# REVERSE MATHEMATICS

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Abstract. In math we typically assume a set of axioms to prove a theorem. In reverse mathematics, the premise is reversed: we start with a theorem and try to determine the minimal axiomatic system required to prove the theorem (over a weak base system). This produces interesting results, as it can be shown that theorems from different fields of math such as group theory and analysis are in fact equivalent. Also, using reverse mathematics we can put theorems into a hierarchy by their complexity such that theorems that can be proven with weaker subsystems are "less complex". This paper will introduce three frequently used subsystems of second-order arithmetic, give examples as to how different theorems would compare in a hierarchy of complexity, and culminate in a proof that subsystem $ACA_0$ is equivalent to the statement that the range of every injective function exists.

Contents



## 1. Introduction

Reverse mathematics is a relatively new program in logic with the aim to determine the minimal axiomatic system required to prove theorems. We typically start from axioms $\mathcal{A}$ to prove a theorem $\tau$. If we could reverse this to show that the axioms follow from the theorem, then this would demonstrate that the axioms were necessary to prove the theorem. However, it is not possible in classical mathematics to start from a theorem to prove a whole axiomatic subsystem. A *weak base theory* $\mathcal{B}$ is required to supplement $\tau$. If $\mathcal{B} + \tau$ can prove $\mathcal{A}$, this proof is called a *reversal*. Then we can conclude that $\mathcal{A}$ and $\tau$ are equivalent over $\mathcal{B}$.

This paper will introduce reverse mathematics at a level accessible to undergraduate mathematics majors. No prior knowledge of logic is needed. This paper will draw heavily from Simpson's reverse mathematics text [2].

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## 2. Second order arithmetic ($Z_2$)

In reverse mathematics, subsystems of second-order arithmetic ($Z_2$) are most often used. $Z_2$ is a formal system consisting of language $L_2$ and some axioms. From these axioms, we can deduce formulas, called theorems of $Z_2$. A subsystem of second-order arithmetic is a formal system consisting of language $L_2$ and axioms that are theorems of $Z_2$; a subsystem consists of some of the theorems of $Z_2$, so it is just a fragment of $Z_2$.

The language of second-order arithmetic $L_2$ is a first-order theory and is two-sorted—that is, there are two kinds of variables. The first kind represent individual natural numbers and is denoted by lowercase letters. The second kind represent sets of natural numbers and is denoted by uppercase letters. The language has constants 0 and 1 and binary operations $\cdot$ and $+$. The atomic formulas of this language are $x = y$, $x < y$, and $x \in A$, where $x$ and $y$ are natural numbers and $A$ is any set variable. Formulas are constructed from atomic formulas using propositional connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$, and quantifiers $\forall$ (for all) and $\exists$ (exists).

The axioms of second-order arithmetic consist of:

1) the axioms of Peano arithmetic (such as the existence of additive and multiplicative identity, associativity and commutativity of addition and multiplication, the distributive law)
2) the induction axiom:

$$(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X) \tag{2.1}$$

3) the comprehension scheme:

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)) \tag{2.2}$$

where $\varphi(n)$ is any $L_2$ formula where $X$ does not occur freely. We use the comprehension scheme to show a set exists in $Z_2$ by defining the set abstractly with $\varphi(n)$. For example, if $X$ is the set of even numbers, then $\varphi(n)$ could be $\exists m(m+m = n)$. Also, for any finite set of natural numbers $X$, there is a formula $\varphi(n)$ that defines it. For the set $X=\{1, 23, 125\}$, one possible formula $\varphi(n)$ that can be used is one that formalizes the statement:

$$(\text{the } n^{th} \text{ prime divides } p^1 \cdot p^{23} \cdot p^{125})$$

where $p^i$ is the $i^{th}$ prime. This works because of the uniqueness of prime power decomposition.

In reverse mathematics, the five most commonly used subsystems in increasing logical strength are $RCA_0$, $WKL_0$, $ACA_0$, $ATR_0$, and $\Pi^1_1$-$CA_0$. The initial three subsystems will be discussed in subsequent sections.

