FREUDENTHAL SUSPENSION THEOREM

TENGREN ZHANG

Abstract. In this paper, I will prove the Freudenthal suspension theorem, and use that to explain what stable homotopy groups are. All the results stated in this paper can be found in Chapter 4 of Algebraic Topology by Hatcher. The reader is expected to be familiar with basic group theory, such as the definitions of group, group homomorphism and exactness, as well as basic algebraic topology, which includes the notions of homotopy, homotopy of pairs, homotopy equivalence, deformation retraction and CW complexes. These can be found in Abstract Algebra by Dummit and Foote, and Algebraic Topology by Hatcher.

Contents

1. Homotopy Groups

Definition 1.1. Let (X, x_0) be a based topological space. The *nth homotopy group* of (X, x_0) is the set of base point preserving homotopy classes of continuous functions $[f]$, where $f : (I^n, \partial I_n) \to (X, x_0)$. We denote this set as $\pi_n(X, x_0)$. Also, we call X *n-connected* if for all $i \leq n$, $\pi_i(X, x_0)$ contains only one element.

As its name suggests, the nth homotopy group of any space X is a group when $n \geq 1$, and is in fact abelian when $n \geq 2$. When $n \geq 1$, we can define an operation $+ : \pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$ such that $[f] + [g] = [f + g]$, where

$$(f+g)(x_1, x_2, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n) & \text{if } 0 \leq x_1 \leq \frac{1}{2}; \\ g(1-2x_1, x_2, \dots, x_n) & \text{if } \frac{1}{2} \leq x_1 \leq 1. \end{cases}$$

We will now check that this operation is well defined. First, observe that $[f+g]$ is in $\pi_n(X, x_0)$. Also, if $[f] = [f']$ and $[g] = [g']$, let F, G be the base point preserving

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homotopies from f to f' and g to g' respectively. One can then easily check that $H : I^n \times I \to X$ given by

$$H(x_1, x_2, \dots, x_n, t) = \begin{cases} F(2x_1, x_2, \dots, x_n, t) & \text{if } 0 \le x_1 \le \frac{1}{2}; \\ G(1 - 2x_1, x_2, \dots, x_n, t) & \text{if } \frac{1}{2} \le x_2 \le 1 \end{cases}$$

is a base point preserving homotopy from $f + g$ to $f' + g'$. Hence, $+$ is well defined.

Under this operation it is easy to check that the identity is the homotopy class of the constant map to x_0 and the inverse of $[f]$ is the homotopy class containing the function $-f$, where $-f(x_1, x_2, \dots, x_n) := f(1 - x_1, x_2, \dots, x_n)$. The associativity of this operation is given by the straight line homotopy H described in the picture below:

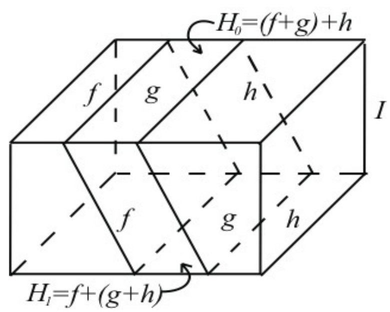


FIGURE 1. Proof of associativity

Also we can see the commutativity of this operation when $n \ge 2$ via the homotopy that is the composition of the homotopies described in the following picture:

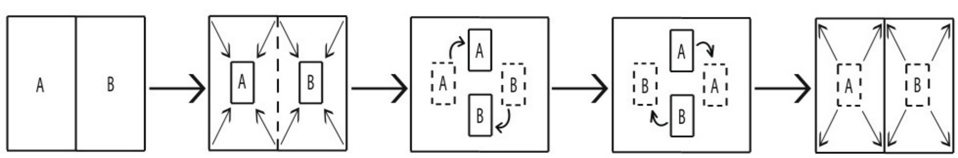


FIGURE 2. Proof of commutativity

The first homotopy shrinks the domain of f and g to proper subsets of I^n, the second and third switches the positions of the domain of f with the domain of g, and the last one enlarges the domains back to their original size. Note that the second and third homotopies can only be defined when $n \ge 2$ because the domains of f and g need to remain disjoint throughout the entire homotopy.

Now that we have associated groups to topological spaces, it is interesting to look at how the maps between topological spaces, i.e. continuous functions, relate to group homomorphisms between their corresponding homotopy groups. Fortunately, this relationship is very straightforward. For any continuous function $f : X \to Y$, we can associate with it a group homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ such that $f_*([g]) = [f \circ g]$. This holds for any n. In fact, we have an even nicer relationship between f_* and f: if f is a homotopy equivalence, then f_* is an isomorphism for any n. For brevity, we will use but not prove some of these facts.

The definition of a homotopy groups is related to that of *relative homotopy groups.* If X is a topological space, $A \subset X$ and $x_0 \in A$, we can define $\pi_n(X, A, x_0)$ to be the set of homotopy classes of continuous functions $[f]$, where $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ and the homotopy is via functions of this form. Here, and in the rest of this paper, $J^{n-1} := \partial I^n \setminus (I^{n-1} \times \{0\})$, where $I^{n-1} \times \{0\}$ is seen as the face of I^n with the last coordinate $x_n = 0$. Note that if we take $A = \{x_0\}$, then the definitions of $\pi_n(X, A, x_0)$ and $\pi_n(X, x_0)$ coincide.

An alternative way to think about homotopy groups is to consider them as the "different" ways to map spheres instead of cubes into X. This means that we think of a representative from a homotopy class in $\pi_n(X, x_0)$ to be of the form $f : (D^n, S^{n-1}) \to (X, x_0)$. Similarly, the corresponding representative for relative homotopy groups is of the form $f : (D^n, S^{n-1}, x_0) \to (X, A, x_0)$. A little thought will convince the reader that these two ways of defining homotopy groups are equivalent. The upshot for using this definition is that when dealing with CW-complexes, the representatives of homotopy classes in the relative homotopy groups look very much like attaching cells to A.

Not surprisingly, the relative homotopy groups turn out to be groups as well. However, the group multiplication in this case is a little trickier to define. Since writing down an explicit formula is both unenlightening and similar to what we have done above, we provide a pictorial definition instead. Although this is done for $\pi_3(X, A, x_0)$, this is true for the general case as well. For the cubes on the left, the right face of the left cube and the left face of the right cube are both sent to x_0 by f and g respectively. Thus, we can define $f + g$ to be the map whose domain is the domain of f and g identified along the faces mentioned above and reparamaterized into a cube. Using a similiar argument as that in the non-relative case, we can check that $+$ here is also well defined.

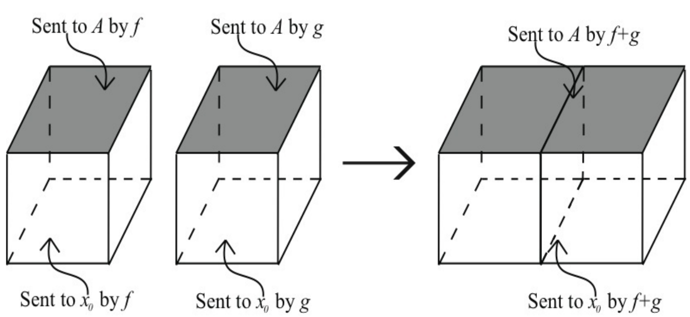


FIGURE 3. Definition of +

Using the proofs above, but with slight modifications, one can show that this multiplication is well defined, and that the nth relative homotopy group of any space is also a group when $n \geq 2$, and is abelian when $n \geq 3$.

