ISOMETRIES OF THE HYPERBOLIC PLANE

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Abstract. In this paper, I will explore basic properties of the group $PSL(2,\mathbb{R})$. These include the relationship between isometries of $\mathbb{H}^2$, Möbius transformations, and matrix multiplication. In addition, this paper will explain a method of characterizing the aforementioned transformations by the trace of their matrices through looking at the number of fixed points of a transformation.

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1. Introduction

A main focus of this paper will be the projective special linear group $PSL(2,\mathbb{R})$. The special linear group $SL(2,\mathbb{R})$, the group of 2×2 matrices with determinant 1 under multiplication, is associated with the set of transformations of the complex upper half-plane $z \longmapsto \frac{az+b}{cz+d}$, called Möbius transformations, where all the variables except z are on the real line and $ad - bc = 1$. The group $PSL(2,\mathbb{R})$ is $SL(2,\mathbb{R})$ quotiented out by the subgroup $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$, so that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ are considered equivalent. Not only do these types of transformations preserve hyperbolic lengths in the upper half-plane but also all orientation preserving isometries in the upper-half plane take this form. Furthermore, these transformations can be represented by matrices, and the traces of these matrices can be used to characterize them. As will be shown in the paper, the absolute value of the traces will either be less than, equal to, or greater than 2, and the corresponding transformations will be denoted as elliptic, parabolic, and hyperbolic, respectively.

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2. Background

We will start out by giving by some basic definitions and properties relating to hyperbolic geometry.

Definition 2.1. A *Möbius transformation* is an invertible map on $\mathbb{C}$ of the form $z \longmapsto \frac{az+b}{cz+d}$. Although the coefficients a, b, c, d can generally be complex numbers, here we will only be concerned with real coefficients such that $ad - bc = 1$.

Definition 2.2. The *upper half-plane* is the set of complex numbers with positive imaginary parts. It is denoted as $\mathbb{H}^2 = \{x + iy \mid y > 0;\ x, y \in \mathbb{R}\}$.

Definition 2.3. The *hyperbolic distance on the upper half-plane*, $d(z_1, z_2)$, where $z_j = x_j + iy_j$, is given by the infimum of $\int_{t_1}^{t_2} \frac{\sqrt{x'(t)^2+y'(t)^2}}{y} dt$ taken over all paths $\gamma(t)$ with $\gamma(t_j) = z_j = x(t_j) + iy(t_j)$.

A computation shows that this implies that the geodesic between points (x_0, y_1) and (x_0, y_2) with $y_1 < y_2$ on the vertical line $x = x_0$ has length $\ln(\frac{y_2}{y_1})$. The geodesic between any two points not on a vertical line is a circular arc where the center is on the real axis. There are no other types of geodesics.

Definition 2.4. A transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ is an *isometry* if for any points $P, Q \in \mathbb{H}^2$, the hyperbolic distance $d(P, Q) = d(A(P), A(Q))$.

Now we will show that Möbius transformations are isometries of the hyperbolic plane.

Theorem 2.5. *Möbius transformations with coefficients in $\mathbb{R}$ preserve hyperbolic lengths.*

Proof. Given a point $z \in \mathbb{H}^2$, let the Möbius transformation $A(z)$ be denoted as $A(z) = w = \frac{az+b}{cz+d}$. We need to show that $\frac{|dw|}{\text{Im}(w)} = \frac{|dz|}{\text{Im}(z)}$, where $z = x + iy$ and $|dz| = \sqrt{dx^2 + dy^2}$, with the analogous definition for $|dw|$. This comes from the definition of hyperbolic distance. The equality is equivalent to showing that $\frac{|dw|}{|dz|} = \frac{\text{Im}(w)}{\text{Im}(z)}$. Starting with the left-hand side, we get

$$\begin{aligned} \left| \frac{(cz+d)a - (az+b)c}{(cz+d)^2} \right| &= \left| \frac{ad - bc}{(cz+d)^2} \right| \\ &= \frac{1}{|cz+d|^2}. \end{aligned}$$