The 0 subscript in the subsystem abbreviations means that induction is restricted. Let $X$ be the set that exists due to the comprehension scheme with $\varphi(n)$. Then the induction axiom becomes the second-order induction scheme:

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\, \varphi(n). \tag{2.3}$$

In the five aforementioned subsystems, we cannot use just any $L_2$ formula $\varphi(n)$ in the induction scheme. The formulas must have a specific form, hence the subsystems have restricted induction schemes.

## 3. Arithmetical Formulas

For each subsystem, induction is restricted to a certain level of arithmetical formula; there are limitations to what $\varphi(n)$ can be used in the induction scheme. Different types of arithmetical formula are denoted by $\Sigma^0_n, \Pi^0_n,$ and $\Delta^0_n$. The 0 superscript indicates that quantifiers ($\exists$ and $\forall$) range over numbers as opposed to sets of numbers.

The following expressions are *bounded number quantifiers*:

$$\forall n < t, \forall n \leq t, \exists n < t, \exists n \leq t.$$

A *bounded quantifier formula* is a formula whose quantifiers are all bounded number quantifiers. For example, $\exists m \leq n(n = m+m)$ is a bounded quantifier formula that asserts $n$ is even. The class of bounded-quantifier formulas is $\Sigma^0_0$ ($\Pi^0_0$).

An $L_2$ formula is $\Sigma^0_n$ if it has form:

$$(\exists y_1)(\forall y_2)(\exists y_3)\cdots(Qy_n)\ \theta$$

where $y_i$ are number variables and $\theta$ is a bounded quantifier formula. The quantifiers alternate between $\exists$ and $\forall$.

An $L_2$ formula is $\Pi^0_n$ if it has form:

$$(\forall y_1)(\exists y_2)(\forall y_3)\cdots(Qy_n)\ \theta$$

where $y_i$ are number variables and $\theta$ is a bounded quantifier formula.

An $L_2$ formula is $\Delta^0_n$ if it is both $\Sigma^0_n$ and $\Pi^0_n$.

We also use this hierarchy to classify sets as $\Sigma^0_n$, $\Pi^0_n$, and/or $\Delta^0_n$. A set is at a particular level of the hierarchy if it is defined by a formula at that level. For example, the set of even numbers $B$ is $\Sigma^0_0$ because a $\Sigma^0_0$ formula can be used to describe the set:

$$n \in B \leftrightarrow \exists m \leq n(n = m + m).$$

The $\Sigma^0_1$ *induction scheme* is the restriction of the second-order induction scheme as in Equation 2.3 to $\varphi(n)$ that are $\Sigma^0_1$. Similarly, the $\Delta^0_1$ *comprehension scheme* is the restriction of the comprehension scheme as in Equation 2.2 to $\varphi(n)$ that are $\Delta^0_1$. If a formula is of form, say, $\Sigma^0_n$, it may or may not be $\Delta^0_n$. We would have to check every $\Pi^0_n$ formula for equivalence to the $\Sigma^0_n$ formula. Thus, the $\Delta^0_1$ comprehension scheme consists of all formulas of form

$$\forall n(\psi(n) \leftrightarrow \varphi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is a $\Sigma^0_1$ formula and $\psi(n)$ is a $\Pi^0_1$ formula.

In all the subsystems, induction is restricted to $\Sigma^0_1$ formulas. The subsystems have different comprehension schemes, so the stronger the subsystem's comprehension scheme, the logically stronger the subsystem is.

## 4. Recursive comprehension axiom ($RCA_0$)

$RCA_0$ is typically used as the weak base theory in reverse math. Most theorems from mathematics are either equivalent to $RCA_0$, or equivalent to $WKL_0$, $ACA_0$, $ATR_0$, or $\Pi^1_1$-$CA_0$ over $RCA_0$. $ATR_0$ and $\Pi^1_1$-$CA_0$ are stronger subsystems that will not be discussed in this paper.

