THE REPRESENTATIONS OF THE SYMMETRIC GROUP

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Abstract. Young tableau is a combinatorial object which provides a convenient way to describe the group representations of the symmetric group, $S_n$. In this paper, we prove several facts about the symmetric group, group representations, and Young tableaux. We then present the construction of Specht modules which are irreducible representations of $S_n$.

CONTENTS

1. The Symmetric Group, $S_n$

Definitions

1.1. The symmetric group, $S_\Omega$, is a group of all bijections from $\Omega$ to itself under function composition. The elements $\pi \in S_\Omega$ are called permutations.

In particular, for $\Omega = \{1, 2, 3, \ldots , n\}$, $S_\Omega$ is the symmetric group of degree $n$, denoted by $S_n$.

Example 1.2. $\sigma \in S_7$ given by

\[
\begin{array}{cccccccc}
i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\sigma(i) & 2 & 5 & 6 & 4 & 7 & 3 & 1 \\
\end{array}
\]

is a permutation.

Definition 1.3. A cycle is a string of integers which represents the element of $S_n$ that cyclically permutes these integers. The cycle $(a_1 \ a_2 \ a_3 \ldots \ a_m)$ is the permutation which sends $a_i$ to $a_{i+1}$ for $1 \leq i \leq m-1$ and sends $a_m$ to $a_1$.

Proposition 1.4. Every permutation in $S_n$ can be written as a product of disjoint cycles.

Proof. Consider $\pi \in S_n$. Given $i \in \{1, 2, 3, \ldots , n\}$, the elements of the sequence $i, \pi(i), \pi^2(i), \pi^3(i), \ldots$ cannot all be distinct. Taking the first power $p$ such that $\pi^p(i) = i$, we have the cycle $(i \ \pi(i) \ \pi^2(i) \ldots \ \pi^{p-1}(i))$. Iterate this process with an element that is not in any of the previously generated cycles until each element of $\{1, 2, 3, \ldots , n\}$ belongs to exactly one of the cycles generated. Then, $\pi$ is the product of the generated cycles. \qed

Definition 1.5. If $\pi \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots , n_r$ such that $n_1 \leq n_2 \leq \ldots \leq n_r$, then the integers $n_1, n_2, \ldots , n_r$ are called the cycle type of $\pi$.

For instance, $\sigma$ in Example 1.2. can be expressed as $\sigma = (4)(3\ 6)(1\ 2\ 5\ 7)$ and its cycle type is $1, 2, 4$. A 1-cycle of a permutation, such as $(4)$ of $\sigma$, is called a fixed point and usually omitted from the cycle notation. Another way to represent the cycle type is as a partition:

\[1\]

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Definition 1.6. A partition of \( n \) is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) where the \( \lambda_i \) are weakly decreasing and \( \sum_{i=1}^{l} \lambda_i = n \). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) is a partition of \( n \), we write \( \lambda \vdash n \).

\( \sigma \) corresponds to the partition \( \lambda = (4, 2, 1) \).

Definitions 1.7. In any group \( G \), elements \( g \) and \( h \) are conjugates if \( g = khk^{-1} \) for some \( k \in G \). The set of all elements conjugate to a given \( g \) is called the conjugacy class of \( g \) and is denoted by \( K_g \).

Proposition 1.8. Conjugacy is an equivalence relation. Thus, the distinct conjugacy classes partition \( G \).

Proof. Let \( a \sim b \) if \( a \) and \( b \) are conjugates. Since \( a = \epsilon a \epsilon^{-1} \) where \( \epsilon \) is the identity element of \( G \), \( a \sim a \) for all \( a \in G \), and conjugacy is reflexive. Suppose \( a \sim b \). Then, \( a = khk^{-1} \Leftrightarrow b = (k^{-1})a(k^{-1})^{-1} \). Hence, \( b \sim a \), and conjugacy is symmetric. If \( a \sim b \) and \( b \sim c \), \( a = khk^{-1} = k(lc^{-1})k^{-1} = (kl)c(kl)^{-1} \) for some \( k, l \in G \), and \( a \sim c \). Thus, conjugacy is transitive. \( \square \)

Proposition 1.9. In \( S_n \), two permutations are in the same conjugacy class if and only if they have the same cycle type. Thus, there is a natural one-to-one correspondence between partitions of \( n \) and conjugacy classes of \( S_n \).

Proof. Consider \( \pi = (a_1 a_2 \ldots a_l)(a_m a_{m+1} \ldots a_n) \in S_n \). For \( \sigma \in S_n \),

\[
\sigma \pi \sigma^{-1} = (\sigma(a_1) \sigma(a_2) \ldots \sigma(a_l)) \cdot (\sigma(a_m) \sigma(a_{m+1}) \ldots \sigma(a_n)).
\]

Hence, conjugation does not change the cycle type. \( \square \)

Definition 1.10. A 2-cycle is called a transposition.

Proposition 1.11. Every element of \( S_n \) can be written as a product of transpositions.

