TEMPERATURE THEORY AND THE THERMOSTATIC STRATEGY

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ABSTRACT. In this paper, we differentiate between cold games, which are easier to analyze and play, and hot games, much more difficult in terms of strategy. We present algorithms to find the left stop, right stop, and mean value of a hot game. We finish with an application of temperature theory and introduce the thermostatic strategy, a strategy that is particularly useful in analyzing sums of hot games.

CONTENTS

1. AN INTRODUCTION TO GAMES

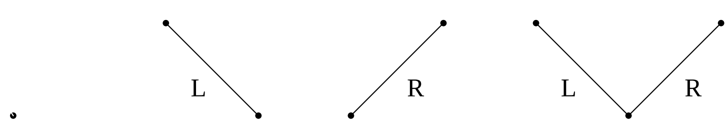
The games that are studied in combinatorial game theory involve only two players, **L**eft and **R**ight, and a finite set of moves before the game comes to an end. Left and Right alternate moves, and the first player who cannot move loses. There are no moves depending on chance, and both players have perfect information.

Definition 1.1. A game is an ordered pair of sets of games. This is denoted by

$$G = \{G^L \mid G^R\},$$

where G^L is the set of games Left can reach in the next move, and G^R is the set of games Right can reach.

Let us look at a few games—the games may be illustrated using trees, with G^L and G^R branching out from the starting node.



In the first game, to the very left, there is a starting position, but no moves for either player. Using the preceding definition, we write $G = \{|\}$. We call this the *zero game*, and, as the first player to move loses, the zero game is a second player win. In other words, the zero game is a game in which the second player has a winning strategy. This will take on greater meaning when we come upon more complicated games in which the second player is only guaranteed a win so long as he plays optimally.

The second game is a game in which Left has a move but Right does not. If Right is offered the chance to move first, he will lose. If Left moves first, however, he has a move to the zero game, which will also force Right, the next player to move, to lose. This game is $\{0 \mid\}$, which we denote by 1. Our sympathies are usually with Left, so we say that a game with a winning strategy for Left is a *positive game*.

Conversely, consider the third game. It is the same game as 1 except the players' moves are flipped. This game is $\{\mid 0\}$, which we denote by -1. We say that a game with a winning strategy for Right is a *negative game*.

Definition 1.2. The *negative* of a game G is given by

$$-G = \{-G^R \mid -G^L\}.$$

The last game that ends in at most one move looks like the fourth game pictured. Both Left and Right can move to the zero game, and so the game is $\{0 \mid 0\}$, which we denote by $*$. In this game, the first player to move wins. Generally we say that games in which the first player to move has a winning strategy are *fuzzy*.

To summarize and introduce useful notation,

- $G = 0$ if there is a winning strategy for the second player.
- $G > 0$ if there is a winning strategy for Left.
- $G < 0$ if there is a winning strategy for Right.
- $G \parallel 0$ if there is a winning strategy for the first player.

Proposition 1.3 ([1], p. 73)**.** *All games fall into one of these four categories.*

We can combine notation to attain some more nuanced comparisons. For example, we say that $G \geq 0$ if either $G > 0$ or $G = 0$. In either case, if Right goes first then Left has a winning strategy. On the other hand, if Right goes first and has a winning strategy himself, then either $G < 0$ or $G \parallel 0$. Therefore, a game G satisfies $G \geq 0$ if and only if Left has a winning strategy when Right goes first. Similarly, we say that $G \mathrel{|\!\rhd} 0$ if either $G > 0$ or $G \parallel 0$. This means that Left has a winning strategy if Left starts.

2. Sums of Games

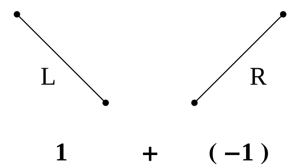
Definition 2.1. The sum of two games G and H is

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}.$$

In gameplay, this is the equivalent of placing two games next to each other, for then our players can make a move in either G or H.

To find the difference between two games G and H, we simply take the negative of the latter game: $G - H = G + (-H)$. We can now use our definition of $-G$ to check that $G + (-G) = 0$.

Example 2.2. The game $1 - 1$, or $1 + (-1)$, is shown here:



We can see that the player who goes first will end up losing, as the second player will then have the last move. Thus this game does turn out to be a zero game.