$RCA_0$ is a subsystem of second-order arithmetic that consists of the axioms of Peano arithmetic, the $\Sigma^0_1$ induction scheme, and the $\Delta^0_1$ comprehension scheme. A set $X$ is the comprehension of a $\Sigma^0_1$ formula if and only if it is recursively enumerable. Similarly, a set $X$ is the comprehension of a $\Delta^0_1$ formula if and only if it is computable (recursive)— equivalently, if and only if there is an algorithm that can determine whether or not a given natural number $n$ is in $X$.

A structure of a set or system describes its variables and non-logic symbols (constants, operations) that yield relations between variables. For example, the structure of the rational numbers is $(\mathbb{Q}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \times_{\mathbb{Q}}, 0_{\mathbb{Q}}, 1_{\mathbb{Q}}, <_{\mathbb{Q}}, =_{\mathbb{Q}})$. A model of a set of formulas is a structure with the same non-logic symbols, and all formulas in the set are in the model as well. Computability theorists define $\omega$-models ($\omega$ denotes the natural numbers, the range of the number variables in an $\omega$-model is the natural numbers) of subsystems. An $\omega$-model $\mathcal{S}$ of $RCA_0$ must satisfy the following properties:

1. $\mathcal{S} \neq \emptyset$
2. $A \in \mathcal{S}$ and $B \in \mathcal{S}$ imply $A \oplus B \in \mathcal{S}$
3. $A \in \mathcal{S}$ and $B \leq_T A$($B$ is Turing reducible to $A$) imply $B \in S$.

   [See [3] for a guide to computability theory.]

Not surprisingly, the minimum $\omega$-model of $RCA_0$ is the computable sets. More precisely, the minimum $\omega$-model has structure:

$$(\underbrace{\omega}_{\substack{\text{range}\\ \text{of the}\\ \text{number}\\ \text{variables}}}, \underbrace{\{X \subseteq \omega : X \text{ is computable}\}}_{\substack{\text{range of}\\ \text{the set}\\ \text{variables}}}, \underbrace{+, \times}_{\substack{\text{number}\\ \text{variable}\\ \text{operations}}}, \underbrace{0, 1}_{\substack{\text{number}\\ \text{variable}\\ \text{constants}}}, \underbrace{<}_{\substack{\text{number}\\ \text{variable}\\ \text{relation}}}).$$

$RCA_0$ says we can assume a set of natural numbers exists only if we can compute the set. Thus, $RCA_0$ cannot prove the existence of non-computable sets.

The following examples are computable, hence in $RCA_0$.

**Examples 4.1.** The following exist in $RCA_0$: (i) constant functions, (ii) function composition, and (iii) characteristic functions.

*Proof.* (i) Define a function $f = \{(x, y) | \varphi(x)\}$. In the tuple $(x, y)$, $x$ is the function's input, $y$ is the output, and $\varphi$ is the formula of the function that relates the two. Let $c \in \mathbb{N}$ be a constant. In the function $f = \{(x, y) | y = c\}$, the formula $y = c$ is $\Sigma^0_0$, so $f$ exists by $\Sigma^0_0$ comprehension.

(ii) Let $f : A \to B$ and $g : B \to C$ be two functions. Define their composition $h : A \to C$ by $h = \{(x, y) | x \in A \wedge \forall z((x, z) \in f \to (z, y) \in g)\}$, which is $\Pi^0_1$. Alternatively we can write $h = \{(x, y) | \exists z \in B((x, z) \in f \wedge (z, y) \in g)\}$, which is $\Sigma^0_1$. Hence, $h$ is $\Delta^0_1$ and exists by $\Delta^0_1$ comprehension.

(iii) A characteristic function of a set outputs 1 if the input is in the set and 0 if the input is not in the set. For a set $A$, the characteristic function

$$\chi_A = \{(x, y) | (x \in A \wedge y = 1) \vee (x \notin A \wedge y = 0)\}$$

exists by $\Sigma^0_0$ comprehension. In future proofs, $a \in A$ or $a \notin A$ is assumed provable in $RCA_0$ since $\chi_A(a)$ exists. □

A variety of mathematical objects can be encoded in $RCA_0$.

**Examples 4.2.** The following can be encoded in $RCA_0$: (i) Finite sets, tuples, numbers in (ii) $\mathbb{Z}$, (iii) $\mathbb{Q}$, and (iv) $\mathbb{R}$, and (v) computable sequences and functions.