Proof. For \( (a_1 a_2 \ldots a_m) \in S_n \),

\[
(a_1 a_2 \ldots a_m) = (a_1 a_m) (a_1 a_{m-1}) \cdots (a_1 a_2)
\]

Since every cycle can be written as a product of transpositions, by Proposition 1.4., every permutation can be expressed as a product of transpositions. \( \square \)

Definition 1.12. If \( \pi = \tau_1 \tau_2 \ldots \tau_k \), where the \( \tau_i \) are transpositions, then the sign of \( \pi \) is \( \text{sgn}(\pi) = (-1)^k \).

Proposition 1.13. The map \( \text{sgn} : S_n \to \{\pm 1\} \) is a well-defined homomorphism. In other words, \( \text{sgn}(\pi \sigma) = \text{sgn}(\pi) \text{sgn}(\sigma) \).

The proof of Proposition 1.13 may be found in [1].

2. Group Representations

Definitions 2.1. \( \text{Mat}_d \), the full complex matrix algebra of degree \( d \), is the set of all \( d \times d \) matrices with entries in \( \mathbb{C} \), and \( \text{GL}_d \), the complex general linear group of degree \( d \), is the group of all \( X = (x_{i,j})_{d \times d} \in \text{Mat}_d \) that are invertible with respect to multiplication.

Definition 2.2. A matrix representation of a group \( G \) is a group homomorphism \( X : G \to \text{GL}_d \).
**Definition 2.3.** For \( V \) a vector space, \( GL(V) \), the **general linear group** of \( V \) is the set of all invertible linear transformations of \( V \) to itself.

In this study, all vector spaces will be over \( \mathbb{C} \) and of finite dimension. Since \( GL(V) \) and \( GL_d \) are isomorphic as groups if \( \text{dim} \ V = d \), we can think of representations as group homomorphisms into the general linear group of a vector space.

**Definitions 2.4.** Let \( V \) be a vector space and \( G \) be a group. Then \( V \) is a **\( G \)-module** if there is a group homomorphism \( \rho : G \to GL(V) \). Equivalently, \( V \) is a **\( G \)-module** if there is an action of \( G \) on \( V \) denoted by \( gv \) for all \( g \in G \) and \( v \in V \) which satisfy:

1. \( gv \in V \)
2. \( g(cv + dw) = c(gv) + d(gw) \)
3. \( (gh)v = g(hv) \)
4. \( \epsilon v = v \)

for all \( g, h \in G; \ v, w \in V; \) and \( c, d \in \mathbb{C} \).

**Proof.** (The Equivalence of Definitions) By letting \( gv = \rho(g)(v) \), (1) means \( \rho(g) \) is a transformation from \( V \) to itself; (2) represents that the transformation is linear; (3) says \( \rho \) is a group homomorphism; and (4) in combination with (3) means \( \rho(g) \) and \( \rho(g^{-1}) \) are inverse maps of each other and, thus, invertible. \( \square \)

When there is no confusion arises about the associated group, the prefix \( G \)- will be dropped from terms, such as shortening **\( G \)-module** to **module**.

**Definition 2.5.** Let \( V \) be a \( G \)-module. A **submodule** of \( V \) is a subspace \( W \) that is closed under the action of \( G \), i.e., \( w \in W \Rightarrow gw \in W \) for all \( g \in G \). We write \( W \leq V \) if \( W \) is a submodule of \( V \).

**Definition 2.6.** A nonzero \( G \)-module \( V \) is **reducible** if it contains a nontrivial submodule \( W \). Otherwise, \( V \) is said to be **irreducible**.

**Definitions 2.7.** Let \( V \) be a vector space with subspaces \( U \) and \( W \). Then \( V \) is the **direct sum** of \( U \) and \( W \), written \( V = U \oplus W \), if every \( v \in V \) can be written uniquely as a sum \( v = u + w, \ u \in U, \ w \in W \). If \( V \) is a \( G \)-module and \( U, W \) are \( G \)-submodules, then we say that \( U \) and \( W \) are **complements** of each other.

**Definition 2.8.** An **inner product** on a vector space \( V \) is a map \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) that satisfies:

1. \( \langle x, y \rangle = \overline{\langle y, x \rangle} \)
2. \( \langle ax, y \rangle = a \langle x, y \rangle \)
3. \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \)
4. \( \langle x, x \rangle \geq 0 \) with equality only for \( x = 0 \)

for \( x, y, z \in V \) and \( a \in \mathbb{C} \).

**Definition 2.9.** For \( \langle \cdot, \cdot \rangle \) an inner product on a vector space \( V \) and a subspace \( W \), the **orthogonal complement** of \( W \) is \( W^\perp = \{ v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W \} \).

Note that \( V = W \oplus W^\perp \).

**Definition 2.10.** An inner product \( \langle \cdot, \cdot \rangle \) on a vector space \( V \) is **invariant** under the action of \( G \) if \( \langle gv, gw \rangle = \langle v, w \rangle \) for all \( g \in G \) and \( v, w \in V \).
Proposition 2.11. Let $V$ be a $G$-module, $W$ a submodule, and $< \cdot, \cdot >$ an inner product on $V$. If $< \cdot, \cdot >$ is invariant under the action of $G$, then $W^\perp$ is also a $G$-submodule.

