ABSTRACTING TONALITY: TRIADS AND UNIFORM TRIADIC TRANSFORMATIONS IN AN ATONAL CONTEXT

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Abstract. Uniform Triadic Transformations, more commonly referred to as UTTs, are certain functions from the set of consonant triads to itself. Conventionally, these consonant triads are major and minor triads, when in fact Hook observed they can be arbitrary sets of pitch classes as well. This paper will examine two tone rows written by the composer Anton Webern and show examples of UTTs acting on them.

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1. Introduction: Triads and UTTs

In this section we recall the UTT formalization of Hook [1]. These provide a convenient nomenclature for functions that occur in music.

Definition 1.1. A *triad* is an ordered pair of the form $\Delta = (r, \sigma)$ where $r \in \mathbb{Z}_{12}$ and σ is designated by $+$ or $-$. Let Γ be the set of all such triads.

Here we say that r is the *root* of the triad and σ is the *mode*. We also say that the triad is *major* when σ is $+$ and *minor* when σ is $-$. For example, we could represent the set of consonant triads by such pairs. However, we follow Hook-Douthett [2] and let the ordered pair represent unspecified chords. Our examples will make this clear. Fiore-Satyendra [3] also defined a generalized contextual group which acts on arbitrary n-tuples satisfying a tritone condition.

Definition 1.2. A *uniform triadic transformation*, or UTT, is a function of the form $U = \langle \sigma_U, t^+, t^- \rangle$ where $t^+, t^- \in \mathbb{Z}_{12}$ and σ_U is $+$ or $-$. Here we call U *mode-preserving* when σ_U is $+$ and *mode-reversing* when σ_U is $-$. When a UTT U acts on a triad $\Delta = (r, \sigma)$, it transposes it by t^+ if Δ is major and by t^- if Δ is minor. That is to say that the root of the resulting triad will be $r + t^+$ or $r + t^-$, respectively. Furthermore, if U is mode-preserving then the mode of Δ will not be changed, whereas if U is mode-reversing, then it will.

For example,

$$\langle +, 4, 3\rangle \text{ acting on } (0, +) \text{ produces } (4, +).$$
$$\langle -, 4, 3\rangle \text{ acting on } (0, +) \text{ produces } (4, -).$$
$$\langle +, 4, 3\rangle \text{ acting on } (0, -) \text{ produces } (3, +).$$

2. Group properties of UTTs

Since UTTs are functions, two of them may be composed with one another to yield a composite UTT. This composite can be found simply by using elementary arithmetic.

$$\langle +, a, b\rangle\langle +, c, d\rangle = \langle +, a + c, b + d\rangle$$
$$\langle -, a, b\rangle\langle +, c, d\rangle = \langle -, a + d, b + c\rangle$$
$$\langle -, a, b\rangle\langle -, c, d\rangle = \langle +, a + d, b + c\rangle$$

Note that if the first UTT is mode-reversing, then t^+ in the product UTT will be equal to the sum of the "outside" integers a and d while t^- will be the sum of the two "inside" numbers b and c. This has to do with the fact that if the first UTT acts on a triad it reverses its sign. In the second example, if $\langle -, a, b\rangle$ acts on a major triad, the triad will be transposed by a and then switched to a minor triad. Then, when $\langle +, c, d\rangle$ is applied, it will become transposed by d since it is now a minor triad.

The set of all UTTs $\mathcal{U}$ with operation given by UTT composition forms a group, where the identity element is $\langle +, 0, 0\rangle$. The inverse of a mode-preserving UTT $\langle +, t^+, t^-\rangle$ is $\langle +, -t^+, -t^-\rangle$ and the inverse of a mode-reversing UTT $\langle -, t^+, t^-\rangle$ is $\langle -, -t^-, -t^+\rangle$.

3. Subgroups of $\mathcal{U}$

The group of UTTs contains a number of interesting subgroups as well. One simple example is the set of all mode-preserving UTTs, which forms a normal subgroup of $\mathcal{U}$. It is normal because it has index 2, since the number of mode-preserving UTTs is exactly half the amount of all UTTs.

