REPRESENTATIONS OF THE SYMMETRIC GROUP

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Abstract. I will introduce the topic of representation theory of finite groups by investigating representations of \( S_3 \) and \( S_4 \) using character theory. Then I will generalize these examples by describing all irreducible representations of any symmetric group on \( n \) letters. Finally, I will briefly discuss how to discover irreducible representations of any group using Schur Functors, which are constructed using the irreducible representations of \( S_n \). This paper assumes familiarity with group theory, \( FG \)-modules, linear algebra, and category theory.

Contents

1. Introduction 1
2. Basic Definitions and Complete Reducibility 2
3. Characters and Hands-on Examples Involving Them 4
4. Representations of \( S_n \) 9
5. Schur Functors 13
Acknowledgments 14
References 14

1. Introduction

A representation of a group \( G \) is a homomorphism \( \rho \) from \( G \) into the general linear group of some vector space \( V \). When such a map exists, if we write \( g \cdot v \) rather than \( \rho(g)(v) \) to denote the image of a vector \( v \) in \( V \) under the automorphism \( \rho(g) \), then we can think of \( V \) as a \( CG \)-module, since for any vectors \( v \) and \( w \), any group elements \( g \) and \( h \), and any scalar \( c \), the following properties hold:

1. \( \rho(g)(v) = g \cdot v \) is in \( V \).
2. \( \rho(1) = I \in GL(V) \) by the homomorphism property, so \( \rho(1)(v) = I(v) = v \), which means \( 1 \cdot v = v \).
3. \( (\rho(g) \circ \rho(h))(v) = \rho(g)(\rho(h)(v)) \), so \( (gh) \cdot v = g \cdot (h \cdot v) \).
4. \( \rho(g)(cv) = c\rho(g)(v) \) by linearity, so \( g \cdot cv = c(g \cdot v) \).
5. \( \rho(g)(v + w) = \rho(g)(v) + \rho(g)(w) \) by linearity, so \( g \cdot (v + w) = g \cdot v + g \cdot w \).

When we know the particular \( CG \)-module structure of \( V \), we often call \( V \) itself a representation of \( G \). A subspace \( W \) of \( V \) is a subrepresentation if it is invariant under the action of \( G \). A representation \( V \) of \( G \) is irreducible if it has no proper nontrivial subrepresentations. Otherwise, it is reducible. As we will soon see, every representation of a finite group over a finite-dimensional complex vector space can be expressed as a direct sum of irreducible representations (over finite-dimensional
complex vector spaces) of the group. Two main goals of representation theory are to describe all irreducible representations of a given group \( G \) and to find a sure-fire method of decomposing any arbitrary representation of \( G \) into a direct sum of irreducible representations. We will accomplish both of these goals for the symmetric group on \( n \) letters \( S_n \).

2. Basic Definitions and Complete Reducibility

We begin by describing some common representations and a few ways of constructing new representations from known ones. Assume that all groups mentioned in this paper (except in Section 5) are finite and that all vector spaces mentioned are finite-dimensional over \( \mathbb{C} \).

Definition 2.1. The trivial representation of a finite group \( G \) is \( \mathbb{C} \) equipped with the trivial action of \( G \): \( gx = x \) for every \( x \) in \( \mathbb{C} \) and for every \( g \) in \( G \). Note that every finite group has the trivial representation, and since \( \mathbb{C} \) has no proper nontrivial subspaces, it is irreducible, as is any one-dimensional representation.

Definition 2.2. Let \( X \) be any finite \( G \)-set. Let \( W \) be the vector space generated by the basis \( \{ e_x \mid x \in X \} \). Define the action of \( G \) on \( W \) by

\[
g \cdot (a_1e_{x_1} + a_2e_{x_2} + \cdots + a_m e_{x_m}) = a_1e_{gx_1} + a_2e_{gx_2} + \cdots + a_m e_{gx_m}.
\]

Such a \( W \) is called a permutation representation of \( G \). Notice that the subspace spanned by the vector \( e_{x_1} + e_{x_2} + \cdots + e_{x_m} \) is invariant under the action of \( G \) because each element of \( G \) simply “shuffles” the addends but does not change the sum. Thus, every permutation representation has a nontrivial subrepresentation and is therefore reducible.