In the same way, for all games G, the second player is able to mirror the first player's moves exactly, and will win the game by making the last move. We find that addition of games is both associative and commutative, with identity 0 and $-G$ the additive inverse of G.

Armed with our newfound knowledge of summing up games, we can now compare two games with each other. We say that $G \geq H$ if $G - H \geq 0$, and define equality by $G = H$ if $G \geq H$ and $G \leq H$.

Definition 2.3. An equivalent definition of inequality is that $G \geq H$ if and only if there is no option H^L such that $H^L \geq G$ and there is no option G^R such that $G^R \leq H$.

3. Numbers

A *number* $x = \{x^L \mid x^R\}$ is a game in which all elements of x^L and x^R are numbers (or the empty set), and no element of x^L is greater than or equal to any element of x^R. The number x is equivalent to the simplest number between x^L and x^R, simplest meaning the one that is created earliest in the order of construction.

For all numbers x, we have $x \ngeq x^R$ and $x^L \ngeq x$. To see this, we consider $x \geq x^R$. This can only be so if there exists no inequality of the form $x^R \geq x^R$ or $x \leq (x^R)^L$, which is impossible, since $x^R \geq x^R$ is always true. The inequality $x^L \ngeq x$ is proved similarly, and from here it is a short step to proving that $x^L < x < x^R$ for all x. Since we already know that $x \ngeq x^R$, we simply have to show that $x^R \geq x$. This is true unless there exists some $x^{RR} \leq x$ or some $x^L \geq x^R$. The latter is prohibited by the definition of a number, and the former implies $x^R < x^{RR} \leq x$, which is clearly false.

Construction 3.1. Before we have any numbers, we have the empty set. Plugging in the empty set as the sole element of x^L and x^R, we get $x = \{|\}$, which we call 0. With the creation of 0, we have two possible elements of x^L and x^R, and $\{|\}$, $\{0\ |\}$, $\{|\ 0\}$, and $\{0 \mid 0\}$ as possible numbers. We have seen $\{|\}$ before, and $\{0 \mid 0\}$ fails to be a number by definition. Recall our earlier notation $\{0\ |\} = 1$ and $\{|\ 0\} = -1$.

Now we have numbers $\{|\ -1\}$, $\{-1 \mid 0\}$, $\{0 \mid 1\}$, and $\{1\ |\}$, which respectively we call -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2. Note that $y = \{1\ |\}$ is equal to $x = \{0, 1\ |\}$. Because $x > x^L$, we have $x > 1$ and $x > 0$, but because $1 > 0$, the 0 is *dominated* by the 1. In gameplay, this means that the 0 will never be picked, for Left will always want to move to the most positive game he is able to play (and Right the most negative).

Proposition 3.2. *With notation as above,* $\frac{1}{2} + \frac{1}{2} = 1$.

Proof. We have by definition of addition $\frac{1}{2}+\frac{1}{2}=\{\frac{1}{2} \mid 1+\frac{1}{2}\}$. It suffices to show that $1 \geq \frac{1}{2}+\frac{1}{2}$ and $1 \leq \frac{1}{2}+\frac{1}{2}$. We have that $1 \geq \frac{1}{2}+\frac{1}{2}$ is true if there does not exist an option $(\frac{1}{2}+\frac{1}{2})^L \geq 1$ and if there does not exist an option $1^R \leq \frac{1}{2}+\frac{1}{2}$. The former is true since $(\frac{1}{2}+\frac{1}{2})^L=\{\frac{1}{2}\}$, and the latter is trivially true, as there are no elements in 1^R. The other inequality, $1 \leq \frac{1}{2}+\frac{1}{2}$, is proved similarly. □

By continuing the above construction, we can obtain all natural numbers, and moreover, all dyadic rationals. We see this as a natural number n is equal to $\{0,1,2,\ldots,n-1 \mid\}$, and a dyadic rational number x with denominator 2^n is equal to $\{x-(\frac{1}{2^n}) \mid x+(\frac{1}{2^n})\}$. If we allow the sets of left and right options to be infinite, then we find that these numbers will contain ordinals and real numbers. For example, an ordinal number w is equal to $\{\alpha : \alpha < w \mid\}$, and $\frac{1}{3}$ is equal to $\{\frac{1}{4}, \frac{1}{4}+\frac{1}{16}, \frac{1}{4}+\frac{1}{16}+\frac{1}{64}, \ldots \mid \frac{1}{2}, \frac{1}{2}-\frac{1}{8}, \ldots\}$. Additionally, numbers are totally ordered: we can think of them as being on an extended number line.

