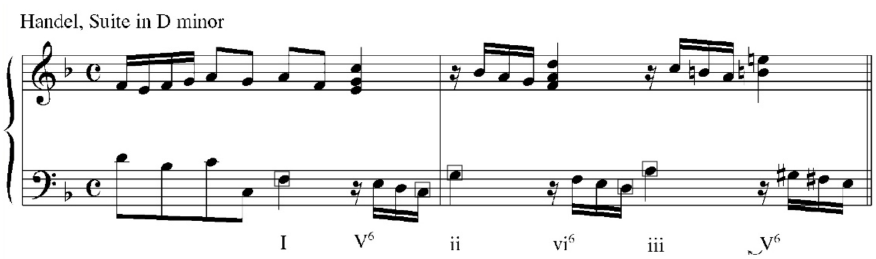

FIGURE 3. In this excerpt from Handel's Suite in D minor, we get another direct folding in an Ascending fifth progression. It should be noted that the octave in which the bass is folding within is not from F to F, but rather from A to A, as the final chord in the second measure alludes to an A-minor tonality through the dominant E major.

beginning on 5, we can add $5+7=0^*$, which is still in the span, so our first element in the folding is x. Second, as $0^*+7=7^*>5^*$, then we need to subtract $12-7=5$ from 0^*, resulting in 7, and our second element is y. Continuing this process we get that sequence of numbers $\{5, 0^*, 7, 2^*, 9, 4^*, 11, \mathbf{6}\}$ and the corresponding sequence of letters $xyxyxyy$. See Figure 4.

We notice that the final note of the sequence is not our starting pitch, but rather off by a half step. This is not a mistake, but rather a result of the generation. If we re-establish our notion of a 5-th to be contained within a diatonic scale, allowing for an approximation of the last step so that the folding remains in the same scale-set, then the last note would indeed result in a return to the beginning.

To further generalize this approximation, we consider the last element of the folding to be the *return to origin*, and denote it x if we need to travel to a higher pitch for the return to the original note in the generation or a y if we need to travel to a lower pitch, corresponding as we are approximating either x - a fifth up, or y - a fourth down.

Example 3.23. Consider the Pentatonic scale, $\{5, 7, 9, 0^*, 2^*\}$. Note that in this mode and transposition we begin on $F = 5$ and span to $F' = 5^*$. The corresponding folding arrives from the first five notes generated from F with T_7, so our folding results:

$T_7(x) =$	5	0	7	2	9
Realignment within Span	5	0^*	7	2^*	9
Distance to next note	$x+7$	$x-5$	$x+7$	$x-5$	$x-4$
Corresponding Folding Letter	x	y	x	y	**y**

Notice that the return to origin from 9 to 5 is neither T_7 nor T_{7-12}. However, since the return moves to a lower pitch, then we still denote it with y.

4. Refined Christoffel Duality

Now that we notice this relation between Christoffel words and their duals we want to express a natural relation between the conjugates of the Christoffel words and the conjugates of its dual, therefore incorporating all of the possible modes of each scale.

Definition 4.1. For every word $w \in \{a, b\}^*$, let $|w|_a$ and $|w|_b$ be the multiplicities of the letters a and b in w, respectively. As before, we let $|w| = n$ be the length of w and w_k be the k-th term.

Definition 4.2 (Definition 2 of [5])**.** Call the function $ev_w : \{a, b\} \to \mathbb{Z}$ given by $ev_w(a) = |w|_b$ and $ev_w(b) = -|w|_a$ the *balanced evaluation* of the alphabet $\{a, b\}$ with respect to w. This induces a *balanced evaluation of the word* w, specifically $\beta_w(k) = ev_w(w_k)$.

Definition 4.3. Call the *balanced accumulation* of w the map $\alpha_w : \{0, 1, \ldots, |w| - 1\} \to \mathbb{Z}$ of partial sums of the sequence $(\beta_w(1), \ldots, \beta_w(|w| - 1)$, namely $\alpha_w(k) := \sum_{l=1}^{k} \beta_w(l)$.

Definition 4.4. A word w is *well-formed* if there exists an integer $m_w \in \{0, \ldots, |w| - 1\}$ such that $\{\alpha_w(0) + m_w, \ldots, \alpha_w(|w| - 1) + m_w\} = \{0, \ldots, |w| - 1\}$.