Definition 2.3. In Definition 2.2, replace the arbitrary \( G \)-set \( X \) by \( G \) itself under the action of left multiplication. In this case, \( W \) is called the regular representation of \( G \).

Definition 2.4. For a symmetric group \( S_n \), the alternating representation is \( \mathbb{C} \) equipped with the action

\[
\sigma \cdot v = \begin{cases} v, & \text{if } \sigma \text{ is an even permutation} \\ -v, & \text{if } \sigma \text{ is an odd permutation} \end{cases}
\]
or equivalently, \( \rho(\sigma) = \text{sgn}(\sigma)I \) for every \( \sigma \) in \( S_n \). Note that any \( S_n \) where \( n \geq 2 \) has the alternating representation, and since this representation is one-dimensional, it is irreducible.

Definition 2.5. For any \( n \), let \( \{ e_1, e_2, \ldots, e_n \} \) be the standard basis for \( \mathbb{C}^n \). Define the action of \( S_n \) on \( \mathbb{C}^n \) to be

\[
\sigma(a_1e_1 + a_2e_2 + \cdots + a_n e_n) = a_1e_{\sigma(1)} + a_2e_{\sigma(2)} + \cdots + a_n e_{\sigma(n)}.
\]

This is a permutation representation of \( S_n \). Again, notice that the one-dimensional subspace of \( \mathbb{C} \) spanned by \( e_1 + e_2 + \cdots + e_n \) is invariant under the action of \( S_n \). Therefore, \( \langle e_1 + e_2 + \cdots + e_n \rangle \) is a subrepresentation of \( \mathbb{C}^n \). Its orthogonal complement \( V = \{(x_1, x_2, \ldots, x_n) \mid x_1 + x_2 + \cdots + x_n = 0 \} \) is also invariant and therefore a subrepresentation. \( V \) is called the standard representation of \( S_n \).
Remark 2.6. It can be shown that given any representation \( V \) of a group \( G \), its dual space, its symmetric powers, and its alternating powers are also representations of \( G \). Furthermore, given two representations \( V \) and \( W \) of \( G \), their tensor product and direct sum are also representations, as is the vector space \( \text{Hom}(V,W) \). In Section 5, we will see how we can construct new representations (of any group) from old using special functors.

The following results show that the irreducible representations of a given group \( G \) are the “building blocks” for all of its other representations.

Lemma 2.7. Any representation \( V \) of a finite group \( G \) can be given a \( G \)-invariant inner product, meaning that for any \( h \) in \( G \) and for any \( v_1, v_2 \in V \),

\[
\langle hv_1, hv_2 \rangle = \langle v_1, v_2 \rangle.
\]

Proof. Let \( \langle \cdot, \cdot \rangle \) be any positive-definite Hermitian inner product on \( V \). Define a new Hermitian inner product on \( V \) in the following way:

\[
\langle v_1, v_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \langle gv_1, gv_2 \rangle, \text{ for any } v_1, v_2 \in V.
\]

Now see that for any \( h \) in \( G \) and for any \( v_1, v_2 \in V \), we have:

\[
\langle hv_1, hv_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \langle ghv_1, ghv_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \langle (gh)v_1, (gh)v_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \langle gv_1, gv_2 \rangle, \text{ because for every } g \in G, gh \text{ is in } G \text{ also}
\]

\[
= \langle v_1, v_2 \rangle.
\]

Lemma 2.8. Let \( V \) be a representation of a finite group \( G \), and let \( W \) be a subrepresentation of \( V \). Then \( W^\perp \), the orthogonal complement to \( W \) inside \( V \) under the \( G \)-invariant inner product described above, is also a subrepresentation of \( V \).