One of the main advantages of looking at relative homotopy groups is that they are related via a long exact sequence as shown in the next theorem. This turns out to be a vital computational tool for higher homotopy groups. To prove the theorem though, we need a useful lemma known as the *compression criterion.*

Lemma 1.2 (Compression Criterion)**.** *A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents the identity in $\pi_n(X, A, x_0)$ if and only if it is homotopic relative S^{n-1} to a map whose image is entirely contained in A.*

Proof. First suppose that $[f]$ is the identity in $\pi_n(X, A, x_0)$. This means that we have a homotopy $H : (D^n \times I, S^{n-1} \times I, \{s_0\} \times I) \to (X, A, x_0)$ from f to the constant map to x_0, so we can define another homotopy G such that G_t maps D^n into X starting with $G_t|_{S^{n-1}} = f|_{S^{n-1}}$, up along the sides of the cylinder defined by H until the boundary of H_t, and ending with H_t. Observe that G_1 is a map whose image is $H(S^{n-1} \times I) \cup \{x_0\} \subset A$, and $G_t|_{S^{n-1}} = f|_{S^{n-1}}$ for all $t \in I$, and f is homotopic to G_1 via G.

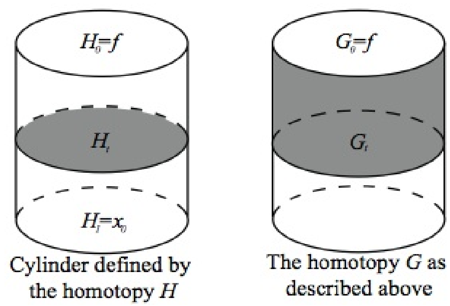


FIGURE 4. Pictorial description of G

To prove the converse, suppose that f is homotopic to g relative S_{n-1} and $g(D^n) \subset A$. This means $[h] = [g]$ in $\pi_n(X, A, x_0)$. Now, D^n is contractible, so let H be a deformation retract of D^n to s_0. Then let $G(x, t) := g(H(x, t))$, and observe that G is a homotopy from g to the trivial map, and $G_t(S^{n-1}) \subset A$ for all $t \in I$. Thus, $[g] = 0$ in $\pi_n(X, A, x_0)$, and since $[h] = [g]$, $[h] = 0$. □

Theorem 1.3 (Exact Sequence for Relative Homotopy Groups)**.** *Let i_* and j_* be the homomorphisms induced by the inclusions $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ respectively, and let $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ be given by $\partial([f]) := [f|_{I^{n-1}\times\{0\}}]$. Then the sequence*

$$\cdots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots$$

is exact.

Proof. First, we will show that $\ker(j_*) = \operatorname{im}(i_*)$. Note that $j_* \circ i_*$ is induced by $j \circ i$, which, being a composition of two inclusion maps, is also an inclusion map. Hence, for any map $[f] \in \pi_n(A, x_0)$, $j_* \circ i_*([f])$ is represented by $j \circ i \circ f = f$, which has image entirely in A, so $j_* \circ i_*([f]) = 0$ by the compression criterion. This shows that $\operatorname{im}(i_*) \subset \ker(j_*)$. Now, pick any $[f] \in \ker(j_*)$, and since $j_*([f]) = [j \circ f]$ is the identity in $\pi_n(X, A, x_0)$, the compression criterion tells us that $j \circ f$ is homotopic relative S^{n-1} to g' for some g' whose image is entirely in A. This homotopy between f and g' is constant on S^{n-1}, so it preserves x_0, which means that $[g'] \in \pi_n(X, x_0)$ and $[g'] = [f]$ in $\pi_n(X, x_0)$. Also, since g' has image entirely in A, $[g'] \in \pi_n(A, x_0)$, and i_* being induced by an inclusion map means $i_*([g']) = [g'] = [f]$, so $[f] \in \operatorname{im}(i_*)$. Hence, $\ker(j_*) \subset \operatorname{im}(i_*)$.

Next, we will show that $\ker(\partial) = \operatorname{im}(j_*)$. For any $[f] \in \pi_n(X, x_0)$, f maps I^n into X so that the boundary goes to x_0. Since j_* is induced by inclusion, $j_*([f]) = [j \circ f] = [f]$, so $\partial \circ j_*([f])$ can be represented by the constant map to x_0. We thus have that $\operatorname{im}(j_*) \subset \ker(\partial)$. Choose any $[f] \in \ker(\partial)$, and $\partial([f]) = 0$ in $\pi_{n-1}(A, x_0)$ means that we have a base point preserving homotopy H from $f|_{I^{n-1}\times\{0\}}$ to the constant map, where the image of H is entirely in A. We can thus define another homotopy G such that $G_0 = f$, $G_t|_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f([I^n])$ unioned with the images of H_s for $0 \le s \le t$. The homotopy G maps S^{n-1} into A at all times, so $[f] = [G_1]$ in $\pi_n(X, A, x_0)$. Moreover, G_1 maps the boundary of I^n to x_0, so $[G_1] \in \pi_n(X, x_0)$. Since j_* is induced by inclusion, this implies that $j_*([G_1]) = [G_1] = [f]$, so $\ker(\partial) \subset \operatorname{im}(j_*)$.

Finally, we need to show that $\ker(i_*) = \operatorname{im}(\partial)$. If $[f] \in \pi_n(X, A, x_0)$, then $i_* \circ \partial([f])$ is the homotopy class in $\pi_{n-1}(X, x_0)$ represented by $f|_{I^{n-1}}$, and this is homotopic relative to J^{n-2} to the constant map to x_0 via f viewed as a homotopy. Hence, $\operatorname{im}(\partial) \subset \ker(i_*)$. Now, take $[f] \in \ker(i_*)$, and since $i_*([f]) = 0$ in $\pi_n(X, x_0)$, we can homotope f to the constant map, through a homotopy H that has image in X and preserves x_0. Since $H_0 = f$ has image in A and H_1 has image $\{x_0\}$ and H_t takes the boundary to $\{x_0\}$ for all $t \in I$, we see that $[H] \in \pi_n(X, A, x_0)$, and $\partial([H]) = [f]$. As such, $[f] \in \operatorname{im}(\partial)$, so $\ker(i_*) \subset \operatorname{im}(\partial)$. □

Equipped with this theorem, we now proceed to explore the homotopy groups of CW complexes.

2. Cellular Approximation

We will start this section by proving an important lemma, a consequence of which is that any map from a lower dimensional cell to a higher dimensional cell can be homotoped to one that is not surjective. Note that non-surjectivity is not automatic for a map from a lower dimensional cell to a higher dimensional cell because of the existence of space filling curves.

Lemma 2.1. *Let $f : I^n \to Z$ be a map, where Z is obtained from a subspace W by attaching a cell e^k. Then f is homotopic relative to $f^{-1}(W)$ to a map g for which there is a union of finitely many simplices $\bigcup_\alpha K_\alpha \subset I^n$ such that*

- *(a) $g(\bigcup_\alpha K_\alpha) \subset e^k$ and $g|_{K_\alpha}$ is linear for all α, and*
- *(b) there exists some nonempty open set $U \subset e^k$ such that $g^{-1}(U) \subset \bigcup_\alpha K_\alpha$.*

Proof. By choosing an identification of e^k to $\mathbb{R}^k$, we have a metric on e^k inherited from the Euclidean metric on $\mathbb{R}^k$. Pick a point $x_0 \in e^k$, and let B_1, B_2 be open balls of radius 1 and 2 centered at x_0, respectively. Then $f^{-1}(\overline{B_1})$ and $f^{-1}(\overline{B_2})$ are closed and are subsets of a compact Hausdorff space I^n, so they are both compact. This means that $f|_{f^{-1}(\overline{B_2})}$ is uniformly continuous, so there is some $\epsilon > 0$ such that for any $x, y \in f^{-1}(\overline{B_2})$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \frac{1}{100}$. Also, $I^n \setminus f^{-1}(B_2)$ is closed and is disjoint from $f^{-1}(\overline{B_1})$, so there is a minimum distance s between them. Choose $\epsilon > 0$ such that $\epsilon < \frac{s}{100}$ and that the uniform continuity condition above holds.

Now, partition I^n into n-cubes with diameter less than ϵ. Let C_1 be the union of the closure of the n-cubes in this partition that have nonempty intersection with $f^{-1}(B_1)$ and let C_2 be the union of the closure of all the cubes that intersect C_1, including cubes in C_1. Observe that $C_2 \subset f^{-1}(B_2)$ because we chose $\epsilon < \frac{s}{100}$. Since we can write cubes as a finite union of simplices[1], C_1 and C_2 are both finite unions of simplices. Let $\bigcup_\alpha K_\alpha$ be C_1, and we will show that C_1 has the properties stated in the theorem.