We also know that

$$\begin{aligned} w &= \frac{az+b}{cz+d} \cdot \frac{\overline{cz+d}}{\overline{cz+d}} \\ &= \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} \\ &= \frac{ac|z|^2 + bd + adz + bc\overline{z}}{|cz+d|^2} \end{aligned}$$

so $\text{Im}(w) = \frac{y}{|cz+d|^2}$. Note that we use the fact that all coefficients are in $\mathbb{R}$ to find $\text{Im}(w)$. Therefore, the right-hand side is $\frac{\text{Im}(w)}{\text{Im}(z)} = \frac{1}{|cz+d|^2}$, which is equal to the left-hand side. ☐

Now we will see that there exist Möbius transformations that map a given geodesic to another given geodesic and a point on that geodesic to another given point.

Theorem 2.6. *Given geodesics in the hyperbolic plane m_1, m_2 and points p_1, p_2 on those geosdesics, there exists a Möbius transformation A such that $A(m_1) = m_2$ and $A(p_1) = p_2$.*

Proof. We start with the special case where m_1 is a circular arc with endpoints x_1, x_2 in $\mathbb{R}$, and m_2 is the imaginary axis. There exists a transformation $A = \frac{az+b}{cz+d}$ where $ax_1 + b = 0$ and $cx_2 + d = 0$ by setting $b = -ax_1$ and $d = -cx_2$. This means that $A(x_1) = 0$ and $A(x_2) = \infty$ so the endpoints of m_1 are mapped onto the endpoints of m_2. We know that geodesics are taken to geodesics because of the existence of a unique geodesic between any two points in $\mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$, which comes from the fact that geodesics in the hyperbolic plane must be either straight lines or circular arcs. Since the transformation takes the endpoints of one geodesic to the endpoints of the other and A preserves lengths, it must map the unique distance-minimizing curve m_1 between x_1 and x_2 to the unique distance-minimizing curve between 0 and ∞. Mapping m_2 to m_1 can be done through inverting the transformation A. The argument where m_1 is a vertical line is similar.

We now know that $A(p_1) = q$ for some point p_1 in m_1 and some point q in m_2. However, it is possible to take q to any desired point p_2 in m_2 through the transformation $A_1(q) = \frac{\lambda^{1/2}q+0}{0q+\lambda^{-1/2}} = \lambda q$ for some $\lambda \in \mathbb{R}$. These findings can be further generalized to arbitrary m_1, m_2 and p_1, p_2. Both geodesics can be mapped to the positive imaginary axis by the above argument. Denoting these as transformations B and C, we can take one geodesic to another by the composition $C^{-1} \circ B$, which takes one geodesic first to the positive imaginary axis and then to the other geodesic. An analogous argument shows that a given point p_1 can be taken to a given point p_2. As shown above, there exist transformations taking both points to any desired point on the positive imaginary axis, which we will denote as r_1 and r_2. Therefore, it is possible to take p_1 to r_1, multiply it by some positive scalar to get to r_2, and take the inverse map to get from r_2 to p_2. □

This implies the following:

Corollary 2.7. *Given points $\{z_1, z_2\}$ and $\{w_1, w_2\}$ such that $d(z_1, z_2) = d(w_1, w_2)$ on the hyperbolic plane, there exists a Möbius transformation taking one set of points to the other.*

Proof. Any points w_1, w_2 lie on a geodesic which we will denote as m, so by theorem 2.6 we can find a Mobius transformation taking z_1 to w_1 and the geodesic between z_1 and z_2 to m. From theorem 2.5, transformations preserve hyperbolic lengths, so z_2 must map to either w_2 or the point on m that is the same distance from w_1 in the opposite direction. In the latter case, composing with an appropriate rotation about w_1 will take the image of z_2 to w_2, giving the transformation we want. □

3. Orientation-preserving isometries of the upper half-plane

We will now see how orientation-preserving isometries in $\mathbb{H}^2$ are related to $PSL(2, \mathbb{R})$. First, we will define orientation-preserving isometry.