Proof. Fix an \( x \) in \( W^\perp = \{ x \in V \mid \langle w, x \rangle = 0 \text{ for every } w \in W \} \). Because our inner product is \( G \)-invariant, \( \langle w, gx \rangle = \langle w, x \rangle = 0 \) for any \( g \) in \( G \) and for any \( w \) in \( W \). Therefore, for any \( x \) in \( W^\perp, gx \) is in \( W^\perp \) also. This shows that \( W^\perp \) is invariant under the action of \( G \).

Exercise 2.9. Using Lemma 2.8, show that any representation of a finite group \( G \) is the direct sum of irreducible representations.

Definition 2.10. Let \( V \) and \( W \) be representations of a finite group \( G \). A map between \( V \) and \( W \) as representations is a vector space map \( \phi : V \to W \) such that \( g \cdot \phi(v) = \phi(g \cdot v) \) for every \( v \) in \( V \) and for every \( g \) in \( G \).

Proposition 2.11. The kernel of \( \phi \) is a subrepresentation of \( V \), and the image of \( \phi \) is a subrepresentation of \( W \).

Proof. Fix a \( g \) in \( G \). Since \( \phi \) is a map between representations, for any \( x \) in \( V \) we have \( \phi(g \cdot x) = g \cdot (\phi(x)) = g \cdot (0) = \rho(g)(0) = 0 \) by linearity of the transformation \( \rho(g) \). Thus, \( \ker \phi \) is invariant under the action of \( G \).
Again, fix a \( g \) in \( G \). Let \( z \) be any vector in the image of \( \phi \). Then \( g \cdot z = g \cdot \phi(x) \)
for some \( x \) in \( V \). And since \( \phi \) is a map between representations, we have \( g \cdot \phi(x) = \phi(g \cdot x) \), which is in the image since for every \( x \) in \( V \), \( g \cdot x \) is also in \( V \). Therefore, \( \text{im} \phi \) is also invariant under the action of \( G \).

\( \square \)

**Lemma 2.12. (Schur’s Lemma)** Let \( X \) and \( Y \) be two distinct, irreducible representations of \( G \), and let \( \phi : X \to Y \) be a map of representations between them. Then \( \phi \) is either an isomorphism or the zero map.

**Proof.** By Proposition 2.11, \( \ker \phi \) and \( \text{im} \phi \) are subrepresentations of \( X \) and \( Y \), respectively. Since \( X \) is irreducible, it can have no proper nontrivial subrepresentations. Therefore, \( \ker \phi \) is either \( \{0\} \) or all of \( X \). If \( \ker \phi = \{0\} \), then \( \phi \) is one-to-one. Since \( Y \) is also irreducible, again, either \( \text{im} \phi = \{0\} \) or \( \text{im} \phi = Y \). If the image is all of \( Y \), then \( \phi \) is onto, making it an isomorphism. If the image is \( \{0\} \), then \( \phi \) is the zero map. If, on the other hand, \( \ker \phi = X \), then \( \phi \) sends every vector in \( X \) to 0, making it the zero map.

\( \square \)

**Theorem 2.13.** Any representation \( V \) of a finite group \( G \) can be written uniquely as a direct sum of the form

\[ V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}, \]

where the \( V_i \)’s are distinct irreducible representations of \( V \) and the multiplicity \( a_i \) of each \( V_i \) is unique.

**Proof.** Suppose \( V \) can also be expressed as

\[ V = W_1^{\oplus b_1} \oplus W_2^{\oplus b_2} \oplus \cdots \oplus W_m^{\oplus b_m}. \]

Let \( \phi : V \to V \) be the identity map. Then by Schur’s Lemma, restricted to each irreducible component \( V_i^{\oplus a_i} \), \( \phi \) is an isomorphism between \( V_i^{\oplus a_i} \) and the component \( W_j^{\oplus b_j} \) for which \( V_i \) is isomorphic to \( W_j \). \( \square \)

### 3. Characters and Hands-on Examples Involving Them

This section introduces characters and how they can be used to find irreducible representations and decompositions of arbitrary representations.