Define a function $\phi : I^n \to I$ such that ϕ is 0 on $I^n \setminus C_2$, is 1 on C_1, and increases piecewise linearly on $C_2 \setminus C_1$. Define another function $h : I^n \to e^k$ that takes the vertices of the simplices in C_2 to their image under f, but is linear on each of these simplices, and is continuous everywhere else. Linearity here means that the composition of h with the identification of e^k with $\mathbb{R}^k$ is linear. Let $H : I^n \times I \to e^k$ given by $H(x,t) = (1-t\phi(x))f(x)+t\phi(x)g(x)$. It is easy to see that H is continuous, $H_0 = f$ and H_1 is linear on each of the simplices in C_1, so if we choose g to be H_1, condition (a) is satisfied.

For each simplex Δ that form the union $C_2 \setminus C_1$, $f(\Delta)$ is contained in a ball B_Δ of radius $\frac{1}{4}$ since for any $x, y \in \Delta$, we have that $d(f(x), f(y)) < \frac{1}{100}$. The function g is linear on Δ, and g agrees with f on the vertices of Δ, so the image of g is the convex hull of the image of the vertices of Δ under f. This means that $g(\Delta) \subset \overline{B_\Delta}$ because $\overline{B_\Delta}$ is convex and contains the vertices of Δ under f. Moreover, $\overline{B_\Delta}$ is not contained in B_1 because $f(\Delta)$ does not intersect B_1, so $x_0 \notin \overline{B_\Delta}$ because the radius of $\overline{B_\Delta}$ is less than half the radius of B_1. This is true for every choice of Δ, and there are only finitely many simplices, so $U := B_1 \setminus \bigcup_\Delta \overline{B_\Delta}$ is open and nonempty since $x_0 \in U$. Note that this implies $U \cap g(C_2 \setminus C_1)$ is empty because $g(C_2 \setminus C_1) \subset \bigcup_\Delta \overline{B_\Delta}$. Also, for all $x \in I^n \setminus C_2$, $g(x) = f(x) \notin B_1$, so $U \cap g(I^n \setminus C_2)$ is empty. Hence, $g^{-1}(U) \subset C_1$, and condition (b) is also satisfied. □

Corollary 2.2. *If $n < k$ then any map $f : I^n \to I^k$ is homotopic relative ∂I^n to a map that is not surjective.*

Proof. Any n-dimensional cell is homeomorphic with I^n, so in the case where the image of a lower dimensional cell under a map f lies in a higher dimensional cell, we have $n < k$ with W in the previous lemma being the boundary of e^k. Since g is linear on each K_α, g has to be piecewise linear on $g^{-1}(U) \subset \bigcup_\alpha K_\alpha$. Also, U is nonempty, so $g^{-1}(U)$ has to have dimension at least k or $g^{-1}(U) = \emptyset$. However, $g^{-1}(U) \subset \bigcup_\alpha K_\alpha$ and each K_α is of dimension n, so the former is impossible. Thus $g^{-1}(U)$ is empty, which means that there is some point in e^k that is not in the image of e^n. □

We can now apply this fact to prove the next theorem, which is one of the main tools used in analyzing homotopy groups of CW complexes. When looking at $\pi_n(X, x_0)$ for a CW complex X, instead of having to keep track of all possible maps of I^n into X, this theorem tells us that we need only to consider the maps that take I^n into the n-skeleton X^n. But first, we need a new definition.

Definition 2.3. A map of CW complexes $f : X \to Y$ is a *cellular map* if it takes the nth skeleton of X to the nth skeleton of Y, i.e. $f(X^n) \subset Y^n$ for all n.

Theorem 2.4 (Cellular Approximation Theorem)**.** *Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. More specifically, if f is already a cellular map on a subcomplex $A \subset X$, then we can take the homotopy to be stationary on A.*

[1]The process through which this is done is called barycentric subdivision. See Hatcher's Algebraic Topology, pg.103.

Proof. We will prove this by induction on the skeletons of X. Suppose that f has been homotoped such that $f|_{X^n \cup A}$ is cellular. Pick any $(n+1)$-cell e^{n+1} in $Y \setminus A$. Since the closure of e^{n+1} is compact, $f(e^{n+1})$ has nonempty intersection with only finitely many cells in Y. Let e^k be the cell of highest dimension that has nonempty intersection with $f(e^{n+1})$, and we can assume that $k > n+1$ since otherwise e^{n+1} is already mapped into Y^{n+1} by h. By Corollary 2.2, we see that there exists some point $x_0 \in e^k$ that is not in the image of e^{n+1} under h.

Now, $e^k \setminus \{x_0\}$ is a punctured disk, so it deformation retracts onto its boundary via a homotopy H. By composing $f|_{e^{n+1}}$ with this deformation retract, we obtain a homotopy between $f|_{e^{n+1}}$ and a map h whose image is entirely in the boundary of e^k, and in particular, h is a map of e^{n+1} into Y^{k-1}. Performing this repeatedly, we have a homotopy G from $f|_{e^{n+1}}$ to a map whose image is entirely in Y^{n+1}. Since the boundary of e^{n+1} is in X^n and we assumed that f is cellular on X^n, this means that G is stationary on the boundary of e^{n+1}.

Because of the invariance of this homotopy on the boundary of e^{n+1}, we may now use the pasting lemma to create a map $H : (X^{n+1} \cup A) \times I \to Y^{n+1} \cup f(A)$ by pasting the constant homotopy on $(X^n \cup A) \times I$ with the homotopy constructed for each $(n+1)$-cell not in A. Observe that H is a homotopy from $f|_{X^{n+1} \cup A}$ to a map g' whose image is entirely in $Y^{n+1} \cup f(A)$. Since (X, A) is a CW pair, it satisfies the homotopy extension property, which means that we can extend g' to g, a map from X to Y that is cellular on $X^{n+1} \cup A$ and is homotopic to f. This completes the inductive step.

The only place we used the inductive hypothesis is to claim that the homotopy G is stationary on the boundary of e^{n+1}. However, e^0 has no boundary, so this same argument, with some trivial modifications, will also prove the base case.

This lets us make a new homotopy $K : X \times I \to Y$ by partitioning I into intervals $[0, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, \frac{7}{8}], \ldots$ and reparameterizing the homotopies obtained from the inductive process above so that the first homotopy occurs in the first interval, the second homotopy in the second interval, and so on. This defines all of K_t except when $t = 1$, but we can define K_1 so that for every n, K_1 takes a point in X^n to where $K_{1-\frac{1}{2^{n+1}}}$ takes it. Observe that this definition of K is continuous, with $K_0 = f$ and K_1 a cellular map. Also, the way that we defined K ensures that for all $t \in I$, $K(a, t) = K(a, 0)$ for all $a \in A$. □

3. Whitehead's Theorem

As mentioned before, any two spaces that are homotopic have isomorphic homotopy groups. The converse is almost true for CW complexes, as we will prove in this section.

Lemma 3.1 (Compression Lemma)**.** *Let (X, A) be a CW pair and let (Y, B) be any pair (not necessarily CW) with $B \neq \emptyset$. Assume further that for each n such that $X \setminus A$ contains an n-cell, $\pi_n(Y, B, y_0)$ is trivial for any choice of base point. Then every map $f : (X, A) \to (Y, B)$ is homotopic relative A to a map whose image lies entirely in B.*

Proof. We will prove this by induction on the skeleta X^n. Suppose that there is a homotopy that is stationary on A which homotopes f to a map g that takes $A \cup X^n$ into B. If there are no $(n+1)$-cells in $X \setminus A$, then g trivially takes $A \cup X^{n+1}$ into B. Otherwise, $\pi_{n+1}(Y, B, y_0)$ is trivial for any choice of y_0. Choose any $(n+1)$-cell e^{n+1}

in $X \setminus A$. Then $g|_{e^{n+1}}$ can be viewed as a representative of some homotopy class in $\pi_{n+1}(Y, B, y_0)$ because its boundary, being part of X^n, is assumed to be mapped into B. Hence, $[g|_{e^{n+1}}] = 0$ in $\pi_{n+1}(Y, B, y_0)$, so by the compression lemma, there is a homotopy G that is stationary on the boundary of e^{n+1} and takes $g|_{e^{n+1}}$ to a map h whose image is entirely in B.