Proof. Suppose $g \in G$ and $u \in W^\perp$. Then, for any $w \in W$,

$$< gu, w > = < g^{-1}gu, g^{-1}w > = < u, g^{-1}w > = 0$$

Hence, $gu \in W^\perp$, and $W^\perp$ is a $G$-submodule. □

Theorem 2.12. (Maschke’s Theorem) Let $G$ be a finite group and let $V$ be a nonzero $G$-module. Then, $V = W^{(1)} \oplus W^{(2)} \oplus \ldots \oplus W^{(k)}$ where each $W^{(i)}$ is an irreducible $G$-submodule of $V$.

Proof. Induction on $d = \dim V$

- Base Case: if $d = 1$, $V$ itself is irreducible. Hence, $V = W^{(1)}$.

- Inductive Case: For $d > 1$, assume true for $d' < d$.

  Suppose $V$ is reducible. Then, $V$ has a nontrivial $G$-submodule, $W$.

  Let $B = \{v_1, \ldots, v_d\}$ be a basis for $V$. Consider the unique inner product on $V$ that satisfies

$$< v_i, v_j > = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for basis elements in $B$.

For any $v, w \in V$, let

$$< v, w >' = \sum_{g \in G} < gv, gw >$$

(1)

$$< v, w >' = \sum_{g \in G} < gv, gw > = \sum_{g \in G} < gw, gv > = < w, v >'$$

(2)

$$< av, w >' = \sum_{g \in G} < g(av), gw > = \sum_{g \in G} a < gv, gw > = a < v, w >'$$

(3)

$$< v + w, z > = \sum_{g \in G} < g(v + w), gz >$$

$$= \sum_{g \in G} < gv, gz > + < gw, gz >$$

$$= < v, z >' + < w, z >'$$

(4)

$$< v, v >' = \sum_{g \in G} < gv, gv > \geq 0 \text{ and } < 0, 0 >' = \sum_{g \in G} < g0, g0 > = 0$$
Hence, \(<\cdot,\cdot>’\) is an inner product on \(V\).
Moreover, since, for \(h \in G\),
\[
< hv, hw >’ = \sum_{g \in G} < ghv, ghw >
= \sum_{k \in G} < kv, kw >
= < v, w >’,
\]
\(<\cdot,\cdot>’\) is invariant under the action of \(G\).
Let \(W^\perp = \{ v \in V : < v, w >’ = 0 \text{ for all } w \in W \}\). Then, \(V = W \oplus W^\perp\), and \(W^\perp\) is a \(G\)-submodule by Proposition 2.11. Since \(W\) and \(W^\perp\) can be written as direct sums of irreducibles by the inductive hypothesis, \(V\) can be expressed as a direct sum of irreducibles.

\[\Box\]

**Definition 2.13.** Let \(V\) and \(W\) be \(G\)-modules. Then a \(G\)-homomorphism is a linear transformation \(\theta : V \to W\) such that
\[
\theta(gv) = g\theta(v)
\]
for all \(g \in G\) and \(v \in V\).

**Definition 2.14.** Let \(V\) and \(W\) be \(G\)-modules. A \(G\)-isomorphism is a \(G\)-homomorphism \(\theta : V \to W\) that is bijective. In this case, we say that \(V\) and \(W\) are \(G\)-isomorphic, or \(G\)-equivalent, denoted by \(V \cong W\). Otherwise, we say that \(V\) and \(W\) are \(G\)-inequivalent.

**Proposition 2.15.** Let \(\theta : V \to W\) be a \(G\)-homomorphism. Then,
1. \(\ker \theta\) is a \(G\)-submodule of \(V\)
2. \(\text{im } \theta\) is a \(G\)-submodule of \(W\)

**Proof:**
1. Since \(\theta(0) = 0\), \(0 \in \ker \theta\) and \(\ker \theta \neq \emptyset\), and if \(v_1, v_2 \in \ker \theta\) and \(c \in \mathbb{C}\), \(\theta(v_1 + cv_2) = \theta(v_1) + c\theta(v_2) = 0 + c0 = 0\) and \(v_1 + cv_2 \in \ker \theta\).
   Hence, \(\ker \theta\) is a subspace of \(V\). Suppose \(v \in \ker \theta\). Then, for any \(g \in G\)
   \[
   \theta(gv) = g\theta(v)
   = g0
   = 0
   \]
   Thus, \(gv \in \ker \theta\) and \(\ker \theta\) is a \(G\)-submodule of \(V\).
2. \(0 \in \text{im } \theta\) and \(\text{im } \theta \neq \emptyset\), and if \(w_1, w_2 \in W\) and \(c \in \mathbb{C}\), there exist \(v_1, v_2 \in V\) such that \(\theta(v_1) = w_1\) and \(\theta(v_2) = w_2\) and \(\theta(v_1 + cv_2) = \theta(v_1) + c\theta(v_2) = w_1 + cw_2\). Thus, \(w_1 + cw_2 \in \text{im } \theta\) and \(\text{im } \theta\) is a subspace of \(W\). Suppose \(w \in \text{im } \theta\). Then, there exists \(v \in V\) such that \(\theta(v) = w\). For any \(g \in G\), \(gw \in \text{im } \theta\) and
   \[
   \theta(gv) = g\theta(v) = gw
   \]
   Hence, \(gw \in \text{im } \theta\) and \(\text{im } \theta\) is a \(G\)-submodule of \(W\).