**Definition 3.1.** Let \( V \) be a representation of a finite group \( G \), and let \( \rho \) be the associated group homomorphism. The **character** \( \chi_V \) of \( V \) is the function from \( G \) into \( \mathbb{C} \) given by

\[ \chi_V(g) = \text{Tr}(\rho(g)) \]

**Remark 3.2.** Since conjugate matrices have the same trace, for a fixed element \( g \) in \( G \), \( \text{Tr}(\rho(g)) = \text{Tr}(\rho(h)\rho(g)\rho(h^{-1})) = \text{Tr}(\rho(hgh^{-1})) \) for any \( h \) in \( G \). This follows from the linearity of \( \rho \). Therefore, the character of any representation \( V \) of \( G \) is, in fact, a **class function**, which is a function that is constant on conjugacy classes.

It turns out that computing the characters of permutation representations is very easy if we use the next theorem:

**Theorem 3.3. (The Fixed Point Formula)** Let \( G \) be a finite group, and let \( X \) be a finite \( G \)-set. Let \( V \) be the associated permutation representation of \( G \) as described in Definition 2.2. Then for every element \( g \) in \( G \), \( \chi_V(g) \) is the number of elements in \( X \) left fixed by the action of \( g \).
Proof. The matrix $M$ associated with the action of $g$ is a permutation matrix. For example, suppose that $X$ has four elements $x_1, x_2, x_3,$ and $x_4$ and that $\rho(g)$ permutes the basis vectors of $V$ by sending $e_{x_1}$ to $e_{x_3}$, $e_{x_2}$ to itself, $e_{x_3}$ to $e_{x_1}$, and $e_{x_4}$ to itself. Then $M$ is the $4 \times 4$ matrix
\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
In general, if $gx_i = x_j$ for some $x_i$ in $X$, then $M$ will have a 1 in the $i$-th column and $j$-th row, and zeroes in all other entries of that column. In particular, if $gx_i = x_i$, then $ge_{x_i} = gx_i = e_{x_i}$, meaning that $M$ has a 1 in the $i$-th row and $i$-th column. Therefore, the trace of $M$ is the number of 1’s along the diagonal, which is exactly the number of points left fixed by $g$. □

Proposition 3.4. For any representations $V$ and $W$ of a group $G$, $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

Proof. This is left as an exercise, but it simply involves counting the number of eigenvalues of any transformation $\rho(g)$. □

The set of class functions on a finite group $G$ is a vector space. We can define the following inner product on this space by:
\[
\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \beta(g).
\]

Theorem 3.5 and Corollaries 3.6 through 3.9 below allow us to decompose any representation of a finite group by applying this inner product to characters. Proofs can be found in [3] on pages 16, 17, and 22 and in [4] on pages 137 through 144.

Theorem 3.5. The set of character functions of the irreducible representations of $G$ is orthonormal with respect to this inner product.

Corollary 3.6. Any representation of $G$ is determined by its character.

Corollary 3.7. A representation $V$ of $G$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Corollary 3.8. Let $W$ be any representation of $G$. As we know, $W$ can be written uniquely as a direct sum of irreducible representations in the form
\[
W = V_1^a_1 \oplus V_2^a_2 \oplus \cdots \oplus V_k^a_k.
\]
In this decomposition,
\begin{itemize}
  \item $a_i = \langle \chi_W, \chi_{V_i} \rangle$ for every $i$,
  \item $\langle \chi_W, \chi_W \rangle = \sum_{i=1}^k a_i^2$,
  \item $\sum_{i=1}^k (\dim(V_i))^2 = |G|$, and
  \item for any $g$ in $G$ that is not the identity, $\sum_{i=1}^k (\dim(V_i)) \cdot \chi_{V_i}(g) = 0$.
\end{itemize}

Corollary 3.9. The number of irreducible representations of a finite group $G$ is equal to the number of conjugacy classes in $G$.