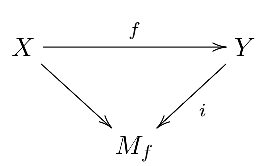
This can be done for every $(n+1)$-cell in $X \setminus A$, so we can create a new homotopy $H : (A \cup X^{n+1}) \times I \to Y$ by pasting the constant homotopy on $A \cup X^n$ to the above homotopy for each $(n+1)$-cell not in A. Observe that H is a homotopy from $g|_{X^{n+1} \cup A}$ to a map h' whose image is entirely in B. Also, $(X, X^{n+1} \cup A)$ is a CW pair, so it satisfies the homotopy extension property, which means we can extend h' to a map $h : X \to Y$. Note that $h|_{X^{n+1} \cup A}$ has image entirely in B, and h is homotopic to g via a homotopy that is stationary on A. Since g is also homtopic to f via a homotopy that preserves A, this proves the inductive step.

For the same reason as that used to justify the base case in the proof of the cellular approximation theorem, the base case is also true here. This inductive process thus gives us a sequence of homotopies that take successive skeletons of X into B. To complete the proof of this theorem, simply use the same construction as in the last paragraph of the proof of the cellular approximation theorem. □

Theorem 3.2 (Whitehead's Theorem). *If a map $f : X \to Y$ between CW complexes induces isomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all n and for every choice of $x_0 \in X$, then f is a homotopy equivalence. In the case where f is the inclusion of X as a subcomplex into Y, X is a deformation retract of Y.*

Proof. First, we will prove the simple case where f is an injection that includes X as a subcomplex into Y. Consider the map $\mathrm{id} : (Y, X) \to (Y, X)$, where X is viewed as a subcomplex of Y via the injection f. By the hypothesis of this theorem, the inclusion map $f : X \to Y$ induces isomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, x_0)$ for all n, so the long exact sequence of homotopy groups tells us that $\pi_n(Y, X, x_0)$ is trivial for all n. Thus, we can use the compression lemma to obtain a homotopy that is stationary on X and homotopes f to a map g whose image is entirely in X. It is easy to see that g deformation retracts Y to X.

To prove the more general case, we need to use M_f, the mapping cylinder[2] of f. We know that Y, as a subspace of M_f, is a deformation retract of M_f, so the inclusion map $i : Y \hookrightarrow M_f$ induces isomorphisms $i_* : \pi_n(Y, x_0) \to \pi_n(M_f, x_0)$. Thus we can replace $\pi_n(Y, x_0)$ with $\pi_n(M_f, x_0)$ in the long exact sequence used in the first paragraph and see that $\pi_n(M_f, X, x_0)$ is trivial for all n. If f is a cellular map then X includes as a subcomplex into M_f, so by what we proved in the first paragraph, X is a deformation retract of M_f, which means that X is homotopy equivalent to M_f. Thus the commuting diagram



tells us that f is a homotopy equivalence from X to Y. In the case where f is not a cellular map, we can use the cellular approximation theorem to obtain a map

[2] This is defined in Hatcher's Algebraic Topology pg.2.

f' that is cellular and homotopic to f. The above proof then tells us that f' is a homotopy equivalence, so f is also a homotopy equivalence. $\square$

4. CW Approximation

By Whitehead's theorem, if we know that a map between two CW complexes induces isomorphisms between their homotopy groups, then we know that these two CW complexes are homotopy equivalent. Thus, to study the topological properties of a CW complex X that are invariant under homotopy equivalence, it is sufficient to study any CW complex Y with a map $f : Y \to X$ such that f_* is a homotopy equivalence. Moreover, if we can choose Y to be simple enough, studying X can be made much easier. This motivates the following definition and theorems.

Definition 4.1. Let A be a CW complex and X a space containing A. An *n-connected CW model for* (X, A) is an n-connected CW pair[3] (Z, A) and a map $f : Z \to X$ such that $f|_A$ is the identity map and for any $x_\gamma \in A$, $f_* : \pi_i(Z, x_\gamma) \to \pi_i(X, x_\gamma)$ is an isomorphism for all $i > n$ and an injection when $i = n$.

Proposition 4.2. *Let A be a CW complex and X a space containing A. If $A \neq \emptyset$, then for all $n \geq 0$, there exists an n-connected CW model (Z, A) of (X, A) with the accompanying map $g : Z \to X$. This model can be chosen with the additional property that Z is obtained from A by attaching only cells of dimension greater than n.*

Proof. First, we shall inductively construct a chain of CW complexes $A = Z_n \subset Z_{n+1} \subset Z_{n+2} \subset \cdots$, where Z_{k+1} is obtained from Z_k by attaching only $(k+1)$-cells. Suppose that we already have a map $f : Z_k \to X$ such that $f|_A$ is the identity on A, $f_* : \pi_i(Z_k, x_\gamma) \to \pi_i(X, x_\gamma)$ is an injection when $n \leq i < k$ and is a surjection when $n < i \leq k$ for all $x_\gamma \in A$. Observe that this is trivially true by cellular approximation when $k = n$, so we have the base case.

To prove the inductive step, choose cellular maps $\phi_{\alpha,\gamma} : S^k \to Z_k$ to represent generators of the kernel of $f_* : \pi_k(Z_k, x_\gamma) \to \pi_k(X, x_\gamma)$. Do this for all $x_\gamma \in A$, and create a new CW complex Y_{k+1} by attaching $(k+1)$-cells to Z_k, one for each pair (α, γ), with boundary map $\phi_{\alpha,\gamma}$. Since $f \circ \phi_{\alpha,\gamma}$ represents zero in $\pi_k(X, x_\gamma)$, it is nullhomotopic. Moreover, (Y_{k+1}, Z^k) is a CW pair, we can extend f to $f' : Y_{k+1} \to X$. The map $f'_* : \pi_i(Y_{k+1}, x_\gamma) \to \pi_i(X, x_\gamma)$ is still injective when $n \leq i < k$ because we only added cells of dimension larger than k, and this does not affect the homotopy groups of dimension less than k. It is also still surjective when $n < i \leq k$ because the diagram

$$\begin{array}{ccc} \pi_i(Z_k, x_\gamma) & \xrightarrow{f_*} & \pi_i(X, x_\gamma) \\ \downarrow j_* & \nearrow f'_* & \\ \pi_i(Y_{k+1}, x_\gamma) & & \end{array}$$

commutes, where j_* is induced by the inclusion map. Moreover, the kernel of $f'_* : \pi_k(Y_{k+1}, x_\gamma) \to \pi_k(X, x_\gamma)$ is trivial because the $(k+1)$-cells we attached act as base point preserving homotopies between the representatives of homotopy classes in the kernel of f_*. Hence, $f'_* : \pi_k(Y_{k+1}, x_\gamma) \to \pi_k(X, x_\gamma)$ is injective for all $x_\gamma \in A$.