*Proof.* (i) Finite sets and tuples can be encoded with prime numbers as discussed earlier (because of the uniqueness of prime power decomposition).

(ii) For $\mathbb{Z}$, each integer can be encoded as an $(\mathbb{N}, \mathbb{N})$ tuple.
Let $b \in \mathbb{N}$ be the tuple $(b, 0)$, so then integer $a = (m, n) = m - n$.
We can have the following operations and relations:

$$\begin{aligned}
(m,n) +_{\mathbb{Z}} (p,q) &= (m+p, n+q) \\
(m,n) -_{\mathbb{Z}} (p,q) &= (m+q, n+p) \\
(m,n) \times_{\mathbb{Z}} (p,q) &= (m \times p + n \times q, m \times q + n \times p) \\
(m,n) <_{\mathbb{Z}} (p,q) &\leftrightarrow m+q < n+p \\
(m,n) =_{\mathbb{Z}} (p,q) &\leftrightarrow m+q = n+p
\end{aligned}$$

In $RCA_0$, we can show $\mathbb{Z}, +, -, \times, 0, 1, <$ is an integral domain.

(iii) For $\mathbb{Q}$, recall that any rational number can be expressed as the quotient of $\frac{m}{n}$ where $m$ is an integer and $n$ is a positive integer.
Then any rational number can be encoded as a tuple $(\mathbb{Z}, \mathbb{Z}^+)$ with the following operations and relations:

$$\begin{aligned}
(m,n) +_{\mathbb{Q}} (p,q) &= (m \times q + n \times p, n \times q) \\
(m,n) -_{\mathbb{Q}} (p,q) &= (m \times q - n \times p, n \times q) \\
(m,n) \times_{\mathbb{Q}} (p,q) &= (m \times p, n \times q) \\
(m,n) <_{\mathbb{Q}} (p,q) &\leftrightarrow m \times q < n \times p \\
(m,n) =_{\mathbb{Q}} (p,q) &\leftrightarrow m \times q = n \times p
\end{aligned}$$

In $RCA_0$, we can show $\mathbb{Q}, +, -, \times, 0, 1, <$ is an ordered field.

(iv) For $\mathbb{R}$, a real number can be expressed as a sequence of rational numbers $\{q_k\}$ $(k \in \mathbb{N})$ such that $\forall k \forall i (|q_k - q_{k+i}| \leq \frac{1}{2^k})$.
With $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ numbers, two numbers $q$ and $q'$ are equal with the $\Delta^0_0$ formula $q = q'$. Two real numbers $\{q_k\}$ and $\{q'_k\}$ are equal if the following is true:

$$\forall k \left( |q_k - q'_k| \leq \frac{2}{2^k} \right)$$

In contrast, this formula is $\Pi^0_1$. An equivalent $\Sigma^0_1$ formula is needed for $\Delta^0_1$ comprehension, so proving two real numbers are equal in $RCA_0$ is problematic.

(v) Sequences and functions that are computable can be encoded in $RCA_0$ by definition of $RCA_0$. □

**Theorem 4.3.** *The following theorems are provable in $RCA_0$:*

*(i) the intermediate value theorem*
*(ii) the Baire category theorem*
*(iii) the Tietze extension theorem for complete separable metric spaces*
*(iv) the soundness theorem*
*(v) Gödel's completeness theorem*
*(vi) the Banach-Steinhaus theorem*

*[Various Authors]*

Although $RCA_0$ is the weakest subsystem, it is sufficient to prove basic properties of numbers as in Examples 4.2(ii-iv), theorems used in calculus, algebra, and analysis, even theorems used in topology such as the Baire category theorem.

Next we will prove primitive recursion and some results that will be used in the final proof.