Example 3.10. Let’s see what these results can tell us about the group $S_3$. We know right away from Corollary 3.9 that $S_3$ has exactly three irreducible representations. This is because in a symmetric group, each equivalence class of permutations
of a certain cycle type constitutes a conjugacy class, and \( S_3 \) has three cycle-types; namely, the identity, transpositions, and 3-cycles. The trivial and alternating representations (let’s denote them by \( U \) and \( U' \), respectively) of \( S_3 \) are irreducible because they are one-dimensional. Since in the trivial representation \( \rho(g) \) is the \( 1 \times 1 \) identity matrix \([1]\) for every \( g \), we know that \( \chi_U(g) = \text{Tr}[1] = 1 \) for every \( g \). In the alternating representation, we have \( \chi_{U'}(\iota) = \text{Tr}[1] = 1 \), \( \chi_{U'}(12) = \text{Tr}[-1] = -1 \), and \( \chi_{U'}(123) = \text{Tr}[1] = 1 \). There now remains only one more irreducible representation for us to find. It cannot be any sort of permutation representation, because those are all reducible. Thus, it is worth checking the standard representation \( V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\} \). By definition, \( C_3 = \langle e_1 + e_2 + e_3 \rangle \oplus V \). Note that \( \langle e_1 + e_2 + e_3 \rangle \) under the permutation action of \( S_3 \) is isomorphic to \( C \) under the trivial action of \( S_3 \), so it is the trivial representation \( U \). Since \( S_3 \) acts on \( C_3 \) by permuting the three standard basis vectors, by the Fixed-Point Formula we have \( \chi_{C_3}(\iota) = 3 \), \( \chi_{C_3}(12) = 1 \), and \( \chi_{C_3}(123) = 0 \). Also, because \( C_3 \cong U \oplus V \), by Proposition 3.4 we have \( \chi_{C_3} = \chi_U + \chi_V \). Therefore, \( C_3 \) has values \( 3 - 1 = 2 \), \( 1 - 1 = 0 \), and \( 0 - 1 = -1 \) on the conjugacy classes of \( \iota \), \( (12) \) and \( (123) \), respectively. Notice that

\[
\langle \chi_{C_3}, \chi_{C_3} \rangle = \frac{1}{|S_3|} (2^2 \cdot 1 + 0^2 \cdot 3 + (-1)^2 \cdot 2) = \frac{1}{6} \cdot 6 = 1.
\]

Therefore, by Corollary 3.7, \( V \) is irreducible. We found it! We can summarize the work we did into the following simple array called the character table for \( S_3 \):

<table>
<thead>
<tr>
<th>Conjugacy Class Sizes</th>
<th>1</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representatives</td>
<td>( \iota )</td>
<td>(12)</td>
<td>(123)</td>
</tr>
<tr>
<td>Trivial Representation</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Alternating Representation ( U' )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Standard Representation ( V )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Remark 3.11. Notice that for any representation \( W \) of a finite group \( G \), \( \chi_W(1) = \text{Tr}(\rho(1)) = \text{Tr}(I) \). This is simply the number of columns of \( I \), which is the dimension of \( W \). Thus, the first column of the character table above shows the dimension of each irreducible representation. We can use this and other handy information from the table to decompose any other representation of \( S_3 \).