Definition 3.1. An *orientation-preserving isometry* is an isometry where, given three noncollinear points a, b, c and their transformations A, B, C, the angles abc and ABC will be equal and have the same sign.

Lemma 3.2. *Matrix multiplication in $SL(2, \mathbb{R})$ is equivalent to the composition of Möbius transformations.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{Az+B}{Cz+D}$, with all coefficients in $\mathbb{R}$. Then

$$\begin{aligned} g(f(z)) &= \frac{A(f(z)) + B}{C(f(z)) + D} \\ &= \frac{Aaz + Ab + Bcz + Bd}{cz + d} \cdot \frac{cz + d}{Caz + Cb + Dcz + Dd} \\ &= \frac{(Aa + Bc)z + (Ab + Bd)}{(Ca + Dc)z + (Cb + Dd)}. \end{aligned}$$

By matrix multiplication,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix}.$$

The coefficents of the above composition are the same as the product of matrix multiplication. □

Lemma 3.3. *For* $z_1, z_2 \in \mathbb{H}^2$, $\sinh(\frac{1}{2} \cdot d(z_1, z_2)) = \frac{|z_1 - z_2|}{2(\mathrm{Im} z_1)^{1/2}(\mathrm{Im} z_2)^{1/2}}$.

Proof. Under Möbius transformations, the left hand side of the equation is invariant since distances are preserved, and the right hand side can be seen to be invariant through a computation. Therefore, if the equation holds for two points on a vertical line, then it will hold for any two points in $\mathbb{H}^2$ since transformations preserve distances and take geodesics to geodesics, which we know from theorems 2.5 and 2.6. Because a vertical line is a geodesic, we can apply a transformation to take the geodesic between any two points to a vertical geodesic. So, without loss of generality, let $z_1 = x + i\lambda y$ and $z_2 = x + iy$, where $\lambda \geq 1, \lambda \in \mathbb{R}$. Starting with the left hand side,

$$\begin{aligned} \sinh\left(\frac{1}{2} \cdot d(z_1, z_2)\right) &= \sinh\left(\frac{\ln(\lambda)}{2}\right) \\ &= \frac{e^{\ln\sqrt{\lambda}} - e^{-\ln\sqrt{\lambda}}}{2} \\ &= \frac{\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}}{2} \\ &= \frac{\lambda - 1}{2\sqrt{\lambda}}. \end{aligned}$$

Now looking at the right hand side,

$$\frac{|\ z_1 - z_2\ |}{2(\mathrm{Im} z_1)^{1/2}(\mathrm{Im} z_2)^{1/2}} = \frac{|\ (\lambda y - y)i\ |}{2\sqrt{\lambda y^2}} = \frac{|\ (\lambda y - y)i\ |}{2y\sqrt{\lambda}} = \frac{|\ (\lambda - 1)i\ |}{2\sqrt{\lambda}} = \frac{(\lambda - 1)}{2\sqrt{\lambda}}.$$

Since $\lambda \geq 1$, both the right and the left hand side are equal. In the second line above, note that the y can be pulled out of the absolute value sign since this is in $\mathbb{H}^2$ and the y would always be positive. □

Theorem 3.4. *All orientation-preserving isometries of* $\mathbb{H}^2$ *are of the form* $z \mapsto \frac{az+b}{cz+d}$*, where* $z \in \mathbb{H}^2$ *and* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$.

Proof. Let T be an orientation preserving isometry of $\mathbb{H}^2$.

Consider the point j in $\mathbb{H}^2$ and $k \in \mathbb{R}$.