**Theorem 4.4.** ***Primitive Recursion***
*The following is provable in $RCA_0$:*
*For any two functions $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^{k+2} \to \mathbb{N}$, there exists a unique function $h : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by*

$$h(0, n_1, \cdots, n_k) = f(n_1, \cdots, n_k)$$

$$h(m+1, n_1, \cdots, n_k) = g(h(m, n_1, \cdots, n_k), m, n_1, \cdots, n_k)$$

*Proof.* Let $\theta(s, m, \langle n_1, \cdots, n_k\rangle)$ be a formula that says: (i) $s \in \mathbb{N}^{<\mathbb{N}}$ is the code of the finite sequence $\langle s(0), \cdots, s(m)\rangle$, which has length $m + 1$; (ii) the sequence that $s$ codes for is recursively defined by $s(0) = f(n_1, \cdots, n_k)$, and for all $i < m$, $s(i + 1) = g(s(i), i, n_1, \cdots, n_k)$.

The formula $\exists s\ \theta(s, m, \langle n_1, \cdots, n_k\rangle)$ is $\Sigma^0_1$. Thus, for each fixed finite sequence $\langle n_1, \cdots, n_k\rangle \in \mathbb{N}^k$, we can prove there exists a sequence that fulfills (i) and (ii): $\exists s\ \theta(s, m, \langle n_1, \cdots, n_k\rangle)$ by $\Sigma^0_1$ induction on $m$.

If $\theta(s, m, \langle n_1, \cdots, n_k\rangle)$ and $\theta(s', m, \langle n_1, \cdots, n_k\rangle)$ hold, then $s(i) = s'(i)$ by induction on $i < m + 1$. In other words, for a fixed $m$ and $\langle n_1, \cdots, n_k\rangle$, the code $s$ for the sequence is unique. It follows from the previous sentence that for a fixed $m$ and $\langle n_1, \cdots, n_k\rangle$, the existence of a sequence that meets (i) and (ii) with last element $j$ is equivalent to the statement that all sequences that meet (i) and (ii) imply that $j$ is the last element. So for all $\langle n_1, \cdots, n_k\rangle \in \mathbb{N}^k$, $m$, and $j$,

$$\exists s(\theta(s, m, \langle n_1, \cdots, n_k\rangle) \wedge s(m) = j) \leftrightarrow \forall s(\theta(s, m, \langle n_1, \cdots, n_k\rangle) \to s(m) = j)$$

The left statement is $\Sigma^0_1$ and the right statement is $\Pi^0_1$, and since the statements are equivalent this is $\Delta^0_1$. Hence by $\Delta^0_1$ comprehension (the comprehension scheme of $RCA_0$), there exists the function $h : \mathbb{N}^{k+1} \to \mathbb{N}$ such that

$$h(m, n_1, \cdots, n_k) = j$$

if and only if $\exists s(\theta(s, m, \langle n_1, \cdots, n_k\rangle) \wedge s(m) = j)$. The function $h$ fulfills the properties of the theorem. □

**Theorem 4.5.** ***Minimization***
*The following is provable in $RCA_0$:*
*Let function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for all $\langle n_1, \cdots, n_k\rangle \in \mathbb{N}^k$ there exists $m \in \mathbb{N}$ such that $f(m, n_1, \cdots, n_k) = 1$. Then there exists $g : \mathbb{N}^k \to \mathbb{N}$ defined by*

$$g(n_1, \cdots, n_k) = \textit{least } m \textit{ such that } f(m, n_1, \cdots, n_k) = 1.$$

*Proof.* Define the function $g$ as a set of tuples ($\mathbb{N}^k$ [input], $\mathbb{N}$ [output]) as follows:

$$g = \{(\langle n_1, \cdots, n_k\rangle, m) | (\langle m, n_1, \cdots, n_k\rangle, 1) \in f \wedge \neg(\exists j < m)((\langle j, n_1, \cdots, n_k\rangle, 1) \in f)\}$$

This set exists by $\Sigma^0_0$ comprehension, so it exists in $RCA_0$. Also, $g$ defined as such fulfills the theorem's conditions. □

**Lemma 4.6.** *The following can be proven in $RCA_0$: for any infinite set $A \subseteq \mathbb{N}$, there exists a function $\pi_A : \mathbb{N} \to \mathbb{N}$ such that $\forall n(n \in A \leftrightarrow \exists m(\pi_A(m) = n))$ and $\forall k \forall m(k < m \to \pi_A(k) < \pi_A(m))$, that is, the function enumerates all the elements of $A$ in order.*