Example 3.12. Since \( S_3 \) acts on the vertices of the equilateral triangle by permuting its three diagonals, there is a permutation representation associated with the resulting action on the set of vertices. If we label our triangle as shown,

then we see that the transposition \( (12) \) acts on the triangle by interchanging diagonals 1 and 2. This is a reflection about diagonal 2, and it interchanges vertices.
A and B but leaves vertex C fixed. Similarly, the other transpositions in $S_3$ each leave exactly one vertex fixed. The element $(123)$ rotates the triangle by 120 degrees clockwise and therefore leaves no vertices fixed. Finally, the identity element leaves all three vertices fixed. By the Fixed Point Formula, then, the character of this representation has values 3, 1, and 0 on the conjugacy classes consisting of the identity, transpositions, and 3-cycles, respectively. Therefore, we have

$$\langle \chi_W, \chi_U \rangle = \frac{1}{|S_3|} (3^2 \cdot 1 + 1^2 \cdot 3 + 0^2 \cdot 2) = \frac{1}{6} (9 + 3 + 0) = \frac{1}{6} \cdot 12 = 2.$$ 

By Corollary 3.8, $\langle \chi_W, \chi_U \rangle$ is the sum of the squares of the multiplicities of the irreducible representations in the decomposition of $W$. Since $2 = 1^2 + 1^2$, and this is the only way to write 2 as the sum of squares of nonnegative integers, it must be that $W$ is the direct sum of exactly two distinct irreducible representations of $S_3$. To find out which two, we simply take the product of $\chi_W$ with the character of each irreducible representation as follows:

$$\langle \chi_W, \chi_V \rangle = \frac{1}{6} (3 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 3 + 0 \cdot 1 \cdot 2) = 1,$n \rangle = \frac{1}{6} (3 \cdot 1 \cdot 1 + (-1) \cdot 3 + 0 \cdot 1 \cdot 2) = 0,$$

$$\langle \chi_W, \chi_{V'} \rangle = \frac{1}{6} (3 \cdot 2 \cdot 1 + 1 \cdot 0 \cdot 3 + 0 \cdot (-1) \cdot 2) = 1.$$

By Corollary 3.8, the trivial representation $U$ appears in the decomposition of $W$ once, the standard representation $V$ appears once, and the alternating representation $U'$ does not appear at all. Therefore, we have the decomposition $W \cong U \oplus V$.

**Example 3.13.** As mentioned before, any tensor power of a known representation of a group is again a representation. Therefore, the $n$-th tensor power of the standard representation $V$ of $S_3$ is another representation of $S_3$. To find its decomposition, first note that Proposition 3.4 implies $\chi_{V^\otimes n} = (\chi_V)^n$. Therefore, we have $\chi_{V^\otimes n}((12)) = 2^n, \chi_{V^\otimes n}((123)) = 0^n = 0$, and $\chi_{V^\otimes n}((123)) = (-1)^n$. Take the product of $\chi_{V^\otimes n}$ with the character of each irreducible:

$$\langle \chi_{V^\otimes n}, \chi_V \rangle = \frac{1}{6} (2^n \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 3 + (-1)^n \cdot 1 \cdot 2) = \frac{1}{6} (2^n + 2(-1)^n) = \frac{1}{3} (2^{n-1} + (-1)^n),$$

$$\langle \chi_{V^\otimes n}, \chi_{V'} \rangle = \frac{1}{6} (2^n \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 3 + (-1)^n \cdot 1 \cdot 2) = \frac{1}{6} (2^n + 2(-1)^n) = \frac{1}{3} (2^{n-1} + (-1)^n),$$

$$\langle \chi_{V^\otimes n}, \chi_{V''} \rangle = \frac{1}{6} (2^n \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 3 + (-1)^n \cdot 1 \cdot 2) = \frac{1}{6} (2^n + 2(-1)^n) = \frac{1}{3} (2^{n-1} + (-1)^n + 1).$$

This gives us the decomposition

$$V^\otimes n \cong U^{\frac{1}{3}}(2^{n-1} + (-1)^n) \oplus U'^{\frac{1}{3}}(2^{n-1} + (-1)^n) \oplus V^{\frac{1}{3}}(2^n + (-1)^{n+1}).$$