Theorem 4.5 (Theorem 1 of [2])**.** *A word w is well formed if and only if it is a Christoffel word or conjugate thereof. It is actually a Christoffel word if and only if its mode m_w is zero.*

Given well-formed word w with mode m_w, Clampitt-Domínguez-Noll call the affine automorphism on $\mathbb{Z}_N$

$$f_w(k) = |w|_y \cdot k - m_w \mod N \tag{4.6}$$

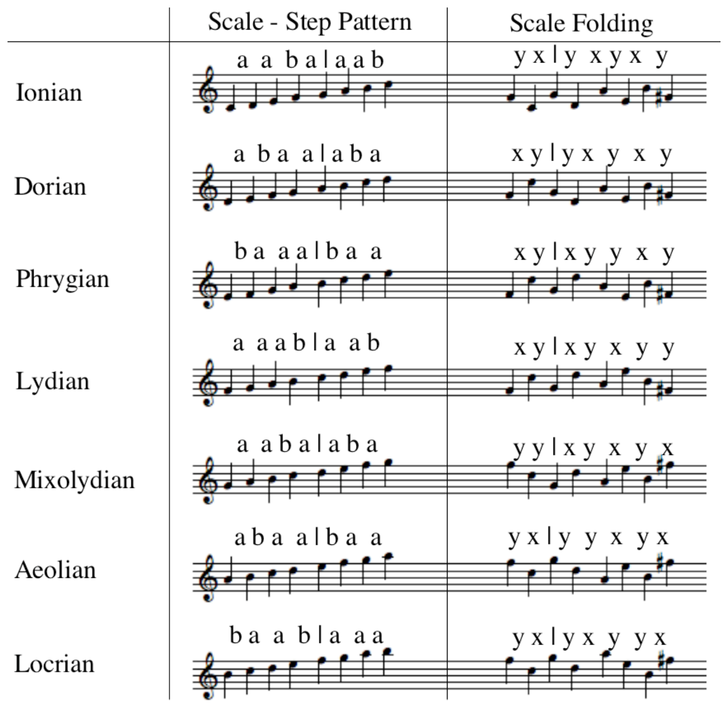

FIGURE 4. (Figure 8 of Noll's paper [5]) The musical folding of each Diatonic mode displayed with their corresponding scale step pattern. Recall a is 2 half-steps, while b is 1 half step, and x is a Major-fifth (7 chromatic steps) up and y is a Major-fourth (5 chromatic steps) down. We will see in section 4 that this table is an instance of Refined Christoffel Duality.

the *plain affinity associated to w.*

Definition 4.7. [2] Given a well-formed word w, we call the *plain adjoint* of w, denoted by $w^{\square}$, the unique word whose associated affinity coincides with the inverse affinity of w. In other words, the plain adjoint $w^{\square}$ is defined by the equation:

$$f_{w^{\square}} = (f_w)^{-1}. \tag{4.8}$$

Examples 4.9.

(1) The following table from Noll and Domínguez shows the relation between conjugates of the Lydian word and their plain adjoints.

w	$aaabaab$	$aabaaba$	$abaabaa$	$baabaaa$	$aabaaab$	$abaaaba$	$baaabaa$
$f_w(k)$	$2k$	$2k-2$	$2k-4$	$2k-6$	$2k-1$	$2k-3$	$2k-5$
$f_{w^{\square}}(k)$	$4k$	$4k-6$	$4k-5$	$4k-4$	$4k-3$	$4k-2$	$4k-1$
$w^{\square}$	$xyxyxyy$	$yyxyxyx$	$yxyyxyx$	$yxyxyyx$	$yxyxyxy$	$xyyxyxy$	$xyxyyxy$

(2) The following table shows the same relation for the Pentatonic word and its modes. This table was calculated from 4.7 and (4.6).

w	$aabab$	$ababa$	$babaa$	$abaab$	$baaba$
$f_w(k)$	$2k$	$2k-3$	$2k-1$	$2k-4$	$2k-2$
$f_{w^\square}(k)$	$3k$	$3k-1$	$3k-2$	$3k-3$	$3k-4$
$w^\square$	$xyxyy$	$yyxyx$	$yxyyx$	$yxyxy$	$xyyxy$

(3) Here is the same table for the Tetractys. Recall that the Tetractys was self-dual, or the dual word to the Tetractys word was itself.

w	abb	bab	bba
$f_w(k)$	$2k$	$2k-2$	$2k-1$
$f_{w^\square}(k)$	$2k$	$2k-2$	$2k-1$
$w^\square$	xyy	yxy	yyx

One can observe that the inverse of an affinity $h(x) = ax + b$ is $h^{-1}(x) = a^*x + (b \cdot -a^*) \pmod{n}$, when $a^* \cdot a = 1 \mod n$.