At this point, we will make the transition from numbers to a different sort of game. Numbers are in actuality very boring games to play. This is because numbers are basically guaranteed wins for one player or another, and neither player will want to move in a number, as all of Left's options, x^L, are less than x, and vice versa for Right. When playing sums of games, players will choose to move in the games which are not numbers.

Theorem 3.3 (Number Avoidance Theorem)**.** *For a game x a number and a game G not a number,*

$$G+x=\{G^L+x \mid G^R+x\}.$$

We will save the proof of this theorem for a later section, after we learn a couple of algorithms that will help us in our proof.

4. Hot Games

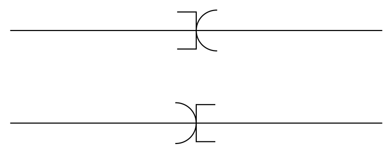
In the previous section, we looked at games we claimed weren't particularly interesting in terms of strategy. We call numbers *cold games*, as there is not much incentive to move in a game that is a number. *Hot games*, on the other hand, hold much incentive to move.

What kind of game is a hot game? Possibly a game like $\{1000 \mid -1000\}$, where whichever player goes first gains a big advantage!

We can't exactly compare hot games to all numbers, as they are not numbers, but we can compare them to some numbers. For any hot game G, we may split the number line into numbers that are greater than or equal to G, numbers that are less than or equal to G, and numbers that are *confused with*, or incomparable, to G. The numbers that we call confused with the hot game are represented by a cloud, flanked by numbers greater than or equal to the game to the right of the cloud, and numbers less than or equal to the game to the left.

Fortunately, we can strip away the confusion to find that the hot game has a mean value above one specific number, a value that the game attains on average when many multiples of the game are played. We are able to find out a lot of information about the hot game, including its mean value and right and left boundaries of the cloud of numbers incomparable with the game.

Definition 4.1. A *Dedekind section* splits the class of all numbers into two subsets at a number x, with x in one of the two subsets. Pictorially, we can represent Dedekind sections as drawn below.



or

We define two Dedekind sections on the number line for a hot game G.

Definition 4.2. For the *Left section* of G, $L(G)$, we place into its right hand side all numbers x such that $x \geq G$ and into its left hand side all numbers x such that $x \triangleleft\!|\, G$.

For the *Right section* of G, $R(G)$, we place into its left hand side all numbers y such that $y \leq G$ and into its right hand side all numbers y such that $y \,|\!\triangleright G$.

Note that these are indeed sections, as proposition 1.3 tells us that every number is in exactly one side of $L(G)$ and one side of $R(G)$.

We can define Left and Right sections for numbers too, with numbers greater than or equal to a number x inhabiting the right side of $L(x)$, and numbers strictly less than x inhabiting the left side of $L(x)$. Likewise, the numbers to the left of $R(x)$ are less than or equal to x, and the numbers to the right of $R(x)$ strictly greater. Intuitively, we can compare Left and Right sections by comparing the points at which they are split. While for numbers $L(x) < R(x)$, it is not so for hot games, where $R(G) < L(G)$. If for a hot game G we had $L(G) < x < R(G)$, we would have $x \leq G \leq x$, and thus $G = x$, a contradiction.

5. The Right and Left Stops

We call the cutoff points of the Left and Right sections the *left stop*, $l(G)$, and *right stop*, $r(G)$, respectively. These and the attached sections may be identified using the following algorithms.

Theorem 5.1. *If G is a number, then $l(G) = r(G) = G$ and we know from above what the sections $L(G)$ and $R(G)$ are equal to. For a game G that is not a number, we have*

$$l(G) = \max_{G^L} r(G^L) \tag{5.2}$$

and

$$r(G) = \min_{G^R} l(G^R), \tag{5.3}$$

and similar equations for the left and right sections.

When playing a game, both players will eagerly move until a number is reached, at which point it would be disadvantageous for either player to move. We can agree to stop a game when a number is reached, and tally up the score of a sum of games by adding up the components when they are numbers.