**Example 3.14.** Now let’s compute the character table of $S_4$. Since there are five cycle-types (i.e., conjugacy classes) in $S_4$, we know right away that $S_4$ has exactly five irreducible representations. The trivial and alternating representations $U$ and $U'$ are irreducible, and we know their characters. The next easy candidate to check is the standard representation $V$. It is left as an exercise to show that $V$ and the representation $V' = V \otimes U$ are both irreducible. (You will need Proposition 3.4 and Corollary 3.7.) It remains now to find the character values of the final irreducible representation, which we will call $W$. Let $a, b, c, d,$ and $e$ denote the values of $\chi_W$ on
the conjugacy classes of \( \iota, (12), (123), (1234) \) and \((12)(34)\), respectively. Corollary 3.8 and the entries in the first column that we know so far give us

\[
\sum_{i=1}^{5} (\dim(V_i))^2 = |S_4| \implies 1^2 + 1^2 + 3^2 + 3^2 + a^2 = 24 \implies a^2 = 4 \implies a = 2.
\]

Now we can use the last part of Corollary 3.8 to compute \( b \):

\[
\sum_{i=1}^{5} (\dim V_i) \chi_{V_i}(g) = 0 \implies 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 3 + (-1) \cdot 3 + b \cdot 2 = 0 \implies 2b = 0 \implies b = 0.
\]

We can compute \( c, d \) and \( e \) in the same way, giving us the complete character table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>6</th>
<th>8</th>
<th>6</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_4 )</td>
<td>( \iota )</td>
<td>(12)</td>
<td>(123)</td>
<td>(1234)</td>
<td>(12)(34)</td>
</tr>
<tr>
<td>( U )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( U' )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( V )</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( V' )</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( W )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Although for now the only thing we know about \( W \) is its character, in Section 4 we will learn the tools for explicitly describing the vector space \( W \) itself.

**Example 3.15.** \( S_4 \) acts on the faces, vertices and edges of the cube by permuting the four long diagonals inside the cube and thus has a permutation representation associated with each action. We will decompose each representation. Label the cube as shown:

![Cube Diagram](image)

**Faces:** Call this permutation representation \( X \). The identity permutation does not move the cube at all, leaving the 6 faces fixed. The transposition \((12)\) rotates the cube 180 degrees about the axis connecting the midpoints of edges \( a \) and \( b \). This motion moves every face to a new location and therefore fixes no faces. The 3-cycle \((123)\) rotates the cube 120 degrees about long diagonal number 4 and also fixes no faces. The 4-cycle \((1234)\) rotates the cube 90 degrees about the axis \( z \), leaving the top and bottom faces fixed but moving the other four around. Finally, \((12)(34)\) rotates the cube 180 degrees about \( z \), again leaving only the top and bottom faces fixed. Therefore, by the Fixed Point Formula, \( \chi_X \) has the values 6, 0, 0, 2, 2 on the conjugacy classes of \( \iota, (12), (123), (1234), \) and \((12)(34)\), respectively. This gives us

\[
\langle \chi_X, \chi_X \rangle = \frac{1}{|S_4|} \cdot (6^2 \cdot 1 + 0^2 \cdot 6 + 0^2 \cdot 8 + 2^2 \cdot 6 + 2^2 \cdot 3) = \frac{1}{24} \cdot (36 + 24 + 12) = \frac{1}{24} \cdot 72 = 3. \]
Since the only way to express 3 as the sum of squares is $3 = 1^2 + 1^2 + 1^2$, we know that $X$ is the direct sum of exactly three irreducible representations. See that

$$\langle \chi_X, \chi_Y \rangle = \frac{1}{|S_4|} (6 \cdot 1 + 0 \cdot 1 \cdot 6 + 0 \cdot 1 \cdot 8 + 2 \cdot 1 \cdot 6 + 2 \cdot 1 \cdot 3) = \frac{1}{24} (6 + 12 + 6) = \frac{1}{24} \cdot 24 = 1,$$