\[\Box\]

**Theorem 2.16.** (Schur’s Lemma) Let \(V\) and \(W\) be irreducible \(G\)-modules. If \(\theta : V \to W\) is a \(G\)-homomorphism, then either
1. \(\theta\) is a \(G\)-isomorphism, or
(2) \( \theta \) is the zero map

Proof. Since \( V \) is irreducible and \( \ker \theta \) is a submodule by Proposition 2.15, \( \ker \theta = \{0\} \) or \( \ker \theta = V \). Similarly, \( \im \theta = \{0\} \) or \( \im \theta = W \). If \( \ker \theta = \{0\} \) and \( \im \theta = W \), \( \theta \) is a G-isomorphism, and if \( \ker \theta = V \) and \( \im \theta = \{0\} \), \( \theta \) is the zero map. \( \square \)

Corollary 2.17. Let \( V \) be an irreducible \( G \)-module. If \( \theta : V \to V \) is a \( G \)-homomorphism, \( \theta = cI \) for some \( c \in \mathbb{C} \), multiplication by a scalar.

Proof. Since \( \mathbb{C} \) is algebraically closed, \( \theta \) has an eigenvalue \( c \in \mathbb{C} \). Then, \( \theta - cI \) has a nonzero kernel. By Theorem 2.16, \( \theta - cI \) is the zero map. Hence, \( \theta = cI \). \( \square \)

Definition 2.18. Given a \( G \)-module \( V \), the corresponding endomorphism algebra is \( \text{End} V = \{ \theta : V \to V : \theta \text{ is a } G\text{-homomorphism} \} \)

Definition 2.19. The center of an algebra \( A \) is \( Z_A = \{ a \in A : ab = ba \text{ for all } b \in A \} \)

Let \( E_{i,j} \) be the matrix of zeros with exactly 1 one in position \( (i,j) \).

Proposition 2.20. The center of \( \text{Mat}_d \) is \( Z_{\text{Mat}_d} = \{ cI_d : c \in \mathbb{C} \} \)

Proof. Suppose that \( C \in Z_{\text{Mat}_d} \). Consider \( CE_{i,i} = E_{i,i}C \)

\( CE_{i,i}(E_{i,i}C, \text{ respectively}) \) is all zeros except for the \( i \)th column(row, respectively) which is the same as that of \( C \). Hence, all off-diagonal elements must be 0.

For \( i \neq j \),

\[
C(E_{i,j} + E_{j,i}) = (E_{i,j} + E_{j,i})C
\]

Then, \( c_{i,i} = c_{j,j} \). Hence, all the diagonal elements must be equal, and \( C = cI_d \) for some \( c \in \mathbb{C} \). \( \square \)

Note that, for \( A, X \in \text{Mat}_d \) and \( B, Y \in \text{Mat}_f \),

\[
(A \oplus B)(X \oplus Y) = AB \oplus XY
\]

Theorem 2.21. Let \( V \) be a \( G \)-module such that \( V \cong m_1V^{(1)} \oplus m_2V^{(2)} \oplus \cdots \oplus m_kV^{(k)} \) where the \( V^{(i)} \) are pairwise inequivalent irreducibles and \( \dim V^{(i)} = d_i \). Then,

1. \( \dim V = m_1d_1 + m_2d_2 + \cdots + m_kd_k \)
2. \( \text{End} V \cong \bigoplus_{i=1}^{k} \text{Mat}_{m_i} \)
3. \( \dim Z_{\text{End} V} = k \).
Proof.

(1) Clear.

(2) By Theorem 2.16. and Corollary 2.17., $\theta \in \text{End} V$ maps each $V^{(i)}$ into $m_i$ copies of $V^{(i)}$ as multiplications by scalars. Hence,

$$\text{End} V \cong \text{Mat}_{m_1} \oplus \text{Mat}_{m_2} \oplus \cdots \oplus \text{Mat}_{m_k}$$

(3) Consider $C \in \mathbb{Z}_{\text{End} V}$. Then,

$$CT = TC$$

for all $T \in \text{End} V \cong \bigoplus_{i=1}^{k} \text{Mat}_{m_i}$

where $T = \bigoplus_{i=1}^{k} M_{m_i}$ and $C = \bigoplus_{i=1}^{k} C_{m_i}$.

$$CT = \bigoplus_{i=1}^{k} C_{m_i} \left( \bigoplus_{i=1}^{k} M_{m_i} \right)$$

$$= \bigoplus_{i=1}^{k} C_{m_i} M_{m_i}$$

Similarly, $TC = \bigoplus_{i=1}^{k} M_{m_i} C_{m_i}$. Hence,

$$C_{m_i} M_{m_i} = M_{m_i} C_{m_i}$$

for all $M_{m_i} \in \text{Mat}_{m_i}$.

By Proposition 2.20., $C_{m_i} = c_i I_{m_i}$ for some $c_i \in \mathbb{C}$. Thus,

$$C = \bigoplus_{i=1}^{k} c_i I_{m_i}$$

and $\dim \mathbb{Z}_{\text{End} V} = k$.

\[\square\]

**Proposition 2.22.** Let $V$ and $W$ be $G$-modules with $V$ irreducible. Then, $\dim \text{Hom}(V, W)$ is the multiplicity of $V$ in $W$.