[3] This is defined in Hatcher's Algebraic Topology pg.7

Now, choose maps $\psi_{\beta,\gamma} : S^{k+1} \to X$ to represent generators of $\pi_{k+1}(X, x_\gamma)$, and do this for each $x_\gamma \in A$. For each pair (β, γ), attach a sphere $S^{k+1}_{\beta,\gamma}$ to Y_{k+1} at x_γ, and call the resulting CW complex Z_{k+1}. Extend f' to a map $f'' : Z_{k+1} \to X$ by mapping each $S^{k+1}_{\beta,\gamma}$ to X via $\psi_{\beta,\gamma}$. The resulting map is continuous by the pasting lemma, since $\psi_{\beta,\gamma}$ and f agree on x_γ. The map $f''_* : \pi_{k+1}(Z_{k+1}, x_\gamma) \to \pi_{k+1}(X, x_\gamma)$ is surjective by construction, and for all $n < i \leq k$, $f''_* : \pi_i(Z_{k+1}, x_\gamma) \to \pi_i(X, x_\gamma)$ is surjective because the diagram

$$\begin{array}{ccc} \pi_i(Y_{k+1}, x_\gamma) & \xrightarrow{f'_*} & \pi_i(X, x_\gamma) \\ {\scriptstyle j_*}\downarrow & \nearrow {\scriptstyle f''_*} & \\ \pi_i(Z_{k+1}, x_\gamma) & & \end{array}$$

commutes. Also, for all $n \leq i \leq k$, it is also injective because the diagram

$$\begin{array}{ccc} \pi_i(Y_{k+1}, x_\gamma) & \overset{f'_*}{\hookrightarrow} & \pi_i(X, x_\gamma) \\ {\scriptstyle j_*}\downarrow & \nearrow {\scriptstyle f''_*} & \\ \pi_i(Z_{k+1}, x_\gamma) & & \end{array}$$

commutes. The map j_* here is surjective by the cellular approximation theorem.

Through this inductive process, we have thus constructed a nested sequence of CW complexes, $\{Z_i\}_i$ and a sequence of maps $\{f_i : Z_i \to X\}_i$. Let $Z := \bigcup_{i \geq n} Z_i$ and define $g : Z \to X$ so that $g|_{Z_i} = f_i$. We know that Z is constructed from A by using only cells of dimension larger than n and (Z, A) is n-connected because if we look at the long exact sequence for (Z, A), the map $i_* : \pi_i(A, x_\gamma) \to \pi_i(Z, x_\gamma)$ is always a surjection by cellular approximation, and is an injection for $i < n$ because we only added cells of dimension larger than n. Moreover, since adding higher dimensional cells do not affect lower homotopy groups, and for each $i > n$, $(f_i)_* : \pi_i(Z_k, x_\gamma) \to \pi_i(X, x_\gamma)$ is an isomorphism for large enough k, we know that $g_* : \pi_i(Z, x_\gamma) \to \pi_i(X, x_\gamma)$ is an isomorphism for $i > n$ and an injection when $i = n$. Furthermore, it is clear from construction that $g|_A$ is the identity map on A, so $g : Z \to X$ is the n-connected CW model we are looking for. $\square$

Corollary 4.3 (CW Approximation Theorem)**.** *If (X, A) is an n-connected CW pair, then there exists a CW pair (Z, A) such that $Z \setminus A$ only contain cells of dimension greater than n and there exists a homotopy equivalence $g : Z \to X$ such that $g|_A = \mathrm{id}$.*

Proof. Let (Z, A) be the n-connected CW model of (X, A) constructed in the previous theorem, and let $g : Z \to X$ be the accompanying map. First, we need to show that g_* is an isomorphism for all i. By the definition of a CW model, for any $a_0 \in A$, we have that $g_* : \pi_i(Z, a_0) \to \pi_i(X, a_0)$ is an isomorphism for $i > n$ and an injection when $i = n$, so we need to show that g_* is an isomorphism when $i < n$

and a surjection when $i = n$. The diagram

$$\begin{array}{ccc} \pi_i(Z, a_0) & \xrightarrow{g_*} & \pi_i(X, a_0) \\ {\scriptstyle j_*}\uparrow & & \uparrow{\scriptstyle j'_*} \\ \pi_i(A, a_0) & \xleftarrow{\mathrm{id}_*} & \pi_i(A, a_0) \end{array}$$

commutes for all i, where j_* and j'_* here are induced by the natural inclusion maps. When $i \leq n$, $\pi_i(X, A, a_0)$ and $\pi_i(Z, A, a_0)$ are trivial, so by their long exact sequences, we see that j_* and j'_* are isomorphisms when $i < n$ and are surjections when $i = n$. The map id_* is also always an isomorphism, so by the above commuting diagram, g_* is an isomorphism when $i < n$ and is a surjection when $i = n$.

This allows us to apply Whitehead's theorem, so $g : X \to Z$ is a homotopy equivalence. To finish the proof, we need only to show that $g|_A$ is the identity map on A. By the cellular approximation theorem, we can assume that g is a cellular map. Let W be M_g quotiented out by the relation $\sim$ defined such that $(x_1, t_1) \sim (x_2, t_2)$ iff $x_1 = x_2 \in A$. Note that since $g|_A = \mathrm{id}_A$ the usual deformation retraction $D : M_g \times I \to M_g$ from M_g to Z is such that for all $a \in A$, $D(a, t, s) = (a, t')$, so this induces a deformation retraction D' from W to Z. For the same reasons, we have a deformation retraction from W to X. These two deformation retractions are stationary on A, so $X \simeq Z$ relative A. □

5. Freudenthal Suspension Theorem

To prove the Freudenthal suspension theorem, we need to know an elementary but very useful algebraic fact known as the *five-lemma*.

Lemma 5.1 (Five-Lemma)**.** *In a commutative diagram of abelian groups*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' & \xrightarrow[k']{} & D' & \xrightarrow[l']{} & E' \end{array}$$

if the rows are exact, then

(a) γ is surjective if β and δ are surjective and ϵ is injective, and
(b) γ is injective if β and δ are injective and α is surjective.

In particular, if $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism.

Proof. First, we will prove (a). Choose any $c' \in C'$. Since δ is surjective, there is some $d \in D$ such that $k'(c') = \delta(d)$, and by the commutativity of the rightmost square, we know $\epsilon(l(d)) = l'(\delta(d)) = l'(k'(c')) = 0$, with the last equality coming from the exactness of the bottom row. Since we assumed that ϵ is injective, this means $l(d) = 0$, so $d \in \ker(l) = \mathrm{im}(k)$, which implies that there is some $c \in C$ such that $k(c) = d$. Now,

$$\begin{aligned} k'(\gamma(c) - c') &= k'(\gamma(c)) - k'(c') \\ &= \delta(k(c)) - \delta(d) \\ &= \delta(d) - \delta(d) \\ &= 0, \end{aligned}$$

so $\gamma(c) - c' = j'(b')$ for some $b' \in B'$ by the exactness of the bottom row. The surjectivity of β tells us that there is some $b \in B$ such that $\beta(b) = b'$. Therefore,

$$\begin{aligned}\gamma(c - j(b)) &= \gamma(c) - \gamma(j(b)) \\ &= \gamma(c) - j'(\beta(b)) \\ &= \gamma(c) - j'(b') \\ &= \gamma(c) - \gamma(c) + c' \\ &= c'.\end{aligned}$$

This shows that γ is surjective.

Next, we will prove (b). Choose any $c \in C$ such that $\gamma(c) = 0$. The commutativity of the second to right square tells us that $\delta(k(c)) = k'(\gamma(c)) = k'(0) = 0$, and since we assumed that δ is injective, this means $k(c) = 0$. Hence, by the exactness of the top row, we know that $c = j(b)$ for some $b \in B$. Now, the second to left square gives us that $j'(\beta(b)) = \gamma(j(b)) = \gamma(c) = 0$, so the exactness of the bottom row implies that there exists some $a' \in A'$ such that $i'(a') = \beta(b)$. The mapα is surjective, so there is some $a \in A$ with $\alpha(a) = a'$, and this lets us write

$$\begin{aligned}\beta(i(a) - b) &= \beta(i(a)) - \beta(b) \\ &= i'(\alpha(a)) - \beta(b) \\ &= i'(a') - \beta(b) \\ &= \beta(b) - \beta(b) \\ &= 0.\end{aligned}$$

Since β is assumed to be injective, $i(a) = b$, so $c = j(b) = j(i(a)) = 0$ by the exactness of the top row. This shows that γ is injective. □

Now we are ready to tackle the Freudenthal suspension theorem. The bulk of the work in proving the theorem will be done in the next proposition, of which the theorem will be a simple corollary. This proposition is the equivalent of the excision theorem for homotopy groups, but it works only with CW complexes that have rather strong connectedness properties.