Consider $A \in SL(2, \mathbb{R})$ such that $A \circ T(j) = j$ and $A \circ T(kj) = kj$. Such an A exists from theorem 2.6 since a given point can be taken to another given point and here we are taking $T(j)$ and $T(kj)$ to j and kj, respectively. $A \circ T$ thus fixes two points on the imaginary axis. Since an isometry that fixes any two points on a geodesic fixes the whole geodesic, and the imaginary axis is a geodesic, the imaginary axis is fixed by $A \circ T$. Therefore, for any $z = x + iy$ and any $t \in \mathbb{R}$, $d(z, it) = d(A \circ T(z), it)$, so $\sinh(\frac{1}{2} \cdot d(z, it)) = \sinh(\frac{1}{2} \cdot d(A \circ T(z), it))$. Denote $A \circ T(z)$ as $u + iv$. Applying lemma 3.3 results in

$$\frac{|\ z - it\ |^2}{4yt} = \frac{|\ u + iv - it\ |^2}{4vt}.$$

Therefore,

$$|\ z - it\ |^2\ v = |\ u + iv - it\ |^2\ y$$

so

$$(x^2 + (y - t)^2)v = (u^2 + (v - t)^2)y.$$

Dividing both sides by t^2 and letting $t \to \infty$ gives $v = y$. Therefore, $x^2 = u^2$ so $x = \pm u$. However, since we are only considering orientation preserving isometries, $x = u$ because otherwise the map would be $z \longmapsto -\overline{z}$, a reflection over the imaginary axis that reverses the sign of the angle and thus the orientation. As a result, $A \circ T$ is the identity so $T = A^{-1}$ and therefore $T \in SL(2, \mathbb{R})$ by the properties of groups. This implies that every orientation preserving isometry is of the form $\frac{az+b}{cz+d}$. □

Theorem 3.5. *Two matrices in* $SL(2, \mathbb{R})$ *induce the same isometry if and only if their matrices differ by a factor of* ± 1.

Proof. Assume two matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ representing Möbius transformations differ by a factor of ± 1. In other words, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where

$\lambda = \pm 1$. If $\lambda = 1$, then $A = a,\ B = b,\ C = c\,, D = d$, implying that for all complex numbers z, $\frac{Az+b}{Cz+d} = \frac{az+b}{cz+d}$. If $\lambda = -1$, then $\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the transformation

$$\begin{aligned} \frac{Az+B}{Cz+D} &= \frac{-az-b}{-cz-d} \\ &= \frac{(-1)(az+b)}{(-1)(cz+d)} \\ &= \frac{az+b}{cz+d}. \end{aligned}$$

Therefore, these two isometries are the same.

Now assume that two isometries are the same. Therefore, $\frac{Az+B}{Cz+D} = \frac{az+b}{cz+d}$ for all $z \in \mathbb{H}^2$. This can also be written as $(Az+B)(cz+d) = (az+b)(Cz+D)$. Multiplying this out results in

$$Acz^2 + Adz + Bcz + Bd = aCz^2 + aDz + bCz + bD.$$

Since these polynomials are equal, monomials of the same degree have the same coefficients. This implies that $Ac = aC$, or $\frac{A}{a} = \frac{C}{c}$, which we denote λ_1, $Bd = bD$ so $\frac{B}{b} = \frac{D}{d}$, which we denote λ_2, and $Ad + Bc = aD + bC$.

Therefore,

$$\lambda_1 ad + \lambda_2 bc = aD + bC = \lambda_2 ad + \lambda_1 bc.$$

This can be rewritten as

$$\lambda_1(ad - bc) = \lambda_2(ad - bc)$$

so $\lambda_1 = \lambda_2$ since $ad - bc = 1$ in $SL(2, \mathbb{Z})$. Therefore, there is a λ such that

$$\frac{A}{a} = \frac{C}{c} = \frac{B}{b} = \frac{D}{d} = \lambda.$$

This can be represented by the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since the determinant of the right hand side, $\lambda^2 \cdot (ad - bc)$, equals 1 and $ad - bc = 1$, $\lambda^2 = 1$ so $\lambda = \pm 1$. □

From the definition of $PSL(2, \mathbb{R})$ and theorems 3.4 and 3.5, it follows that $PSL(2, \mathbb{R})$ is isomorphic to the group of orientation-preserving isometries of $\mathbb{H}^2$.