**Proof.** Let $m$ be the multiplicity of $V$ in $W$. By Theorem 2.16. and Corollary 2.17., $\theta \in \text{Hom}(V, W)$ maps $V$ into $m$ copies of $V$ in $W$ as multiplications by scalars. Hence,

$$\dim \text{Hom}(V, W) = m$$

\[\square\]

**Definition 2.23.** For a group $G = \{g_1, g_2, \ldots, g_n\}$, the corresponding **group algebra** of $G$ is a $G$-module

$$\mathbb{C}[G] = \{c_1 g_1 + c_2 g_2 + \cdots + c_n g_n : c_i \in \mathbb{C} \text{ for all } i\}$$

**Proposition 2.24.** Let $G$ be a finite group and suppose $\mathbb{C}[G] = \bigoplus_{i=1}^{k} m_i V^{(i)}$ where the $V^{(i)}$ form a complete list of pairwise inequivalent irreducible $G$-modules. Then,

$$\text{number of } V^{(i)} = k = \text{number of conjugacy classes of } G$$
Proof. For \( v \in \mathbb{C}[G] \), let the map \( \phi_v : \mathbb{C}[G] \to \mathbb{C}[G] \) be right multiplication by \( v \).

In other words,
\[
\phi_v(w) = wv \quad \text{for all } w \in \mathbb{C}[G]
\]

Since \( \phi_v(gw) = (gw)v = g(wv) = g\phi_v(w) \), \( \phi_v \in \text{End} \mathbb{C}[G] \).

Claim: \( \mathbb{C}[G] \cong \text{End} \mathbb{C}[G] \).

Consider \( \psi : \mathbb{C}[G] \to \text{End} \mathbb{C}[G] \) such that \( \psi(v) = \phi_v \).

If \( \psi(v) = \phi_v \) is the zero map, then
\[
0 = \phi_v(\epsilon) = \epsilon v = v.
\]

Hence, \( \psi \) is injective.

Suppose \( \theta \in \text{End} \mathbb{C}[G] \) and let \( v = \theta(\epsilon) \in \mathbb{C}[G] \). For any \( g \in G \),
\[
\theta(g) = \theta(g\epsilon) = g\theta(\epsilon) = gv = \phi_v(g)
\]

Since \( \theta \) and \( \phi_v \) agree on a basis \( G \), \( \theta = \phi_v \) and \( \psi \) is surjective. Thus, \( \psi \) is an anti-isomorphism, and \( \mathbb{C}[G] \cong \text{End} \mathbb{C}[G] \).

By (3) of Theorem 2.21, \( k = \dim Z_{\text{End} \mathbb{C}[G]} = \dim Z_{\mathbb{C}[G]} \).

Consider \( z = c_1g_1 + c_2g_2 + \cdots + c_ng_n \in Z_{\mathbb{C}[G]} \).

For all \( h \in G \), \( zh = hz \iff z = hzh^{-1} \iff \)
\[
c_1g_1 + c_2g_2 + \cdots + c_ng_n = c_1h_{g_1}h^{-1} + c_2h_{g_2}h^{-1} + \cdots + c_nh_{g_n}h^{-1}
\]

Since \( h_{g_i}h^{-1} \) runs over the conjugacy class of \( g_i \), all elements of each conjugacy class have the same coefficient. If \( G \) has \( l \) conjugacy classes \( K_1, \ldots, K_l \), let
\[
z_i = \sum_{g \in K_i} g \quad \text{for } i = 1, \ldots, l.
\]

Then, any \( z \in Z_{\mathbb{C}[G]} \) can be written as
\[
z = \sum_{i=1}^{l} d_iz_i.
\]

Hence,
\[
\text{number of conjugacy classes} = \dim Z_{\mathbb{C}[G]} = k = \text{number of } V^{(i)}.
\]

3. Young Tableaux

**Definition 3.1.** Suppose \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n \). The **Young diagram**, or **shape**, of \( \lambda \) is a collection of boxes arranged in \( l \) left-justified rows with row \( i \) containing \( \lambda_i \) boxes for \( 1 \leq i \leq l \).

**Example 3.2.** \[
\begin{array}{ccc}
\hline
& & \\
& & \\
\hline
\end{array}
\]
is the Young diagram of \( \lambda = (4, 2, 1) \).

**Definition 3.3.** Suppose \( \lambda \vdash n \). **Young tableau of shape** \( \lambda \) is an array \( t \) obtained by filling the boxes of the Young diagram of \( \lambda \) with the numbers \( 1, 2, \ldots, n \) bijectively.

Let \( t_{i,j} \) stand for the entry of \( t \) in the position \( (i, j) \) and \( sh \ t \) denote the shape of \( t \).
Example 3.4. $t = \begin{array}{ccc} 2 & 5 & 6 \\ 7 & 3 & 1 \\ 4 &  &  \\ \end{array}$ is a Young tableau of $\lambda = (4, 2, 1)$, and $t_{1,3} = 6$.