Left will try to reach as large a number as possible, while Right will try to reach as small a number as possible. In this way we can find the left and right stops by following intelligent play. We play the game until a number is reached, taking note of the next unfortunate player to move. The game will fall slightly to the left or right of the stopping value, depending on the next player to move.

Example 5.4. The game $\{9 \mid \{7 \mid 2\}\}$. If Left goes first, Left will move to the game 9, with Right about to move. Thus we say that $l(G) = r(9)$. If Right starts he will move to $\{7 \mid 2\}$, and Left to 7, so $r(G) = r(7)$. The cloud of confusion extends from just to the right of 7 to just to the right of 9, and thus excludes 7 while including 9.

Now we know enough to prove the Number Avoidance Theorem. Recall the statement of the theorem that $G + x = \{G^L + x \mid G^R + x\}$.

Proof of theorem 3.3. We want to show that $G + x \leq \{G^L + x \mid G^R + x\}$ and $G + x \geq \{G^L + x \mid G^R + x\}$. We will only prove the former inequality since the latter inequality is proved similarly.

We have $G + x = \{G^L + x, G + x^L \mid G^R + x, G + x^R\}$ by definition. Let us suppose for contradiction that $G + x \not\leq \{G^L + x \mid G^R + x\}$. This means that either there exists $(G+x)^L$ such that $(G+x)^L \geq \{G^L + x \mid G^R + x\}$, or $G^R + x \leq G + x$. The latter is impossible, as $G^R + x$ is a right option of $G + x$. Similarly, if there exists $(G + x)^L$ such that $(G + x)^L \geq \{G^L + x \mid G^R + x\}$, it is not $G^L + x$. The only possibility remaining is $G + x^L \geq \{G^L + x \mid G^R + x\}$. We assume that we have already proven the theorem for all numbers simpler than x, and so let $H = \{G^L + x \mid G^R + x\} \leq \{G^L + x^L \mid G^R + x^L\} = H'$. We will show that this is impossible, using the algorithms shown above.

The left stop of H is given by our algorithm to be $l(H) = \max(r(G^L + x))$, where the maximum is taken over the set G^L. Similarly, the left stop of H' is given by $l(H') = \max(r(G^L + x^L))$ taken over the same set. For any G^L we have $G^L + x^L < G^L + x$, and thus $l(H') = \max(r(G^L + x^L)) < \max(r(G^L + x)) = l(H)$, a contradiction. □

6. The Mean Value Theorem

Since we found the right and left stops following intelligent play, we know that a hot game G will result on its right stop when Right plays first, and its left stop when Left plays first. We can further say that G finds on average its center of mass at a value we call its mean value, $m(G)$. We propose that for any finite n, the game nG is very close to the game $nm(G)$.

To prove this, we start with some inequalities about the right and left stops.

Theorem 6.1. *For games G, H, we have*

$$r(G) + r(H) \leq r(G + H) \leq r(G) + l(H) \leq l(G + H) \leq l(G) + l(H).$$

Proof. These inequalities may be proved using playing strategies, and are proved similarly. For example, we will prove the first inequality, that $r(G) + r(H) \leq r(G + H)$. Left player plays second, and if he does not have a better strategy, may simply respond in the same component that Right plays, continuing the strategies that yield $r(G)$ and $r(H)$. In this way, Left can always ensure a stopping value that is at least equal to $r(G) + r(H)$. □

We may come up with many more inequalities which are in essence the same. For example, $r(G) = r(G + H - H) \geq r(G + H) + r(-H) = r(G + H) - l(H)$.

Theorem 6.2 (Mean Value Theorem). *For every game G there is a number $m(G)$ and a real number t such that*

$$nm(G) - t \leq nG \leq nm(G) + t,$$

for all finite integers n.

Proof. It will suffice to show that the difference between $l(nG)$ and $r(nG)$ is bounded independently of n. By the previous theorem, we have $nr(G) \leq r(nG)$ and $l(nG) \leq nl(G)$, and thus $r(G) \leq \frac{1}{n}r(nG) \leq \frac{1}{n}l(nG) \leq l(G)$. Our proof would then show $\frac{1}{n}r(nG)$ and $\frac{1}{n}l(nG)$ to converge to the common value $m(G)$.