Proposition 5.2 (Homotopy Excision Theorem)**.** *Let X be a CW complex decomposed as the union of subcomplexes A and B such that $A \cap B =: C$ is connected and nonempty. If (A, C) is m-connected, (B, C) is n-connected and $j : (A, C) \to (X, B)$ is the inclusion map, then for any $x_0 \in C$, $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is an isomorphism when $i < m + n$ and is a surjection when $i = m + n$.*

We will prove this proposition in several lemmas.

Lemma 5.3. *Suppose that A is built from C by attaching $(m+1)$-cells, e_α^{m+1}, and B is built from C by attaching a single $(n+1)$-cell, e^{n+1}. Then $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is an isomorphism when $i < m+n$ and is a surjection when $i = m+n$.*

Proof. First, we need to show that $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is a surjection when $i \leq n$ and an injection when $i < n$. Suppose $i \leq n$ and choose any map $f : (D^i, S^{i-1}) \to (X, B)$ such that $x_0 \in f(S^{i-1})$. Since $f|_{S^{i-1}}$ is entirely in B, by the cellular approximation theorem, we can homotope it to a cellular map via a homotopy H, and the image of this cellular map has to be entirely in C because B is constructed from C by adding e^{n+1}. By the homotopy extension property, H extends to some $H' : (D^i \times I, S^{i-1} \times I) \to (X, B)$. This allows us to assume that $f|_{S^{i-1}}$ is cellular, and by the cellular approximation theorem again, there is a homotopy that is stationary on S^{i-1} taking f to a cellular map f'. Note that f' has image in A and $f'|_{S^{i-1}}$ has image in C, so f' represents a homotopy class in $\pi_i(A, C, s_0)$ as well, with $j_*([f']) = [f'] = [f]$. Hence, j_* is surjective when $i \leq n$.

To prove injectivity, suppose $i < n$ and choose any $[f_0], [f_1] \in \pi_i(A, C, x_0)$ such that $j_*([f_0]) = j_*([f_1])$. This means that there is a homotopy H with $H_0 = f_0$, $H_1 = f_1$ and such that the image of H is in X and the image of $H_t|_{f^{-1}(C)}$ is in C for all $t \in I$. Since $i < n$, H can be viewed as a map of (D^{i+1}, S^i) into (X, A), with $i + 1 \leq n$, so by the cellular approximation theorem, we can assume that H takes (D^{i+1}, S^i) into (A, C) because both $X \setminus A$ and $B \setminus C$ are single $(n+1)$-cells. Thus, $[f_0] = [f_1]$, and this shows that $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is injective when $i < n$.

To finish the proof of this simple case, we now need to show that $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is an injection when $n \leq i < m+n$ and a surjection when $n < i \leq m+n$. For surjectivity, we choose any representative f of a homotopy class in $\pi_i(X, B, x_0)$, and show that there is some $[g'] \in \pi_i(A, C, x_0)$ such that $j_*([g']) = [f]$. The map f has compact image, so it intersects with only finitely many of e_α^{m+1} and e^{n+1}. By Lemma 2.1, f is homotopic relative $f^{-1}(A)$ to a map g such that there are simplices $\Delta_\alpha^{m+1} \subset e_\alpha^{m+1}$, $\Delta^{n+1} \subset e^{n+1}$ where $g^{-1}(\Delta_\alpha^{m+1})$ and $g^{-1}(\Delta^{n+1})$ are finite unions of complex polyhedra on which g is piecewise linear. A consequence of this is that $[f] = [g]$ in $\pi_i(X, B, x_0)$. If g does not surject onto Δ^{n+1} then there is some $y_0 \in e^{n+1}$ not in the image of g. This means that we have a deformation retraction of $X \setminus \{y_0\}$ onto A, which when composed with g gives a homotopy relative $g^{-1}(X \setminus \overline{e^{n+1}})$ from g to a map $g' : (D^i, S^{i-1}) \to (A, C)$, so $[g'] \in \pi_i(A, C, x_0)$ and $j_*([g']) = [g'] = [f]$. Hence, we need only consider the maps f where the corresponding map g surjects onto e^{n+1}.

Also, if g does not surject onto Δ_α^{m+1}, then by a similar argument as in the previous three lines, g is homotopic relative $g^{-1}(X \setminus \overline{e_\alpha^{m+1}})$ to a map with image entirely in $X \setminus e_\alpha^{m+1}$. This allows us to assume that if g does not surject onto Δ_α^{m+1}, then the image of g does not intersect e_α^{m+1}. Thus, for the rest of this case, we can assume that g surjects onto Δ^{n+1} and Δ_α^{m+1} for all α.

We claim that if $n < i \leq m + n$, then there exists points $p_\alpha \in \Delta_\alpha^{m+1}$ and $q \in \Delta^{n+1}$, and a map $\phi : I^{i-1} \to [0, 1)$ such that

(a) $g^{-1}(q)$ lies below the graph of ϕ in $I^{i-1} \times I = I^i$,
(b) $g^{-1}(p_\alpha)$ lies above the graph of ϕ for each α with e_α^{m+1} having nonempty intersection with the image of g, and
(c) $\phi = 0$ on ∂I^{i-1}.

Before we prove this claim, we shall see how it helps us prove surjectivity of j_* when $n < i \leq m + n$. Let $H : I^{i-1} \times I \times I \to I$ be a homotopy given by $H(x, y, t) = g(x, y + (1 - y)t\phi(x))$. Pictorially, $H_0 = g$ and H_1 is the restriction of g to the region above ϕ in the diagram below.

Since $g^{-1}(q)$ lies below the graph of ϕ, it is not in the domain of H_1, so q is not in the image of H_1. Also, since $g^{-1}(p_\alpha)$ lies above the graph of ϕ for the finitely many α's that we are considering, all of these p_α's are in the image of H_t for all t. Moreover, $g|_{S^{i-1}}$ has image in C and the p_α's are all in $A \setminus C$, so the p_α's are not in the image of $g|_{S^{i-1}}$. The way we defined H then ensures that the p_α's are not in $H_t|_{S^{i-1}}$ for all $t \in [0, 1]$.

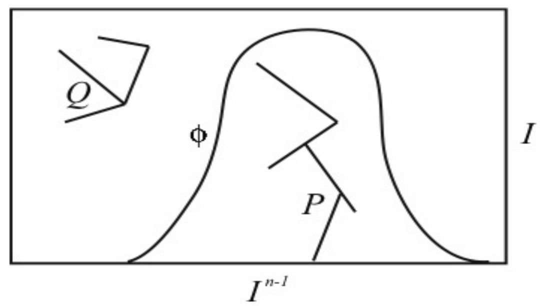


FIGURE 5. Excision

Let $P = \bigcup_\alpha g^{-1}(p_\alpha)$, let $Q = g^{-1}(q)$ and observe that the diagram

$$\begin{array}{ccc} \pi_i(A, C, x_0) & \xrightarrow{j_*} & \pi_i(X, B, x_0) \\ \cong \downarrow i'_* & & \cong \downarrow i_* \\ \pi_i(X \setminus Q, (X \setminus Q) \setminus P, x_0) & \xrightarrow{i''_*} & \pi_i(X, X \setminus P, x_0) \end{array}$$

commutes, where i'_*, i_* and i''_* are induced by the obvious inclusion maps. The map i_* is an isomorphism because B is a deformation retract of $X \setminus P$ and i'_* is an isomorphism because A and C are deformation retracts of $X \setminus Q$ and $(X \setminus Q) \setminus P$ respectively. Now, $[g] \in \pi_i(X, B)$, when viewed as an element in $\pi_i(X, X \setminus P, x_0)$ is equal to $[H_1]$ as argued in the preceding paragraph. Also, the image of H_1 does not intersect Q and the image of $H_1|_{\partial I^i}$ does not intersect P or Q, so H_1 also represents a homotopy class in $\pi_i(X\setminus Q, (X\setminus Q)\setminus P, x_0)$, with $i''_*([H_1]) = [H_1]$. The isomorphism i'_* then implies that there is some $[g'] \in \pi_i(A, C, x_0)$ such that $i'_*([g']) = [H_1]$, and the commutativity of the above diagram implies that $j_*([g']) = [g] = [f]$, so j_* is a surjection when $n < i \leq m + n$.