4. Characterization of isometries

Now, we will see how Möbius transformations can be characterized by the trace of their matrices. One method is through the Jordan normal form of a real 2 by 2 matrix, which implies that all matrices of $SL(2, \mathbb{R})$ are conjugate to $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ for $\theta \in [0, 2\pi)$, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in \mathbb{R}$, or $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for $a, b \in \mathbb{R}$ [1]. Since the determinants of these matrices must equal 1, we can see that the absolute value of the traces of the matrices will be respectively less than 2, called elliptic, greater than 2, called hyperbolic, and equal to 2, called parabolic if it is not the identity transformation. However, another way to reach this same conclusion is through the use of fixed points.

Lemma 4.1. *Any Möbius transformation $A \in PSL(2,\mathbb{R})$ has at least one fixed point in $\mathbb{C} \cup \{\infty\}$.*

Proof. For a fixed point $z \in \mathbb{C} \cup \{\infty\}$, the transformation can be expressed as $A(z) = \frac{az+b}{cz+d} = z$. This can be rewritten as

$$az + b = cz^2 + dz$$

so

$$cz^2 + (d-a)z - b = 0.$$

From the quadratic formula, the roots of this expression are

$$\frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

if $c \neq 0$. If the determinant $(d-a)^2 - 4bc$ is greater than 0, there will be 2 real fixed points, if it is less than 0 then there will be 2 complex fixed points, and if it equals 0, there will be 1 real fixed point. In the case that $c = 0$, then let $\frac{az+b}{d} = z$ so $(a-d) \cdot z + b = 0$. If $a \neq d$, then $\frac{-b}{a-d}$ will be a fixed point, as will ∞. If $a = d$, then ∞ will be the only fixed point. $\square$

Lemma 4.2. *If a Möbius transformation fixes three or more points, then it is the identity transformation.*

Proof. From the previous lemma, we see that the fixed points of a Möbius transformation can be expressed as the roots of a quadratic equation. If there are three or more roots of the quadratic, then this implies that all coefficients of the quadratic are 0. In particular, $c = b = 0$ and $d - a = 0$ so $d = a$. The Möbius transformation then becomes $\frac{az}{d} = z$, which is the identity transformation. $\square$

Lemma 4.3. *It is possible to move any point on $\mathbb{R} \cup \{\infty\}$ to 0 or ∞ and it is possible to move any point on $\mathbb{H}^2$ to the point i with Möbius transformations.*

Proof. Given a point p on $\mathbb{R} \cup \{\infty\}$, we want to find a Möbius transformation A such that $A(p) = \frac{wp+x}{yp+z} = \infty$ where all the variables are real numbers. This is true only if the denominator, $yp + z$, equals 0. Therefore, $z = -yp$. Let $y = 1$ which implies that $z = -p$. This can be represented by the matrix $\begin{bmatrix} w & x \\ 1 & -p \end{bmatrix}$. Since the determinant of this matrix, $-wp - x$, must equal 1, we can let $w = 0$, so that $x = -1$. This Möbius transformation, represented by $\begin{bmatrix} 0 & -1 \\ 1 & -p \end{bmatrix}$, takes p to ∞. Using similar steps, you can take a point p on $\mathbb{R} \cup \{\infty\}$ to 0 through a transformation such as $\begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix}$ or a point $c = a + bi$ in $\mathbb{C} \cup \{\infty\}$ to the point i through a transformation such as $\begin{bmatrix} 1 & -a \\ 0 & b \end{bmatrix}$. $\square$

Given a Möbius transformation A with one or two fixed points in $\mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$, it follows from the preceding lemma that we may assume these points are $\{\infty\}$, $\{0, \infty\}$, or $\{i\}$ by conjugating A. Note that matrices that conjugate to one another retain the same trace.