We have $l(nG) = r((n-1)G+G^L)$ assuming G^L is the left option that satisfies the maximum in equation 5.2. Then $l(nG) = r((n-1)G+G^L) \leq r(nG)+l(G-G^L)$. □

7. Thermographs

We find the mean value of a hot game G by imposing a real-valued tax, t, on its left and right options to reduce the incentive to move. We call such a tax temperature, and refer to this process as "cooling" G.

We define a new *cooled game*, G_t, that takes into account such a tax for the game G, adding a fee of t to every move that a player makes.

Definition 7.1. For a game G and t a real number ≥ 0,

$$G_t = \{{G^L}_t - t \mid {G^R}_t + t\},$$

unless G_t is a number.

Note that G_t will be a number for all t that are sufficiently large. Just before G_t becomes a number, it will be *infinitesimally close* to a number. Infinitesimals are games which have right and left stops both equal to 0: for example, $* = \{0 \mid 0\}$, $\uparrow = \{0 \mid *\}$, and $\heartsuit = \{* \mid 0\}$[1]. We say that G_t is infinitesimally close to a number if G_t is the sum of a number with an infinitesimal. We specify t_0 as the temperature of G, or the smallest t for which G_t is infinitesimally close to a number, and we denote this number by x. For all $t > t_0$, we let $G_t = x$.

Thus we may slightly refine the previous definition, and say that

$$G_t = \begin{cases} \{{G^L}_t - t \mid {G^R}_t + t\} & 0 \leq t \leq t_0 \\ x & t > t_0. \end{cases}$$

For example, take $G = \{1000 \mid -1000\}$. When $t = 1000$ we have $G_t = \{0 \mid 0\}$, which is infinitesimally close to the number 0. Therefore, $G_t = \{1000-t \mid -1000+t\}$ for $t < 1000$, $G_{1000} = 0 + *$, and $G_t = 0$ for $t > 1000$.

Let us define new values $l_t(G)$ and $r_t(G)$ that are $l(G_t)$ and $r(G_t)$.

Theorem 7.2. *For a game G and a real number $t \geq 0$, we have*

$$l_t(G) = \max r_t(G^L) - t \tag{7.3}$$

and

$$r_t(G) = \min l_t(G^R) - t, \tag{7.4}$$

[1] Stuffy old men know this game to be $\downarrow$.

unless $l_t(G) < r_t(G)$. *If* $l_t(G) < r_t(G)$, *then* G_t *is the simplest number* x *at the lowest temperature at which this occurs, and we have* $l_t(G) = l(x)$ *and* $l_t(G) = r(x)$.

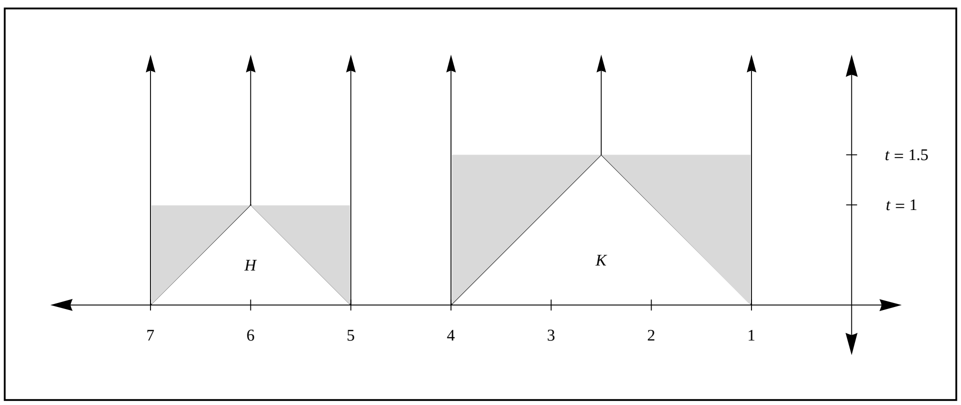
For games with more complicated right and left options, we may draw a *thermograph* to illustrate the game's mean value. A thermograph is the region between a game's left and right boundaries, $l_t(G)$ and $r_t(G)$. We plot games on the same number line as their right and left options, and temperature on the vertical axis.