There are three specific UTTs which are of particular interest to music theorists because of their prominence in Western classical music. Namely, they are

$$P = \langle -, 0, 0\rangle,$$
$$L = \langle -, 4, 8\rangle,$$
$$R = \langle -, 9, 3\rangle.$$

The set of UTTs generated by these three is called the *neo-Riemannian group*, and it is the group of all UTTs of the form $\langle \sigma_U, n, -n\rangle$. A UTT is said to be *Riemannian* if it belongs to this group. The neo-Riemannian group is a subgroup of $\mathcal{U}$, and one of its interesting properties is that every mode-reversing Riemannian UTT is the inverse of itself, since

$$\langle -, n, -n\rangle\langle -, n, -n\rangle = \langle +, n + (-n), (-n) + n\rangle = \langle +, 0, 0\rangle.$$

The set of Riemannian UTTs can be broken down into two subsets called *Schritts* and *Wechsels*. The *Schritts* are of the form $\langle +, n, -n\rangle$ while the *Wechsels* take the form $\langle -, n, -n\rangle$. *Schritts* also form a subgroup of $\mathcal{U}$, however it is clear that the *Wechsels* do not as they do not contain the identity element $\langle +, 0, 0\rangle$.

4. UTTs as a Group Action

Definition 4.1. Recall that a *group action of a group G on a set X* is a map
$G \times X \to X$
such that

- $ex = x \; \forall x \in X$
- $(g_1 g_2)x = g_1(g_2 x) \; \forall g_1, g_2 \in G, \; \forall x \in X.$

Essentially what we have dealt with thus far is a group action of the group $\mathcal{U}$ on the set $\{(r, \sigma) \mid r \in \mathbb{Z}_{12}, \; \sigma \in \{\pm 1\}\}$ of triads such that

$$(u, \Delta) \longmapsto u(\Delta).$$

Conventionally, Δ is thought of as a consonant major or minor triad. In fact, Δ need not be consonant at all, so long as the triad and the group action remain well defined. That is to say, Δ might in fact be an arbitrary collection of pitches, so long as the notion of what it means for Δ to have root r and sign σ remains well defined.

With this in mind, it becomes clear that UTTs are not just applicable in tonal settings. They can be used to analyze atonal music and tone rows as well.

Definition 4.2. A *tone row* is a sequence of all 12 elements of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

The following two examples are tone rows written by Anton Webern.

5. Example 1: Concerto for Nine Instruments Op. 24

The tone row of this piece is $\langle 5, 4, 8, 9, 1, 0, 2, 10, 11, 6, 7, 3 \rangle$. In the paper "Uniform Triadic Transformations and the Twelve-Tone Music of Webern," [2] Julian Hook and Jack Douthett did an extensive study of the characteristics of this row. These characteristics will be briefly outlined here.

We can divide the tone row into four triads, yielding the following.

$$\langle 5, 4, 8 \rangle, \; \langle 9, 1, 0 \rangle, \; \langle 2, 10, 11 \rangle, \; \langle 6, 7, 3 \rangle$$

The most interesting property of this tone row is that it can be generated by applying inversions and transpositions to the first triad. To show this, define the first three notes to be a triad Δ. We will say that any other triad has a positive sign if it is related to Δ by transposition or negative sign if it is related to Δ by inversion. Furthermore, we define the root of $\langle 5, 4, 8 \rangle$ to be 4. Then Figure 1 displays all of the transpositions and inversions of Δ.

From this table it is clear that the tone row consists of the triads $(4, +), \; (1, -), \; (10, +)$, and $(7, -)$.