Proposition 4.10. *For Christoffel words, the plain adjoint $w^\square$ is precisely the dual word w^*.*

Proof. One can check from the tables that for the three Christoffel words discussed this holds. Since for a Christoffel word has mode 0, then it's plain affinity is just, $f_w(k) = |w|_y \cdot k$, so it is clear that when $k = 0$, $f_w(0) = 0$. Therefore the inverse function $f_w^{-1}(0) = 0$, but recalling $f_w^{-1}(x) = a^*x + (b \cdot -a^*) \pmod{n}$ we see that $(b \cdot -a^*) \pmod{n}$ must be zero. This is the mode of the inverse, and therefore the plain adjoint $w^\square$ of w has a mode of zero and by 4.5 must be Christoffel. □

The plain adjoints allow for correspondence with the hinted at in 4 while maintaining Christoffel duality.

We will soon see a shorter way to calculate certain plain adjoints using special Sturmian Morphisms.

5. Sturmian Morphisms

Christoffel words and their conjugates can naturally be extended to infinite words (In either a two-sided or one-sided sense). The endomorphisms on these infinite words provide a group of morphisms that allow for another generation of the diatonic modes and a unique preference for the Ionian.

Definition 5.1. A *Sturmian morphism* is a monoid homomorphism $\{a, b\}^* \to \{a, b\}^*$ which sends every Christoffel word to a conjugate of a Christoffel word.

Remark 5.2. Berstel et. al. call this a *Christoffel Morphism.* [7]

Remark 5.3. The set of Sturmian morphisms form a monoid under function composition. We denote the monoid of Sturmian morphisms by St.

Theorem 5.4. *The Sturmian Morphisms are precisely the morphisms generated by the following monoid homomorphisms from $\{a, b\}^* \to \{a, b\}^*$.*

Generating Sturmian Morphism	a	b
G	a	ab
$\tilde{G}$	a	ba
D	ba	b
$\tilde{D}$	ab	b
$\mathbf{E}$	b	a

5.1. **Infinite analogue.** While much can be said using solely this definition of a *Sturmian morphism*, they can be seen more generally as the endomorphisms on *Sturmian words*, a class of infinite words that hold many "Christoffel" traits.

Remark 5.5. Any endomorphism $f : \{a, b\}^* \to \{a, b\}^*$ defines a function $\bar{f}$: {infinite words in alphabet {a,b}} $\to$ {infinite words in alphabet {a,b}} by defining $\bar{f}(w)$ to be the infinite word obtained from w by replacing a by $f(a)$ and b by $f(b)$.

Definition 5.6. [5] Let w denote an infinite word over the alphabet $\{a, b\}$. For any $n \in \mathbb{N}$, let $Factors_n(w) \subset \{a, b\}$ denote the set of finite words which occur as factors of length n within the infinite word w. The infinite word w is called a *Sturmian word*, if the cardinality $|Factors_n(w)|$ is equal to $n + 1$ for every $n > 0$.

Example 5.7. Consider an infinite repetition of the Lydian word, $aaabaab \circ aaabaab \circ \ldots$. For $n = 1$ there are two factor words, a and b, and thus $|Factors_1(w)| = 2$. For $n = 2$, there are three possible factor words, aa, ab, and ba, and thus $|Factors_2(w)| = 3$. Check now for $n = 5$, the possible factor words are $aaaba$, $aabaa$, $abaab$, $baaba$, $abaaa$, and $baaab$ and therefore $|Factors_5(w)| = 6$. For $n = 8$, there are factor words, $aaabaaba$, $aabaabaa$, $abaabaaa$, $baabaaab$, $aabaaaba$, $abaaabaa$, $baaabaab$, $baaabaab$. There are only 8 solutions, so this infinite repetition does not yield a Sturmian word.

Thus, we see from the example that a word comprised of constant infinite repetitions will not be Sturmian.

Example 5.8. The sequence arising from the substitution map is a Sturmian Word. That is start a sequence on 0 and map every $0 \to 01$, and $1 \to 0$, leaving a sequence.

$$0 \to 01 \to 010 \to 01001 \to 01001010 \to 0100101001001 \to \ldots. \tag{5.9}$$

The resulting infinite chain 0100101001001... is a Sturmian word. Take note that this, like all Sturmian words, was generated by a Sturmian morphism on $\{0, 1\}^*$.