$\pi \in S_n$ acts on a tableau $t = (t_{i,j})$ of $\lambda \vdash n$ as follows:

$$\pi t = (\pi t_{i,j}) \text{ where } \pi t_{i,j} = \pi(t_{i,j})$$

Definitions 3.5. Two $\lambda$-tableaux $t_1$ and $t_2$ are row equivalent, $t_1 \sim t_2$, if corresponding rows of the two tableaux contain the same elements. A tabloid of shape $\lambda$, or $\lambda$-tabloid, is then $\{t\} = \{t_1 : t_1 \sim t\}$ where $sh t = \lambda$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, then the number of tableaux in a $\lambda$-tabloid is

$$\lambda_1!\lambda_2!\ldots\lambda_l! \overset{def}{=} \lambda!.$$  
Hence, the number of $\lambda$-tabloids is $n!/\lambda!$.

Example 3.6. For $s = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \end{array}$, we have $\{s\} = \begin{array}{cccc} 1 & 2 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ \end{array}$ and $\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \end{array} \overset{def}{=} \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \end{array}$

Definition 3.7. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ are partitions of $n$. Then $\lambda$ dominates $\mu$, written $\lambda \succeq \mu$, if $\lambda_1 + \lambda_2 + \ldots + \lambda_l \geq \mu_1 + \mu_2 + \ldots + \mu_m$ for all $i \geq 1$. If $i > l$ (i.e., $i > m$, respectively), then we take $\lambda_i$ (i.e., $\mu_i$, respectively) to be zero.

Lemma 3.8. (Dominance Lemma for Partitions) Let $t^\lambda$ and $s^\mu$ be tableaux of shapes $\lambda$ and $\mu$, respectively. If for each index $i$, the elements of row $i$ in $s^\mu$ are all in different columns of $t^\lambda$, then $\lambda \succeq \mu$.

Proof. Since the elements of row 1 in $s^\mu$ are all in different columns of $t^\lambda$, we can sort the entries in each column of $t^\lambda$ so that the elements of row 1 in $s^\mu$ all occur in the first row of $t^\lambda_{(1)}$. Then, since the elements of row 2 in $s^\mu$ are also all in different columns of $t^\lambda$ and, thus, $t^\lambda_{(1)}$, we can re-sort the entries in each column of $t^\lambda_{(1)}$ so that the elements of rows 1 and 2 in $s^\mu$ all occur in the first two rows of $t^\lambda_{(2)}$.

Inductively, the elements of rows 1, 2, $\ldots$, $i$ in $s^\mu$ all occur in the first $i$ rows of $t^\lambda_{(i)}$. Thus,

$$\lambda_1 + \lambda_2 + \ldots + \lambda_i = \text{number of elements in the first } i \text{ rows of } t^\lambda_{(i)}$$

$$\geq \text{number of elements in the first } i \text{ rows of } s^\mu$$

$$= \mu_1 + \mu_2 + \ldots + \mu_i \hspace{1cm} \square$$

Definition 3.9. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ are partitions of $n$. Then $\lambda > \mu$ in lexicographic order if, for some index $i$,

$$\lambda_j = \mu_j \text{ for } j < i \text{ and } \lambda_i > \mu_i$$

Proposition 3.10. If $\lambda, \mu \vdash n$ with $\lambda \succeq \mu$, then $\lambda \geq \mu$.

Proof. Suppose $\lambda \neq \mu$. Let $i$ be the first index where they differ. Then, $\sum_{j=1}^{i-1} \lambda_j = \sum_{j=1}^{i-1} \mu_j$ and $\sum_{j=1}^{i} \lambda_j > \sum_{j=1}^{i} \mu_j$. Hence, $\lambda_i > \mu_i$. $\square$
4. REPRESENTATIONS OF THE SYMMETRIC GROUP

Definition 4.1. Suppose \( \lambda \vdash n \). Let \( M^\lambda = \mathbb{C}\{\{t_1\}, \ldots, \{t_k\}\} \), where \( \{t_1\}, \ldots, \{t_k\} \) is a complete list of \( \lambda \)-tabloids. Then \( M^\lambda \) is called the permutation module corresponding to \( \lambda \).

\( M^\lambda \) is indeed an \( S_n \)-module by letting \( \pi \{t\} = \{\pi t\} \) for \( \pi \in S_n \) and \( t \) a \( \lambda \)-tableau.

In addition, \( \dim M^\lambda = n!/\lambda ! \), the number of \( \lambda \)-tabloids.

Definition 4.2. Any \( G \)-module \( M \) is cyclic if there is a \( v \in M \) such that \( M = CGv \) where \( Gv = \{gv : g \in G\} \). In this case, we say that \( M \) is generated by \( v \).

Proposition 4.3. If \( \lambda \vdash n \), then \( M^\lambda \) is cyclic, generated by any given \( \lambda \)-tabloid.

Definition 4.4. Suppose that the tableau \( t \) has rows \( R_1, R_2, \ldots, R_l \) and columns \( C_1, C_2, \ldots, C_k \). Then,

\[
R_t = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_l}
\]

and

\[
C_t = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_k}
\]

are the row-stabilizer and column-stabilizer of \( t \), respectively.