What is interesting about these triads is that the set of UTTs that map between them forms a group. There are actually two possible groups of UTTs that generate this tone row.

$$M = \{\langle +, 0, 0 \rangle, \; \langle -, 3, 9 \rangle, \; \langle +, 6, 6 \rangle, \; \langle -, 9, 3 \rangle\}$$
$$N = \{\langle +, 0, 0 \rangle, \; \langle -, 3, 3 \rangle, \; \langle +, 6, 6 \rangle, \; \langle -, 9, 9 \rangle\}$$

The following diagrams show the relationships between the triads in the tone row. The first diagram uses UTTs from the group M and the second uses UTTs from the group N.

	$\sigma = +$	$\sigma = -$
$r = 0$	$\langle 1, 0, 4\rangle$	$\langle 11, 0, 8\rangle$
$r = 1$	$\langle 2, 1, 5\rangle$	$\langle 0, 1, 9\rangle$
$r = 2$	$\langle 3, 2, 6\rangle$	$\langle 1, 2, 10\rangle$
$r = 3$	$\langle 4, 3, 7\rangle$	$\langle 2, 3, 11\rangle$
$r = 4$	$\langle 5, 4, 8\rangle$	$\langle 3, 4, 0\rangle$
$r = 5$	$\langle 6, 5, 9\rangle$	$\langle 4, 5, 1\rangle$
$r = 6$	$\langle 7, 6, 10\rangle$	$\langle 5, 6, 2\rangle$
$r = 7$	$\langle 8, 7, 11\rangle$	$\langle 6, 7, 3\rangle$
$r = 8$	$\langle 9, 8, 0\rangle$	$\langle 7, 8, 4\rangle$
$r = 9$	$\langle 10, 9, 1\rangle$	$\langle 8, 9, 5\rangle$
$r = 10$	$\langle 11, 10, 2\rangle$	$\langle 9, 10, 6\rangle$
$r = 11$	$\langle 0, 11, 3\rangle$	$\langle 10, 11, 7\rangle$

FIGURE 1. The set of triads generated by transpositions and inversions of $\langle 1, 0, 4\rangle$.

$$\begin{array}{ccc} (4,+) & \xrightarrow{\langle -,9,3\rangle} & (1,-) \\ {\scriptstyle\langle -,3,9\rangle}\big\downarrow & & \big\downarrow{\scriptstyle\langle -,3,9\rangle} \\ (7,-) & \xrightarrow[\langle -,9,3\rangle]{} & (10,+) \end{array}$$

$$\begin{array}{ccc} (4,+) & \xrightarrow{\langle -,9,9\rangle} & (1,-) \\ {\scriptstyle\langle -,9,9\rangle}\big\downarrow & & \big\downarrow{\scriptstyle\langle -,9,9\rangle} \\ (7,-) & \xrightarrow[\langle -,9,9\rangle]{} & (10,+) \end{array}$$

Recall that for this tone row, every triad in it is related to the others by transposition and inversion. This is not always the case however, nor does it need to be. This next example shows that it is possible to have a group that defines the relationship between triads in a tone row where the triads are not related to one another by transposition and inversion.

6. Example 2: Symphony Op. 21

The tone row of this piece is $\langle 5, 8, 7, 6, 10, 9, 3, 4, 0, 1, 2, 11\rangle$. It has a number of interesting characteristics as well. For example, taking the retrograde of the first six notes, $\langle 9, 10, 6, 7, 8, 5\rangle$, and transposing it by 6 yields the last six notes of the tone row, $\langle 3, 4, 0, 1, 2, 11\rangle$.

Just as before we can divide the row into four triads as follows.

$$\langle 5, 8, 7\rangle\ \langle 6, 10, 9\rangle\ \langle 3, 4, 0\rangle\ \langle 1, 2, 11\rangle$$

Note that each triad contains two notes that are "next" to each other in the sense that the interval between them is 1. Define the root of the triad to be the note that is not one of these two notes. Next, we will say that a triad is major if the interval between the root and the furthest note from the root is a major third