Now, we will prove the claim. Choose any $q \in \Delta^{n+1}$. Since g is a linear map from an i dimensional object onto Δ^{n+1}, an $(n+1)$-dimensional object, $g^{-1}(q)$ has dimension at most $i - (n + 1)$. Let $T = \pi^{-1}(\pi(g^{-1}(q)))$, where $\pi : I^i \to I^{i-1}$ is the projection map. Observe that T has dimension at most $i - n$. Hence, for all α, $f(T) \cap \Delta^{m+1}_\alpha$ is also of dimension at most $i - n$ since $T \cap g^{-1}(\Delta^{m+1}_\alpha)$ is also of dimension at most $i - n$, and $g|_{T\cap g^{-1}(\Delta^{m+1}_\alpha)}$ is linear. Thus if $i - n < m + 1$, there is some $p_\alpha \in \Delta^{m+1}_\alpha$ that is not in $g(T)$, which implies that $g^{-1}(p_\alpha) \cap T$ is empty. Choose all the p_α's this way. Since T and $g^{-1}(p_\alpha)$ for all α are closed subspaces of I^i, a compact Hausdorff space, they all have to be compact. Thus we can choose an ϵ-neighborhood around T, call it T', that has empty intersection with $f^{-1}(p_\alpha)$ for all α.

Since J^{i-1} is sent to the base point x_0 by g and j being the inclusion map preserves the basepoint, we know that $g|_{J^{i-1}}$ has image in C, and in particular, $g|_{I^{i-1}\times\{1\}}$ has image in C. Thus $g^{-1}(q)\cap(I^{i-1}\times\{1\})$ is empty. Moreover, $I^{i-1}\times\{1\}$ and $g^{-1}(q)$ are closed subsets of a compact Hausdorff space, and therefore are compact, so there is an ϵ-neighbourhood around $I^{i-1} \times \{1\}$ that has non-empty intersection with $g^{-1}(q)$. Define $\phi : I^{i-1} \to [0, 1)$ by letting ϕ be zero outside of T', $1 - \epsilon$ on T and linear on $T' \setminus T$. It is easy to that ϕ has the required properties.

We have thus proven that j_* is surjective in this simple case, so now we need to show that it is also injective when $n \leq i < m+n$. Suppose $j_*([f_0]) = j_*([f_1])$ for some $[f_0], [f_1] \in \pi_i(A, C, x_0)$. This means that there is a homotopy $H : (I^i, \partial I^i, J_{I^i}) \times I \to (X, B, x_0)$ such that $H_0 = f_0$ and $H_1 = f_1$. Replacing f in the injectivity proof with H, we can repeat the same arguments, obtaining a map H' that is homotopic to H relative to $H^{-1}(A)$ such that there are simplices $\Delta_\alpha^{m+1} \subset e_\alpha^{m+1}$, $\Delta^{n+1} \subset e^{n+1}$ onto which H' surjects, and where $H'^{-1}(\Delta_\alpha^{m+1})$ and $H'^{-1}(\Delta^{n+1})$ are finite unions of complex polyhedra on which H is piecewise linear.

In the same way as before, we can create a homotopy G such that the image of G_1 does not intersect $q \in e^{n+1}$ and $G_0 = H'$. Also, as before, G_t is stationary on ∂I^{i+1} for all $t \in I$. This means that G_1 is also a homotopy between $H_0 = f_0$ and $H_1 = f_1$, and the image of G_1 can be assumed to be entirely in A via a deformation retraction of $X \setminus \{q\}$ onto A, so $[f_0] = [f_1]$. In the surjectivity proof, the domain of f is the i-cube, while the domain of H is the $(i+1)$-cube, so since surjectivity holds when $n < i \leq m+n$, injectivity holds when $n < i+1 \leq m+n$, i.e. $n \leq i < m+n$. □

Lemma 5.4. *Suppose that A is still built from C by attaching $(m+1)$-cells, e_α^{m+1}, but B is built from C by adding cells of dimension at least $n+1$. Then $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is an isomorphism when $i < m+n$ and is a surjection when $i = m+n$.*

Proof. For any $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$ the image of f is compact, so it intersects only finitely many of the cells in $B \setminus C$. Since these cells all have dimension at least $n+1$, if $i \leq m+n$, we can repeat the proof for surjectivity in lemma 5.3 finitely many times, one for each cell in $B \setminus A$, to show that f is homotopic relative $f^{-1}(A)$ to a map $f' : (I^i, \partial I^i, J^{i-1}) \to (A, C, x_0)$. Thus, f' represents a homotopy class in $\pi_i(A, C, x_0)$ and $[f] = [f'] = j_*([f'])$, which shows that j_* is surjective. Similarly, if $j_*([f_0]) = j_*([f_1])$ for some $[f_0], [f_1] \in \pi_i(A, C, x_0)$, then the image of the homotopy $H : (I^i, \partial I^i, J_{I^i}) \times I \to (X, B, x_0)$ such that $H_0 = f_0$ and $H_1 = f_1$ is compact, so it too intersects only finitely many cells in $B \setminus C$. When $i < m+n$, we can repeat the proof for injectivity in lemma 5.3 finitely many times, to get a homotopy that is stationary on ∂I^i from H to $H' : (I^i, \partial I^i, J_{I^i}) \times I \to (A, C, x_0)$. This shows that $[f_0] = [f_1]$ in $\pi_i(A, C, x_0)$, because H' is a homotopy from f_0 to f_1, which proves the injectivity of j_*. □

Lemma 5.5. *Suppose that A is constructed from C by attaching cells of dimension at least $(m+1)$ instead of only $(m+1)$-cells and B is constructed from C in the same way as in the previous lemma.Then $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ is an isomorphism when $i < m+n$ and is a surjection when $i = m+n$.*

Proof. Before we start, note that we may assume the cells in $A \setminus C$ have dimension at most $m+n+1$ because adding cells of dimension greater than $m+n+1$ will have no effect on π_i for $i \leq m+n+1$.

Let $A_k \subset A$ be the union of C with the all cells in A that are at most k-dimensional and let $X_k = A_k \cup B$. We will prove by induction on k that $(j|_{A_k})_* : \pi_i(A_k, C) \to \pi_i(X_k, B)$ is an injection when $i < m+n$ and a surjection when $i \leq m+n$. Since we can assume that $A \setminus C$ contains only cells of dimension at most $m+n+1$, this inductive prove is sufficient to prove this case.

By our assumptions in this case, $k \geq m+1$ so our base case of $k = m+1$ is exactly lemma 5.4. We thus need only to prove the inductive step, and this will be done by applying the five-lemma. Consider the diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(A_k, A_{k-1}) & \longrightarrow & \pi_i(A_{k-1}, C) & \longrightarrow & \pi_i(A_k, C) & \longrightarrow & \pi_i(A_k, A_{k-1}) & \longrightarrow & \pi_{i-1}(A_{k-1}, C) \\
\Big\downarrow {\scriptstyle (i_1)_*} & & \Big\downarrow {\scriptstyle (j|_{A_{k-1}})_*} & & \Big\downarrow {\scriptstyle (j|_{A_k})_*} & & \Big\downarrow {\scriptstyle (i_4)_*} & & \Big\downarrow {\scriptstyle (j|_{A_{k-1}})_*} \\
\pi_{i+1}(X_k, X_{k-1}) & \longrightarrow & \pi_i(X_{k-1}, B) & \longrightarrow & \pi_i(X_k, B) & \longrightarrow & \pi_i(X_k, X_{k-1}) & \longrightarrow & \pi_{i-1}(X_{k-1}, B)
\end{array}$$

where $(i_1)_*$ and $(i_4)_*$ are induced by the obvious inclusion maps, and the rows of this diagram come from the exact sequence of the homotopy groups of CW subcomplexes $C \subset A_k \subset A_{k+1}$ and $B \subset X_k \subset X_{k+1}$ for all k. In this diagram, I have also excluded the basepoints because it is clear what they are. That these sequences are exact can be proven using the proof we gave in theorem 1.3, but with slight modifications. It is also clear that this is a commutative diagram.