Theorem 4.4. *If the Möbius transformation $A \in PSL(2,\mathbb{R})$ has one fixed point in $\mathbb{R} \cup \{\infty\}$, then the absolute value of the trace of its matrix is 2.*

Proof. First, note that if there is one fixed point, it must be the case that the fixed point is on $\mathbb{R} \cup \{\infty\}$. Otherwise, in $\mathbb{C}$, the conjugate of the fixed point would also be a root of $A(z) = \frac{az+b}{cz+d} = z$ because the roots take the form $\frac{-(d-a)\pm\sqrt{(d-a)^2+4bc}}{2c}$. From the preceding lemma, without loss of generality, let the fixed point be the point at ∞. Since $A(z) = \frac{az+b}{cz+d} = z$ for $z = \infty$, this implies that $c = 0$, because otherwise $A(z) = \frac{a}{c} \neq \infty$, since $a, b, c, d \in \mathbb{R}$. The equation $\frac{az+b}{d} = z$ can be rewritten as

$$dz = az + b,$$

which implies

$$z = \frac{b}{d-a}.$$

Since $z = \infty$, $d = a$. This transformation can be represented by the matrix $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, where $a^2 = 1$, so $a = \pm 1$. Therefore, $|\mathrm{tr}(A)| = 2$. □

Theorem 4.5. *If the Möbius transformation $A \in PSL(2,\mathbb{R})$ has two fixed points on $\mathbb{R} \cup \{\infty\}$, then $|\mathrm{tr}(A)| > 2$.*

Proof. Without loss of generality, let the two fixed points be ∞ and 0. From the preceding theorem, having a fixed point at infinity implies that for a transformation $A(z) = \frac{az+b}{cz+d} = z$, $c = 0$. Now consider a fixed point at 0. This implies

$$A(z) = \frac{az+b}{cz+d} = \frac{b}{d} = 0,$$

which implies $b = 0$. This transformation can therefore be represented as $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where $ad = 1$, so $d = a^{-1}$. The matrix can then be rewritten as $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$. Therefore, $|\mathrm{tr}(A)| > 2$ except if $a = \pm 1$, in which case this becomes the identity map. We can ignore this case when considering only two fixed points. □

Note that it is impossible to fix exactly two points in $\mathbb{H}^2$ since given one fixed point in $\mathbb{H}^2$, the other fixed point in $\mathbb{C}$ would be its conjugate, which is not in $\mathbb{H}^2$.

Theorem 4.6. *If the Möbius transformation $A \in PSL(2,\mathbb{R})$ has one fixed point in $\mathbb{H}^2$, then $|\mathrm{tr}(A)| < 2$.*

Proof. Without loss of generality, let the point i be the fixed point. Then

$$A(i) = \frac{ai+b}{ci+d} = i.$$

This can be written as

$$ai + b = ci^2 + di = di - c,$$

which implies

$$b + c + (a-d)i = 0.$$

This means that $a - d = 0$ or $a = d$, and $b + c = 0$ or $b = -c$. Since $ad - bc = a^2 + b^2 = 1$, $a = \cos(\theta)$ and $b = \sin(\theta)$ for some $\theta \in [0, 2\pi)$. This can be represented by the matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Therefore, $|\mathrm{tr}(A)| < 2$ except if $\theta = k\pi$ for $k \in \mathbb{Z}$. In that case, the matrix becomes the identity transformation in $PSL(2, \mathbb{R})$, which we can ignore here for the same reason given in the previous proof. □

5. Concluding Remarks

We have now seen the connections between Möbius transformations, isometries of the hyperbolic plane, and $PSL(2, \mathbb{R})$. Additionally, by equating the group operation of matrix multiplication with the group operation of composition of Möbius transformations, we have developed a way to characterize such transformation by the traces of their matrices.

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References

[1] Agricola, Ilka, and Thomas Friedrich. 2008. Elementary Geometry. Providence: American Mathematical Society.

[2] Stahl, Saul. 2008. A gateway to modern geometry : the Poincaré half-plane. Sudbury, Mass.: Jones and Bartlett Publishers.