Convention dictates that the number line be backwards, so that positive values may be further left than negative, and the left stop to the left of the right stop.

Example 7.5. The game $G = \{\{7 \mid 5\} \mid \{4 \mid 1\}\}$. Let us name for clarity $H = \{7 \mid 5\}$ and $K = \{4 \mid 1\}$. Since G_t has a recursive definition, we study recursively the left and right options until we reach options that are numbers. We do not tax numbers, and so draw vertical arrows above the games 7, 5, 4, and 1.

We shall start with H: $H_t = \{7 - t \mid 5 + t\}$ becomes a number at a temperature of $t > 1$. We tax the options of H with a tax t until both left and right boundaries meet above the number 6. Thus for $t > 1$ we draw a vertical arrow above 6, denoting it as the mean value.

Similarly with K, $K_t = \{4 - t \mid 1 + t\}$ becomes a number at temperature $t > 1\frac{1}{2}$. We tax the options of K with a tax t until we reach the mean value of K, $2\frac{1}{2}$.

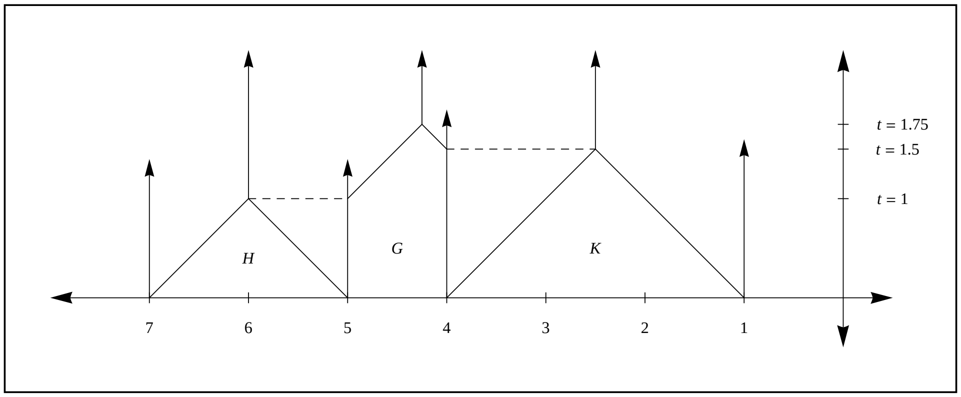


Now, the left boundary of G is found by taxing the right boundary of H with a tax of $-t$, and the right boundary of G is found by taxing the left boundary of H with a tax of t. We continue until the mean value of G is reached.

We have

$$G_t = \begin{cases} \{\{7-t \mid 5+t\} - t \mid \{4-t \mid 1+t\} + t\} & 0 \le t \le 1 \\ \{6-t \mid \{4-t \mid 1+t\} + t\} & 1 < t \le 1\frac{1}{2} \\ \{6-t \mid 2\frac{1}{2} + t\} & 1\frac{1}{2} < t \le 1\frac{3}{4} \\ 4\frac{1}{4} & t > 1\frac{3}{4}. \end{cases}$$

Note that at temperature $t < 1$, $G_t = \{\{7-t \mid 5+t\} - t \mid \{4-t \mid 1+t\} + t\} = \{\{7 - 2t \mid 5\} \mid \{4 \mid 1 + 2t\}\}$, and thus the left and right boundaries of G_t at such temperatures are vertical lines above 5 and 4.



There are some rules and patterns that apply to all thermographs.

Theorem 7.6. *For any game G, its left boundary is a line in periods either vertical or diagonal to the right with a slope of -1. The right boundary of G is a line in periods either vertical or diagonal to the left with a slope of 1. The two boundaries meet and form a mast above a dyadic rational.*

Proof. Observe that subtracting t from a right boundary, a line that is vertical or diagonal to the left with a slope of 1, will result in a line that is diagonal to the right with a slope of -1 or vertical, a left boundary. Adding t to a left boundary will result in a right boundary with the specifications we desire. The games that we start such a process with are always numbers. Two boundaries that are aiming toward each other at the same rate must meet at a number that may be found by dividing by 2. □

8. The Thermostatic Strategy

When playing combinatorial games, things may become complicated very quickly when you are faced with sums of hot games. We can use temperature to determine the most advantageous move to make. The thermostatic strategy provides a generally optimal analysis of sums of hot games, telling the player in which component in the sum to move. By following the thermostatic strategy, the player ensures that he can reach a stopping position for the sum of games that differs from the optimal one by no more than the temperature of the hottest game.