Example 4.5. For \( t \) in Example 3.4., \( R_t = S_{\{2,4,5,6\}} \times S_{\{3,7\}} \times S_{\{1\}} \) and \( C_t = S_{\{1,2,7\}} \times S_{\{3,5\}} \times S_{\{6\}} \times S_{\{4\}} \).

Given a subset \( H \subseteq S_n \), let \( H^+ = \sum_{\pi \in H} \pi \) and \( H^- = \sum_{\pi \in H} sgn(\pi)\pi \) be elements of \( \mathbb{C}[S_n] \). If \( H = \{\pi\} \), then we denote \( H^- \) by \( \pi^- \).

For a tableau \( t \), let \( \kappa_t = C_t^- = \sum_{\pi \in C_t} sgn(\pi) \pi \). Note that if \( t \) has columns \( C_1, C_2, \ldots, C_k \), then \( \kappa_t = \kappa_{C_1} \kappa_{C_2} \ldots \kappa_{C_k} \).

Definition 4.6. If \( t \) is a tableau, then the associated polytabloid is \( e_t = \kappa_t\{t\} \).

Example 4.7. For \( s \) in Example 3.6,

\[
\kappa_s = \kappa_{C_1} \kappa_{C_2} = (\epsilon - (1,3))(\epsilon - (2,4))
\]

Thus,

\[
e_t = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}
\]

Lemma 4.8. Let \( t \) be a tableau and \( \pi \) be a permutation. Then,

\( 1 \) \( R_{\pi t} = \pi R_t \pi^{-1} \)
\( 2 \) \( C_{\pi t} = \pi C_t \pi^{-1} \)
\( 3 \) \( \kappa_{\pi t} = \pi \kappa_t \pi^{-1} \)
\( 4 \) \( e_{\pi t} = \pi e_t \)

Proof.

(1)

\[
\sigma \in R_{\pi t} \iff \sigma\{\pi t\} = \{\pi t\}
\]

\[
\iff \pi^{-1}\sigma\{t\} = \{t\}
\]

\[
\iff \pi^{-1}\sigma \in R_t
\]

\[
\iff \sigma \in \pi R_t \pi^{-1}
\]
(2) and (3) can be shown analogously to (1).

(4) \[ e_{\pi t} = \kappa_{\pi t} \{ \pi t \} = \pi \kappa_{t} \pi^{-1} \{ \pi t \} = \pi \kappa_{t} \{ t \} = \pi e_{t} \]

□

**Definition 4.9.** For a partition \( \lambda \vdash n \), the corresponding Specht module, \( S^{\lambda} \), is the submodule of \( M^{\lambda} \) spanned by the polytabloids \( e_{t} \), where \( \text{sh} \ t = \lambda \).

**Proposition 4.10.** The \( S^{\lambda} \) are cyclic modules generated by any given polytabloid.

Given any two \( \lambda \)-tabloids \( t_{i}, t_{j} \) in the basis of \( M^{\lambda} \), let their inner product be

\[ < \{ t_{i} \}, \{ t_{j} \} > = \delta_{\{ t_{i} \}, \{ t_{j} \}} = \begin{cases} 1 & \text{if } \{ t_{i} \} = \{ t_{j} \} \\ 0 & \text{otherwise} \end{cases} \]

and extend by linearity in the first variable and conjugate linearity in the second to obtain an inner product on \( M^{\lambda} \).

**Lemma 4.11.** **(Sign Lemma)** Let \( H \leq S_{n} \) be a subgroup.

(1) If \( \pi \in H \), then \( \pi H^{-} = H^{-} \pi = \text{sgn}(\pi) H^{-} \)

(2) For any \( u, v \in M^{\lambda} \),

\[ < H^{-} u, v > = < u, H^{-} v > \]

(3) If the transposition \( (b c) \in H \), then we can factor

\[ H^{-} = k(\epsilon - (b c)) \]

where \( k \in \mathbb{C}[S_{n}] \).

(4) If \( t \) is a tableau with \( b, c \) in the same row of \( t \) and \( (b c) \in H \), then

\[ H^{-} \{ t \} = 0 \]

**Proof.**

(1)

\[
\pi H^{-} = \sum_{\sigma \in H} \text{sgn}(\sigma) \pi \sigma
\]

\[
= \sum_{\sigma \in H} \text{sgn}(\sigma) \pi \sigma
\]

\[
= \sum_{\tau \in H} \text{sgn}(\pi^{-1} \tau) \tau
\]

(by letting \( \tau = \pi \sigma \))

\[
= \sum_{\tau \in H} \text{sgn}(\pi^{-1}) \text{sgn}(\tau) \tau
\]

\[
= \text{sgn}(\pi^{-1}) \sum_{\tau \in H} \text{sgn}(\tau) \tau
\]

\[
= \text{sgn}(\pi) H^{-}
\]

\( H^{-} \pi = \text{sgn}(\pi) H^{-} \) can be proven analogously.
Corollary 4.12. Let \( t \) be a \( \lambda \)-tableau and \( s \) be a \( \mu \)-tableau, where \( \lambda, \mu \vdash n \). If \( \kappa_t\{s\} \neq 0 \), then \( \lambda \geq \mu \). Moreover, if \( \lambda = \mu \), then \( \kappa_t\{s\} = \pm e_t \)

Proof. Suppose \( b \) and \( c \) are two elements in the same row of \( s \). If they are in the same column of \( t \), then \( (bc) \in C_t \) and \( \kappa_t\{s\} = 0 \) by (4) of Sign Lemma. Hence, the elements in each row of \( s \) are all in different columns in \( t \), and \( \lambda \geq \mu \) by Dominance Lemma.