*Proof.* Define $\nu_A : \mathbb{N} \to \mathbb{N}$ by

$$\nu_A(m) = \text{least } n \text{ such that } n \in A \text{ and } n \geq m$$

Define $\pi_A : \mathbb{N} \to \mathbb{N}$ using primitive recursion (Theorem 4.4):

$$\begin{aligned} \pi_A(0) &= \nu_A(0) \\ \pi_A(m+1) &= \nu_A(\pi_A(m)+1) \end{aligned}$$

Using $\Sigma^0_0$ induction, we can see that $\pi_A$ fulfills the lemma's conditions. □

**Lemma 4.7.** *Let $\varphi(n)$ be a $\Sigma^0_1$ formula in which $X$ and function $f$ do not occur freely. The following is provable in $RCA_0$: either*
*(i) there exists a finite set $X$ such that*

$$\forall n(n \in X \leftrightarrow \varphi(n)),$$

*that is, there are only finitely many $n$ such that $\varphi(n)$ is true, or*
*(ii) there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall n(\varphi(n) \leftrightarrow \exists m(f(m) = n)),$$

*that is, there is a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ whose range is the $n$ that satisfy $\varphi(n)$.*

*Proof.* Suppose (i) is false, then we will show (ii) must be true. Formula $\varphi(n)$ is $\Sigma^0_1$, so we can rewrite it as $\exists j\ \theta(j,n)$ where $\theta(j,n)$ is $\Sigma^0_0$. Define a set

$$Y = \{(j,n) | \theta(j,n) \wedge \neg(\exists i < j)\theta(i,n)\},$$

which exists by $\Sigma^0_0$ comprehension. Since (i) is false, there are infinitely many $n$ such that $\varphi(n)$ holds, so the set $Y$ is infinite. By Lemma 4.6, there is a function $\pi_Y : \mathbb{N} \to \mathbb{N}$ that enumerates the elements of $Y$ in strictly increasing order. We define the second projection function $p_2 : \mathbb{N} \to \mathbb{N}$ as such:

$$\text{for all } j, n \in \mathbb{N}, \ \ p_2((j,n)) = n$$

Function $p_2$ exists by $\Sigma^0_0$ comprehension. Finally, let $f : \mathbb{N} \to \mathbb{N}$ be defined as the composition function $f(m) = p_2(\pi_Y(m))$, which meets the criterion in (ii). □

## 5. Weak König's lemma ($WKL_0$)

$WKL_0$ consists of the axioms of $RCA_0$ and also Weak König's Lemma, which states that *every infinite binary tree has an infinite path.*

$$WKL_0 = RCA_0 + \text{Weak König's Lemma}$$

After some definitions, the implications of this lemma will be expanded on below.

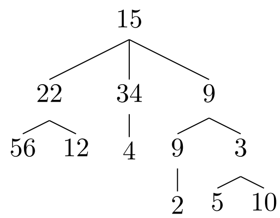
The set of all finite strings of natural numbers is denoted by $\mathbb{N}^{<\mathbb{N}}$. For example,

$$\langle 3, 35, 264, 6, 3, 2 \rangle \text{ and}$$
$$\langle 3, 264, 35, 6, 3, 2 \rangle$$

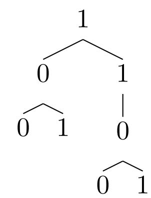
are two different strings of six natural numbers.

**Definition 5.1.** A set of strings of natural numbers $T$ is a *tree* if it is closed under initial segments, that is, for all $\sigma \in T$ and for all $\tau \preceq \sigma$ ($\tau$ is an initial segment of $\sigma$), we have $\tau \in T$.

**Definition 5.2.** The set of all finite strings of 0's and 1's is denoted by $\{0,1\}^{<\mathbb{N}}$. A *binary tree* is a subset of $\{0,1\}^{<\mathbb{N}}$.



A Tree



A Binary Tree

These finite trees have four levels. An infinite tree has infinitely many levels.