Suppose we are playing a sum of games $A+B+C+\cdots$ as Left player, and that we know the thermographs of each of the individual components in the sum. Let us define a *compound thermograph* as having a right boundary that is the sum of all of the right boundaries of the individual components

$$r_t(A) + r_t(B) + r_t(C) + \cdots$$

and a left boundary that is

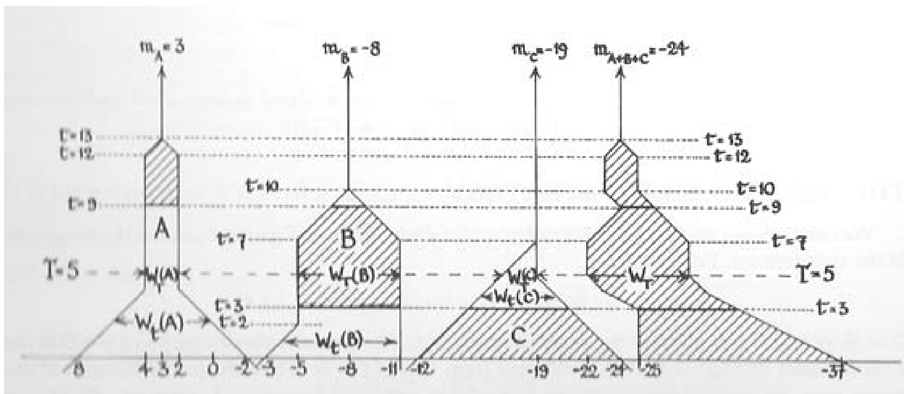
$$r_t(A) + r_t(B) + r_t(C) + \cdots + W_t$$

where W_t is the maximum width of any of the components $A + B + C + \cdots$ at temperature t.

In other words, $W_t = \max\{W_t(A), W_t(B), W_t(C), \ldots\}$ and we can see that the compound thermograph of the sum $A + B + C + \cdots$ will have a width of W_t at temperature t.

The *ambient temperature* is the least temperature such that the left boundary, $r_t(A)+r_t(B)+r_t(C)+\cdots+W_t$, is maximal. If the Left player moves in a component that is widest—with a width of W_t—at the ambient temperature, he is guaranteed a stopping position that is at least $r_t(A) + r_t(B) + r_t(C) + \cdots + W_t$.

Example 8.1. Let us see an example.



In this case, the left boundary of the compound thermograph is maximal between the temperatures of 5 and 7, and thus the ambient temperature is 5. The thermostatic strategy recommends that the player move in component B, which is the widest component at temperature $t = 5$, with a width of W_t.

Let us introduce a few theorems before we prove the thermostatic strategy.

Theorem 8.2. *The difference between the right or left stop of a game and its mean value is no more than the temperature of the game.*

Proof. This follows from the observation that the right and left boundaries of a thermograph are always either vertical or diagonal with a slope of ± 1. The maximal difference occurs only if a boundary is completely diagonal. □

Theorem 8.3. *The temperature of any sum is no more than the largest temperature of any component in the sum.*

Proof. The right and left boundaries of the compound thermograph meet when the maximal width, W_t, is zero. □

Corollary 8.4. *When playing the sum of a large number of games, the difference between the stopping position recommended by the thermostatic strategy and the optimal strategy is bounded by the largest temperature.*

Proof. This follows from the previous two theorems. □

Theorem 8.5. *The mean value of a sum of games is equal to the sum of the mean values of the games.*