If \( \lambda = \mu \), then \( \{s\} = \pi\{t\} \) for some \( \pi \in C_t \). Then, by (4) of Sign Lemma,

\[
\kappa_t\{s\} = \kappa_t\pi\{t\} = sgn(\pi)\kappa_t\{t\} = \pm e_t
\]

\( \square \)

Corollary 4.13. If \( u \in M^\mu \) and \( \text{sh} t = \mu \), then \( \kappa_t u \) is a multiple of \( e_t \).

Proof. Let \( u = \sum_{i \in I} c_i s_i \) where \( c_i \in \mathbb{C} \) and \( s_i \) are \( \mu \)-tableaux. By Corollary 4.12., \( \kappa_t u = \sum_{i \in J} \pm c_i e_t = (\sum_{j \in J} \pm e_t) e_t \) for some \( J \subseteq I \).

\( \square \)

Theorem 4.14. (Submodule Theorem) Let \( U \) be a submodule of \( M^\mu \). Then,

\[
U \supseteq S^\mu \quad \text{or} \quad U \subseteq S^\mu^\perp
\]

Thus, \( S^\mu \) is irreducible.

Proof. For \( u \in U \) and a \( \mu \)-tableau \( t \), \( \kappa_t u = c e_t \) for some \( c \in \mathbb{C} \) by Corollary 4.13.. Suppose that there exists a \( u \) and \( t \) such that \( c \neq 0 \). Then, since \( U \) is a submodule, \( c e_t = \kappa_t u \in U \). Hence, \( e_t \in U \) and \( S^\mu \subseteq U \) since \( S^\mu \) is cyclic.

Otherwise, \( \kappa_t u = 0 \) for all \( u \in U \) and all \( \mu \)-tableau \( t \). Then, by (2) of Sign Lemma,

\[
< u, e_t >=< u, \kappa_t\{t\} >=< \kappa_t u, \{t\} >=< 0, \{t\} >= 0.
\]

Since \( e_t \) span \( S^\mu \), \( u \in S^\mu^\perp \) and \( U \subseteq S^\mu^\perp \).

\( \square \)

Proposition 4.15. If \( \theta \in \text{Hom}(S^\lambda, M^\mu) \) is nonzero, then \( \lambda \geq \mu \). Moreover, if \( \lambda = \mu \), then \( \theta \) is multiplication by a scalar.
Proof. Since $\theta \neq 0$, there exists a basis element $e_t \in S^\lambda$ such that $\theta(e_t) \neq 0$. Because $M^\lambda = S^\lambda \oplus S^\lambda_\perp$, we can extend $\theta$ to an element of $\text{Hom}(M^\lambda, M^\mu)$ by letting $\theta(S^\lambda_\perp) = \{0\}$. Then,

$$0 \neq \theta(e_t) = \theta(\kappa_t\{t\}) = \kappa_t\theta(\{t\}) = \kappa_t\left(\sum_i c_i\{s_i\}\right)$$

where $c_i \in \mathbb{C}$ and $s_i$ are $\mu$-tableaux. Hence, by Corollary 4.12, $\lambda \supseteq \mu$.

If $\lambda = \mu$, $\theta(e_t) = ce_t$ for some $c \in \mathbb{C}$ by Corollary 4.12. For any permutation $\pi$,

$$\theta(e_{\pi t}) = \theta(\pi e_t) = \pi\theta(e_t) = \pi(ce_t) = c\pi e_t$$

Thus, $\theta$ is multiplication by $c$.  

Theorem 4.16. The $S^\lambda$ for $\lambda \vdash n$ form a complete list of irreducible $S_n$-modules.

Proof. Since the number of irreducible modules equals the number of conjugacy classes of $S_n$ by Proposition 2.24, it suffices to show that they are pairwise inequivalent. Suppose $S^\lambda \cong S^\mu$. Then, there exists a nonzero $\theta \in \text{Hom}(S^\lambda, M^\mu)$ since $S^\lambda \subseteq M^\mu$. Thus, by Proposition 4.15, $\lambda \supseteq \mu$. Analogously, $\lambda \subseteq \mu$. Hence, $\lambda = \mu$.  

Corollary 4.17. The permutation modules decompose as

$$M^\mu = \bigoplus_{\lambda \supseteq \mu} m_{\lambda\mu} S^\lambda$$

where the diagonal multiplicity $m_{\mu\mu} = 1$.

Proof. If $S^\lambda$ appears in $M^\mu$ with nonzero multiplicity, then there exists a nonzero $\theta \in \text{Hom}(S^\lambda, M^\mu)$ and $\lambda \supseteq \mu$ by Proposition 4.15. If $\lambda = \mu$, then $m_{\mu\mu} = \dim \text{Hom}(S^\mu, M^\mu) = 1$ by Propositions 2.22 and 4.15.

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References