Now, if we take C to be A_{k-1}, A to be A_k and B to be X_{k-1}, by lemma 5.4, we see that $(i_1)_*$ and $(i_2)_*$ are isomorphisms when $i+1 < k+n$. Since $k \geq m+1$, this means that the first and fourth maps counting from the left are isomorphisms when $i+1 < m+1+n$, or when $i < m+n$. Also, by the inductive hypothesis, the second and fifth maps from the left are also isomorphisms when $i < m+n$, so by the five-lemma, the middle map is an isomorphism when $i < m+n$. This proves that $(j|_{A_k})_* : \pi_i(A_k, C, x_0) \to \pi_i(X_k, B, x_0)$ is an injection when $i < m+n$, and to prove that it is a surjection when $i \leq m = n$, we need now only to show it for $i = m+n$. When $i = m+n$, the second map from the left is a surjection by the inductive hypothesis. Also, $i = m+n$ implies $i-1 < m+n$, so the fifth map from the left is an isomorphism by the inductive hypothesis. Moreover, taking C to be A_{k-1}, A to be A_k and B to be X_{k-1}, lemma 5.4 tells us that the fourth map from the left is a surjection when $i = m+n$, so the five lemma again gives us that the middle map is a surjection when $i = m+n$. This proves the inductive step.

Note that when $i = 2$, we cannot apply the five-lemma as stated, because the six groups in the middle of the commutative above might not be abelian, and $\pi_{i-1}(A_{k-1}, C, x_0)$ and $\pi_{i-1}(X_{k-1}, B, x_0)$ are not groups. However, if we go through the proof of the five-lemma, it is easy to see that it holds in this more general case with trivial modifications. When $i = 0, 1$, the five-lemma does not work as well, but we can directly prove that $j_* : \pi_1(A, C, x_0) \to \pi_1(X, B, x_0)$ and $j_* : \pi_0(A, C, x_0) \to \pi_0(X, B, x_0)$ satisfy the conditions of the proposition. For $j_* : \pi_1(A, C, x_0) \to \pi_1(X, B, x_0)$, if $n \geq 1$, then we can apply the argument used to deal with the situation where $i \leq n$ in lemma 5.3 to obtain the desired result. On the other hand, if $n = 0$, it is trivial that $j_* : \pi_1(A, C, x_0) \to \pi_1(X, B, x_0)$ is an isomorphism. The map $j_* : \pi_0(A, C, x_0) \to \pi_0(X, B, x_0)$ is also clearly an isomorphism because (A, C) and (X, B) are at least 0-connected. □

Proof of proposition 5.2. Now, all we need to do is to reduce the proposition to what we proved in lemma 5.5. Since we are assuming that (A, C) is m-connected and (B, C) is n-connected, by corollary 4.3, we have homotopy equivalences of CW pairs $g_A : (Z_A, C) \to (A, C)$ relative C and $g_B : (Z_B, C) \to (B, C)$ relative C, where $Z_A \setminus C$ contains only cells of dimension at least $m+1$ and $Z_B \setminus C$ contains only cells of dimension at least $n+1$. Since both of these homotopy equivalences are relative $C = A \cap B$, by the pasting lemma we can create a homotopy equivalence $f : (Z_A \cup Z_B, Z_B) \to (A \cup B, B)$, so $f_* : \pi_i(Z_A \cup Z_B, Z_B) \to \pi_i(A \cup B, B)$ is an isomorphism for all i. Lemma 5.5 tells us that $j'_* : \pi_i(Z_A, C) \to \pi_i(Z_A \cup Z_B, Z_B)$, induced by the natural inclusion map, is an injection when $i < m+n$ and a surjection when $i \leq m+n$, so the same has to be true for $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$ by the following commuting diagram:

$$\begin{array}{ccc} \pi_i(Z_A, C, x_0) & \xrightarrow[(g_A)_*]{\cong} & \pi_i(A, C, x_0) \\ \downarrow{\scriptstyle j'_*} & & \downarrow{\scriptstyle j_*} \\ \pi_i(Z_A \cup Z_B, Z_B, x_0) & \xrightarrow[f_*]{\cong} & \pi_i(A \cup B, B, x_0) \end{array}$$

□

Corollary 5.6 (Freudenthal Suspension Theorem)**.** *Let X be an $(n-1)$-connected CW complex. Then there is a map $j : \pi_i(X, x_0) \to \pi_{i+1}(SX, x_0)$ that is an isomorphism when $i < 2n-1$ and is a surjection when $i = 2n-1$.*

Proof. First, observe for any X, $CX := (X \times I)/(X \times \{0\})$ is contractible by the homotopy $H : CX \times I \to CX$ given by $H([x, s], t) = [x, (1-t)s]$, so $\pi_i(CX)$ is trivial for all i.

We can think of SX as two copies of CX, which we call C_+X and C_-X, identified along their bases. Define j to be the composition of the following three maps drawn in the diagram below, each of which is induced by the obvious inclusion maps.

$$\pi_i(X, x_0) \xrightarrow{\cong} \pi_{i+1}(C_+X, X, x_0) \longrightarrow \pi_{i+1}(SX, C_-X, x_0) \xrightarrow{\cong} \pi_{i+1}(SX, x_0)$$

For any i, the left most map and right most maps are isomorphisms because of the long exact sequences of the CW pairs (C_+X, X) and (SX, C_-X), respectively, since $\pi_i(C_\pm X)$ is always trivial. Also, when X is $(n-1)$-connected, the CW pair $(C_\pm X, X)$ is n-connected by the long exact sequence of the pair $(C_\pm X, X)$. This allows us to apply the previous proposition, so the middle map in the diagram above is an isomorphism when $i+1 < n+n$ and a surjection when $i+1 = n+n$. Hence, j is an isomorphism when $i < 2n-1$ and a surjection when $i = 2n-1$. □

The map j here is also known as the *suspension map*.

6. Stable Homotopy Groups

One of the most important implications of the Freudenthal Suspension Theorem is that it proves the existence of stable homotopy groups for all CW complexes.

Suppose we have an n-connected CW complex X. Choose any $i \in \mathbb{N}$, and consider the chain of suspension maps

$$\pi_i(X, x_0) \longrightarrow \pi_{i+1}(SX, x_0) \longrightarrow \pi_{i+2}(S^2X, x_0) \longrightarrow \cdots$$

Since X is an n-connected CW complex, the Freudenthal suspension theorem tells us that the map $\pi_i(X, x_0) \to \pi_{i+1}(SX, x_0)$ is an isomorphism when $i < 2n+1$. In particular, the suspension map is an isomorphism when $i \leq n$, so $\pi_i(SX, x_0)$ is trivial when $0 < i \leq n+1$. Moreover, X is 0-connected means that SX also has to be 0-connected, so X being n-connected implies that SX is $(n+1)$-connected. Repeating this argument, we have that S^kX is $(n+k)$-connected for any k. Now, let $N(n, i) = i-1-2n$, and observe that when $k > N(n, i)$ we have $i+k < 2(n+k)+1$, so $\pi_{i+k}(S^kX)$ are isomorphic for all $k > N(n, i)$. Let $N = N(n, i)$, and we call $\pi_{i+N}(S^NX, x_0)$ the *ith stable homotopy group* of the space X. Observe that since SX is always connected, we do not actually need X to be connected, so every space X has ith stable homotopy groups for all i.

More formally, we have the following definition.

Definition 6.1. The *ith stable homotopy group* of a space X,

$$\pi_i^s(X) := \varinjlim_q \pi_{q+i}(S^q X, x_0).$$

The Freudenthal suspension theorem proves that for any space X, this colimit is realized after finitely many elements along the sequence $\{\pi_{q+i}(S^q X, x_0)\}_i$.

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References

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