Example 5.10. [5] All Sturmian words can be explicitly written as *mechanical words* with irrational slope. Given two real numbers α such that $0 \leq \alpha \leq 1$ and $\rho \in \mathbb{R}$, a translation, we define the lower mechanical word of slope α and intercept ρ as

$$s(n) := \lfloor (n + 1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor \tag{5.11}$$

Lemma 5.12 (Lemma 4.1 in Berthé et. al. [3])**.** *A morphism $f : \{a, b\}^* \to \{a, b\}^*$ is Sturmian if and only if $\bar{f}$ maps Sturmian words to Sturmian words.*

5.2. **Generation of Scales.** Recall from Theorem 5.4 that the monoid St of Sturmian morphisms is generated by $G, \tilde{G}, D, \tilde{D}$, and E. An important sub-monoid of this, St_0 is the monoid generated by $G, \tilde{G}, D, \tilde{D}$ (note the absence of E). It is called the collection of *special Sturmian Morphisms* [4], and these play a distinguished role in the Divider Incidence Theorem, as we now explain.

In the conjugacy class of a Christoffel word of length n, there are $n-1$ words that can be obtained as images $f(ab) = f(a)(b) = f(a)f(b)$ of the initial word ab where $f \in St_0$. [5] Noll separates this word $f(ab)$ into factors giving us a *divided word* $(f(a)|f(b))$. The following table gives the six possible diatonic words which can be obtained through special Sturmian Morphisms on this divided word.

Mode	**Sturmian Representation on (ab)**
Ionian	$GGD(ab) = GGD(a)(b) = (aaba)\|(aab)$
Dorian	$G\tilde{G}D(ab) = G\tilde{G}D(a)(b) = (abaa)\|(aba)$
Phrygian	$\tilde{G}\tilde{G}D(ab) = \tilde{G}\tilde{G}D(a)(b) = (baaa)\|(baa)$
Lydian	$GG\tilde{D}(ab) = GG\tilde{D}(a)(b) = (aaab)\|(aab)$
Mixolydian	$G\tilde{G}\tilde{D}(ab) = G\tilde{G}\tilde{D}(a)(b) = (aaba)\|(aba)$
Aeolian	$\tilde{G}\tilde{G}\tilde{D}(ab) = \tilde{G}\tilde{G}\tilde{D}(a)(b) = (abaa)\|(baa)$

One should notice that there is one conjugate missing from this list, and that is the Locrian, represented by *baabaaa*. This is the only conjugate which cannot be generated by $f(ab)$ with $f \in St_0$ and therefore is what Noll calls a "bad conjugate." This surprisingly coincides with the historical exclusion of this scale, which was not used in the medieval chant where these scales initially appeared.

Though there are no common names for the pentatonic in Western Art Music, there have been instances of the scale corresponding to *aabab* being called the Major Pentatonic Scale and the scale corresponding to *baaba* the Minor Pentatonic Scale, mostly due to their relation to corresponding Diatonic scales. However, I will choose to call our previously stated Pentatonic scale *aabab* Mode I, *ababa* Mode II and so forth.

Mode	**Sturmian Representation on (ab)**
Mode I	$G\tilde{D}(ab) = (aab)\|(ab)$
Mode II	$\tilde{G}\tilde{D}(ab) = (aba)\|(ba)$
Mode IV	$GD(ab) = (aba)\|(ab)$
Mode V	$\tilde{G}D(ab) = (baa)\|(ba)$

As in the case with the Diatonic, we are left with one "bad conjugate" or a scale which cannot be generated by special Sturmian Morphisms applied to ab and that is Mode III or *babaa*.

Proposition 5.13 (Prop. 5 and 6 in [2])**.** *If $w = f(ab)$ where $f \in \langle G, D\rangle$ or $f \in \langle G, \tilde{D}\rangle$ then the plain adjoint, $w^{\square} = f^{rev}(ab)$ where f^{rev} is the application of special Sturmian generators in reverse order.*

While the proof of this proposition would be much too exhaustive for this paper and need a lot more background material it can be demonstrated in Section 3.2 of [7] that every Christoffel word can be constructed from a generation $f(ab)$ such that $f \in \{G, \tilde{D}\}$ and every such generation yields a Christoffel word. Further Clampitt, Domínguez, and Noll show in [9] that in the case of Christoffel Dual words this holds. Further in [2] they extend this formula to incorporate any conjugates generated by $f \in \{G, D\}$ as well. However, it does not always hold for any conjugate and their respective plain-adjoint but only in these special cases.