**Definition 5.3.** An *infinite path* through an infinite tree $T$ is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, the initial string $f[k] = \langle f(0), f(1), ...f(k-1)\rangle$ is in $T$. A path through a binary tree is a function $g : \mathbb{N} \to \{0,1\}$.

An interesting equivalence to Weak König's Lemma is $\Sigma^0_1$ separation, which states that given two $\Sigma^0_1$ formulas of number variable $n$ that are exclusive, there exists a set containing all $n$ satisfying one formula and none satisfying the other. Reversals in $WKL_0$ frequently make use of $\Sigma^0_1$ separation.

**Theorem 5.4.** *Over $RCA_0$, the following are equivalent:*
*1. $WKL_0$.*
*2. $\Sigma^0_1$ separation*
*Let $\varphi_0(n)$ and $\varphi_1(n)$ be $\Sigma^0_1$ formulas. If $\neg\exists n(\varphi_0(n) \wedge \varphi_1(n))$ then*

$$\exists X \forall n\, ((\varphi_0(n) \to n \in X) \wedge (\varphi_1(n) \to n \notin X)).$$

The proof can be found in [2]. This is a full reversal theorem, in that $WKL_0$ proves $\Sigma^0_1$ separation, and $\Sigma^0_1$ separation implies $WKL_0$ over $RCA_0$ (the reversal).

**Examples 5.5.** Over $RCA_0$, the following are equivalent:
- (i) $WKL_0$
- (ii) some properties of continuous real-valued functions on [0, 1] and compact metric spaces such as uniform continuity, the maximum principle, Riemann integrability, and Weierstrass approximation
- (iii) the completeness and compactness theorems in mathematical logic
- (iv) the existence of real closure for countable formally real fields
- (v) the uniqueness of algebraic closure of countable fields
- (vi) the existence of prime ideals and countable commutative rings
- (vii) the Brouwer and Schauder fixed point theorems
- (viii) the Peano existence theorem for solutions of ordinary differential equations
- (ix) the separable Hahn/Banach theorem
- (x) the Heine/Borel theorem for [0,1] and compact metric spaces

[Various Authors]

The statements in Examples 5.5 are equivalent to Weak König's Lemma, hence the statements are also equivalent to $WKL_0$ over $RCA_0$. All theorems that can be proven in $RCA_0$ can be proven in $WKL_0$ since $WKL_0$ consists of $RCA_0$ plus Weak König's Lemma. The addition of Weak König's Lemma to $RCA_0$ allows for the existence of non-computable sets.

The next logically stronger subsystem, $ACA_0$, is equivalent over $RCA_0$ to König's Lemma. König's Lemma states that every infinite, finitely branching tree has a path, so this is stronger than Weak König's Lemma which adds the condition that the trees must be binary trees. Hence, all theorems that can be proven in $WKL_0$ can be proven in $ACA_0$.

## 6. Arithmetical comprehension axiom ($ACA_0$)

$ACA_0$ is defined similarly to $RCA_0$, but is stronger. $ACA_0$ also consists of the axioms of Peano arithmetic, but has a comprehension scheme for all arithmetical formulas (formulas with no set quantifiers), not just $\Delta^0_1$ formulas. The relationship between $RCA_0$ and $ACA_0$ can be represented by the following equation:

$$ACA_0 = RCA_0 + \text{Arithmetical Comprehension}$$

**Examples 6.1.** Over $RCA_0$, the following are equivalent:
- (i) $ACA_0$
- (ii) sequential compactness of [0,1] and compact metric spaces
- (iii) the existence of the strong algebraic closure of a countable field
- (iv) every countable vector space over $\mathbb{Q}$ has a basis
- (v) every countable commutative ring has a maximal ideal
- (vi) the uniqueness of the divisible closure of a countable Abelian group
- (vii) König's lemma for subtrees of $\mathbb{N}^{\mathbb{N}}$
- (viii) Ramsey's theorem for colorings of $[\mathbb{N}]^3$
- (ix) the least upper bound principle for sequences of real numbers

[Various Authors]

Finally, we will show an example of how to do a reverse mathematics proof.