	$\sigma = +$	$\sigma = -$
$r = 0$	$\langle 0, 4, 3\rangle$	$\langle 0, 3, 2\rangle$
$r = 1$	$\langle 1, 5, 4\rangle$	$\langle 1, 4, 3\rangle$
$r = 2$	$\langle 2, 6, 5\rangle$	$\langle 2, 5, 4\rangle$
$r = 3$	$\langle 3, 7, 6\rangle$	$\langle 3, 6, 5\rangle$
$r = 4$	$\langle 4, 8, 7\rangle$	$\langle 4, 7, 6\rangle$
$r = 5$	$\langle 5, 9, 8\rangle$	$\langle 5, 8, 7\rangle$
$r = 6$	$\langle 6, 10, 9\rangle$	$\langle 6, 9, 8\rangle$
$r = 7$	$\langle 7, 11, 10\rangle$	$\langle 7, 10, 9\rangle$
$r = 8$	$\langle 8, 0, 11\rangle$	$\langle 8, 11, 10\rangle$
$r = 9$	$\langle 9, 1, 0\rangle$	$\langle 9, 0, 11\rangle$
$r = 10$	$\langle 10, 2, 1\rangle$	$\langle 10, 1, 0\rangle$
$r = 11$	$\langle 11, 3, 2\rangle$	$\langle 11, 2, 1\rangle$

FIGURE 2. The set of all possible triads generated by the method used in Example 2: Symphony Op. 21

(interval length 4) and minor if the interval is a minor third (interval length 3). For example, the first triad is $(5, -)$ because the difference between 5 and 8 is a minor third. The rest of the triads in this row are $(6, +)$, $(0, +)$, and $(11, -)$. Figure 2 shows all of the possible major and minor triads using this method.

The set of UTTs that map the triads in this tone row to one another also forms a group, defined as follows.

$$P = \{\langle +, 0, 0\rangle,\ \langle +, 6, 6\rangle,\ \langle -, 11, 1\rangle,\ \langle -, 5, 7\rangle\}$$

Just as with the group M in the Op. 24 example, the group P has the interesting property that every element in it is its own inverse. The relationship between P and the triads in the row can also be diagrammed as follows.

$$\begin{array}{ccc} (5,-) & \xrightarrow{\langle -,11,1\rangle} & (6,+) \\ {\scriptstyle\langle +,6,6\rangle}\downarrow & & \downarrow{\scriptstyle\langle +,6,6\rangle} \\ (11,-) & \xrightarrow[\langle -,11,1\rangle]{} & (0,+) \end{array}$$

What sets this example apart from the Op. 24 example is that here the triads in the tone row do not all belong to the same T/I class. That is to say, given one triad from the row, it is impossible to generate the other triads in the row by applying transpositions and inversions to the given one. The diagram above is actually a special case of the Fiore-Satyendra Theorem, which states the following.

Theorem 6.1 (Fiore-Satyendra)**.** *The generalized contextual group is dual to the T/I group, i.e.* $ab = ba$ *when* a *is in the contextual group and* b *is in the T/I group, more precisely each group is the centralizer of the other.*

Here the contextual group is the same as the neo-Riemannian group mentioned earlier. The T/I group is the group containing all transposition and inversion operations. In this group each inversion operation is the inverse of itself, and the inverse of the n transposition is the $12 - n$ transposition. The identity element is the 0 transposition.

In the above diagram, $\langle -, 11, 1\rangle$ belongs to the neo-Riemannian group, while $\langle +, 6, 6\rangle$ is transposition by 6. From the diagram it is clear that $\langle -, 11, 1\rangle\langle +, 6, 6\rangle = \langle +, 6, 6\rangle\langle -, 11, 1\rangle$, and this can be checked using simple arithmetic.

$$\langle -, 11, 1\rangle\langle +, 6, 6\rangle = \langle -, 11 + 6, 1 + 6\rangle = \langle -, 5, 7\rangle$$
$$\langle +, 6, 6\rangle\langle -, 11, 1\rangle = \langle -, 6 + 11, 6 + 1\rangle = \langle -, 5, 7\rangle$$

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