Proof. We have

$$(A+B+C+\cdots)_t = A_t + B_t + C_t + \cdots$$

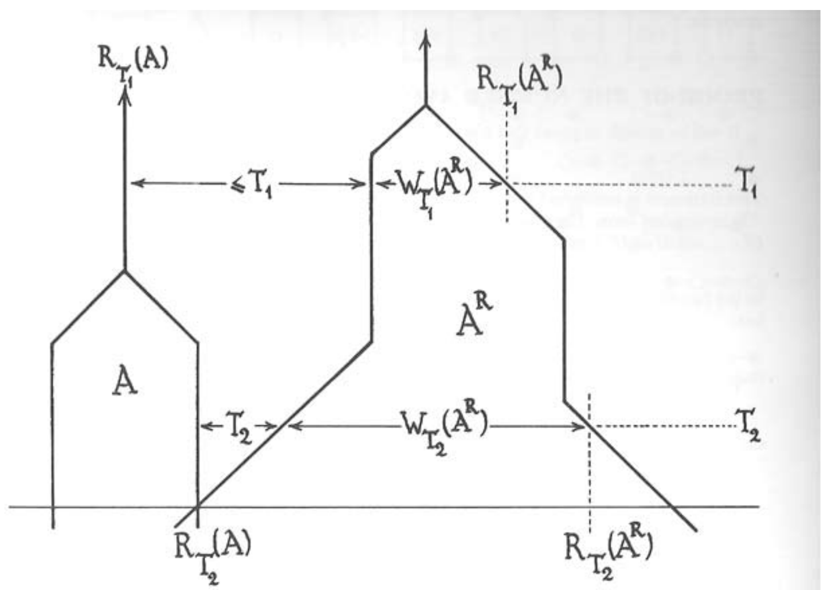
so for $t > \max\{t(A), t(B), t(C), \ldots\}$ we have

$$(A+B+C+\cdots)_t = m(A) + m(B) + m(C) + \cdots .$$

□

Theorem 8.6 (The Thermostatic Strategy)**.** *Given a sum of games* $A+B+C+\cdots$*, suppose Left moves in the widest component at the ambient temperature. Then for any given temperature* t*, he is guaranteed a stopping position that is at least* $r_t(A)+r_t(B)+r_t(C)+\cdots-t$ *if Right starts, and at least* $r_t(A)+r_t(B)+r_t(C)+\cdots+W_t$ *if he starts himself.*

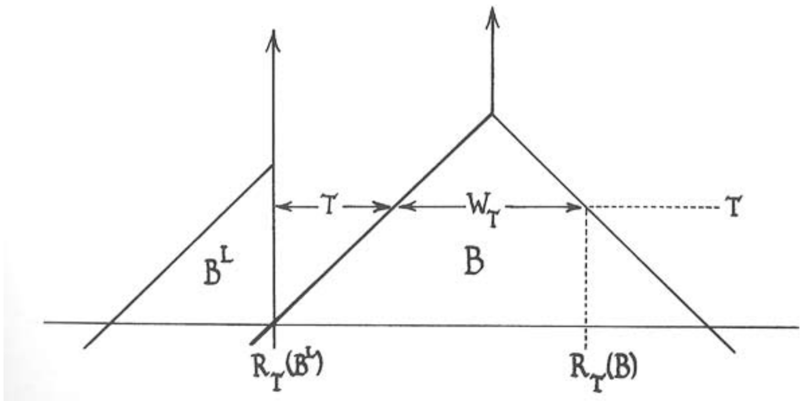
Proof. Suppose Right starts, and let him move from $A+B+C+\cdots$ to $A^R+B+C+\cdots$. Left is then inductively guaranteed at least $r_t(A^R)+r_t(B)+r_t(C)+\cdots+W_t(A^R)$.



Using the illustration, we can see that no matter the temperature t, we have $r_t(A^R)+W_t(A^R) \geq r_t(A)-t$. This is because if $t < t(A)$, then $r_t(A^R)+W_t(A^R) = l_t(A^R) = r_t(A)-t$ through the process of taxing we have defined. If $t \geq t(A)$, then we have $r_t(A^R)+W_t(A^R) \geq r_t(A)-t$ due to the fact that the distance between $r_t(A)$ and $r_t(A^R)+W_t(A^R)$ has stopped growing after $r_t(A)=m(A)$.

If Left starts, he will play at the ambient temperature t, at which some component of the sum will have the maximal width, W_t. Such a component has a temperature that is greater than t.

Suppose the component mentioned is B, and Left moves from $A+B+C+\cdots$ to $A+B^L+C+\cdots$ He is inductively guaranteed at least $r_t(A)+r_t(B^L)+r_t(C)+\cdots-t$.



Since we made sure to move at a temperature $t < t(B)$, we have $r_t(B^L) - t = r_t(B) + W_t$. $\square$

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References

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