**Theorem 6.2.** *$ACA_0$ over $RCA_0$ is equivalent to the following:*
- *(i) $\Sigma^0_1$ comprehension*
- *(ii) The range of every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ exists*

*Proof.* Recall the discussion at the beginning of the section on $RCA_0$ as to how reverse mathematics is performed. First, we want to show the axioms $\mathcal{A} = ACA_0$ prove theorem $\tau$, where $\tau$ is statements (i) and (ii). We use weak base theory $\mathcal{B} = RCA_0$. To do the reversal proof we will show $RCA_0 + \tau$ imply $ACA_0$. This will allow us to conclude that Theorem 6.2 is true.

**Proof that** $ACA_0 \to \tau$
$ACA_0 \to$ **(i)**: By definition, $ACA_0$ implies (i).
**(i)** $\to$ **(ii)**: Recalling Equation 2.2, (i) $\Sigma^0_1$ comprehension is the same as $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$ where $\varphi(n)$ is restricted to a $\Sigma^0_1$ formula in which $X$ does not occur freely.
The existence of the range is equivalent to saying:
there exists a set $X \subseteq \mathbb{N}$ ($X$ is the range) such that $\forall n(n \in X \leftrightarrow \exists m(f(m) = n))$; $\varphi(n)$ as in the $\Sigma^0_1$ comprehension formula is $\exists m(f(m) = n)$, which is a $\Sigma^0_1$ formula.

**Proof that** $RCA_0 + \tau \to ACA_0$
**(ii)** $\to$ **(i)**: This follows from Lemma 4.7.
**(i)** $\to ACA_0$: We need to show that $\Sigma^0_1$ comprehension implies arithmetical comprehension. Every arithmetical formula consists of alternating $\forall$ and $\exists$ quantifiers (we do not know if the first quantifier is a $\forall$ or if it is a $\exists$) followed by a $\Sigma^0_0$ bounded quantifier formula. If the formula starts with a $\forall$ quantifier, then we can just put

a dummy variable with an existential quantifier at the beginning of the formula. Then we can write each arithmetical formula as a $\Sigma^0_k$ formula for some $k \in \mathbb{N}$, so it is sufficient to prove $\Sigma^0_1$ comprehension implies $\Sigma^0_k$ comprehension. We can do this using induction on $k \in \mathbb{N}$.
Base case: this is trivial for $k = 0, 1$.
Inductive step: Assuming $\Sigma^0_k$ comprehension, we need to show $\Sigma^0_{k+1}$ comprehension. Let $\varphi(n)$ be $\Sigma^0_{k+1}$ for $k \geq 1$. Then we can write $\varphi(n)$ as $\exists j\ \theta(n, j)$ where $\theta(n, j)$ is $\Pi^0_k$. Let set $Y = \{(n, j) | \neg\theta(n, j)\}$, which exists by $\Sigma^0_k$ comprehension. By $\Sigma^0_1$ comprehension, let set $X = \{n | \exists j\ ((n, j) \notin Y)\}$. Then $n \in X$ if and only if $\exists j\ \theta(n, j)$, which is equivalent to $\varphi(n)$. □

## 7. Conclusion

An interesting aside is that the five most commonly used subsystems in reverse mathematics, $RCA_0$, $WKL_0$, $ACA_0$, $ATR_0$, and $\Pi^1_1$-$CA_0$, correspond to philosophically motivated programs in foundations of mathematics: Bishop's constructivism, Hilbert's finitistic reductionism, Weyl's predicativity as developed by Feferman, predicative reductionism as developed by Friedman and Simpson, and impredicativity, respectively. By studying reverse mathematics, we gain insight into mathematical philosophy and the implications of using different programs.

Reverse mathematics provides a new lens with which we may examine theorems. We can prove that theorems from different fields of math have the same logical strength. We can also show that as we go from theorems provable in $RCA_0$ (Theorem 4.3) to those in $WKL_0$ (Examples 5.5) to those in $ACA_0$ (Examples 6.1), the theorems are increasing in logical strength. An hierarchy of logical strength exists, and can be extended by other subsystems such as the logically stronger $ATR_0$ and $\Pi^1_1$-$CA_0$.

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