Example 5.14.

- We have that the Ionian word $w = aabaaab = GGD(ab)$. Its plain adjoint is $w^{\square} = yx|yxyxy = DGG(xy)$.
- If $w = GG\tilde{D}(ab) = aaabaab$ then $w^{\square} = \tilde{D}GG(xy) = xy|xyxyy$.
- If $w = G\tilde{D}(ab) = aabab$ then $w^{\square} = \tilde{D}G(xy) = xy|xyy$.
- If $w = GD(ab) = abaab$ then $w^{\square} = DG(xy) = yx|yxy$.

We see that these examples all match our previously attained dual words.

The possible importance of the generation of these scales lies in the natural divider of the generation of special Sturmian morphisms. We see that each Christoffel conjugate has an adjoint and the adjoint represents an important conceptual folding of the scale, though it does not give any indication of a reason for preference. However, we notice that only a select number of scales qualifies for the ‘nice’ result of Proposition 5.13. Still, in the diatonic sense, there is no apparent reason why the Lydian is not as popular as the Ionian. Scholars have often struggled with why the Ionian has been preferred over the Lydian, as the Lydian is the scale in which the generation and the scale both begin on the same note. Further the Christoffel nature of the Lydian would also lend itself to this preference. However, the concept of the *divider incidence* of these begins to once again point towards the Ionian.

Definition 5.15. The final tone of the first factor and the initial tone of the second factor of the divided word is called the *divider*. In our example of the F-Ionian mode in 4, G is the divider for both the scale and its folding and is shown clearly in Figure 5.2.

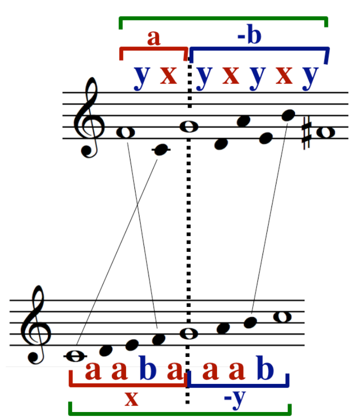

Figure 5. (Figure 11 from Noll [5]) The Ionian mode and its folding

Clampitt-Domínguez-Noll call this case, when the divider is the same for both the folding and the scale, *divider incidence*. Another interesting result demonstrated in

Figure 4 is that of the positioning of the divider within the scale. Notice that the distance between F and G in the folding is one whole step, or a distance relation of a, and the distance from G to the final note of the generation $F\sharp$ is $-b$ or one half step down. Similarly in the scale, the difference between C and G is a fifth up or x and the distance between G and C above is a fourth up, or $-y$. This relation also holds for the Pentatonic Mode IV. Considering the scale this mode on $0, 2, 5, 7, 9$ with $7 = G$ the dividing tone. The folding again has a dividing tone of G and if we extend the folding $5, 0, 7, 2, 9, \mathbf{4}$. Again the scale goes up a major fifth x to the divider and up a fourth after the divider, $-y$, while the folding goes up a whole step a from the beginning to the divider and down a minor third or $-b$ to the final note of the generation. We see in both of these cases this 'Ionian' mode which is generated by $\langle G, D\rangle$ produces this unique result.

6. Conclusion

The special properties of scales generated by the perfect fifth seem to provide a mathematical foundation for why the collections of pitches which constitute the Tetractys, the Pentatonic, and the Ionian may lend themselves to preference. While the word theory representations of scales in Christoffel words and their conjugates provide an astoundingly apt classification of these chords, it does not yet seem to point towards a preferred mode, other than the natural Christoffel word itself. However, when this divider incidence is introduced we find a uniqueness in the Ionian mode, which can now be considered in a possible natural cause for the preference which coincides with history. Further the embedding of the Guidonian hexachord and it's placement within the scale may lend itself to argument for preference and is satisfied by its role as the central palindrome in the Christoffel words which generate the diatonic mode. A possible test for the validity of the assertion of importance of divider incidence is to look through music featuring the Pentatonic and see if a similar preference arises in this Mode IV which shares the divider incidence property.

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