$$\langle \chi_X, \chi_Y' \rangle = \frac{1}{|S_4|} (6 \cdot 1 + 0 \cdot (-1) \cdot 6 + 0 \cdot 1 \cdot 8 + 2 \cdot (-1) \cdot 6 + 2 \cdot 1 \cdot 3) = \frac{1}{24} (6 - 12 + 6) = \frac{1}{24} \cdot 0 = 0,$$

$$\langle \chi_X, \chi_V \rangle = \frac{1}{|S_4|} (6 \cdot 3 \cdot 1 + 0 \cdot 1 \cdot 6 + 0 \cdot 0 \cdot 8 + 2 \cdot (-1) \cdot 6 + 2 \cdot (-1) \cdot 3) = \frac{1}{24} (18 - 12 - 6) = \frac{1}{24} \cdot 0 = 0,$$

$$\langle \chi_X, \chi_V' \rangle = \frac{1}{|S_4|} (6 \cdot 3 \cdot 1 + 0 \cdot (-1) \cdot 6 + 0 \cdot 0 \cdot 8 + 2 \cdot 1 \cdot 6 + 2 \cdot (-1) \cdot 3) = \frac{1}{24} (18 + 12 - 6) = \frac{1}{24} \cdot 24 = 1.$$

At this point, we know that $X$ must contain a copy of $W$. Therefore, we have the decomposition $X = U \oplus V' \oplus W$.

**Vertices:** Call this permutation representation $Y$. The identity permutation fixes all 8 vertices. Transpositions fix no vertices. Each 3-cycle fixes 2 vertices. Each 4-cycle fixes no vertices. Finally, elements in the conjugacy class of $(12)(34)$ do not fix any vertices, either. Therefore, $\chi_Y$ has the values 8, 0, 2, 0, 0 on the respective conjugacy classes, which gives us

$$\langle \chi_Y, \chi_Y \rangle = \frac{1}{|S_4|} (8^2 \cdot 1 + 0^2 \cdot 6 + 2^2 \cdot 8 + 0^2 \cdot 6 + 0^2 \cdot 3) = \frac{1}{24} (64 + 32) = \frac{1}{24} \cdot 96 = 4.$$

Since the possible ways of writing 4 as the sum of squares of nonnegative integers are $4 = 2^2$ and $4 = 1^2 + 1^2 + 1^2$, $Y$ is either the direct sum of four irreducible representations or the direct sum of two copies of one irreducible representation. To find out which case it is, we simply compute:

$$\langle \chi_Y, \chi_U \rangle = \frac{1}{|S_4|} (8 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 6 + 2 \cdot 1 \cdot 8 + 0 \cdot 1 \cdot 6 + 0 \cdot 1 \cdot 3) = \frac{1}{24} (8 + 16) = \frac{1}{24} \cdot 24 = 1,$$

$$\langle \chi_Y, \chi_U' \rangle = \frac{1}{|S_4|} (8 \cdot 1 \cdot 1 + 0 \cdot (-1) \cdot 6 + 2 \cdot 1 \cdot 8 + 0 \cdot (-1) \cdot 6 + 0 \cdot 1 \cdot 3) = \frac{1}{24} (8 + 16) = \frac{1}{24} \cdot 24 = 1,$$

$$\langle \chi_Y, \chi_V \rangle = \frac{1}{|S_4|} (8 \cdot 3 \cdot 1 + 0 \cdot 1 \cdot 6 + 2 \cdot 0 \cdot 8 + 0 \cdot (-1) \cdot 6 + 0 \cdot (-1) \cdot 3) = \frac{1}{24} \cdot 24 = 1,$$

$$\langle \chi_Y, \chi_V' \rangle = \frac{1}{|S_4|} (8 \cdot 3 \cdot 1 + 0 \cdot (-1) \cdot 6 + 2 \cdot 0 \cdot 8 + 0 \cdot 1 \cdot 6 + 0 \cdot (-1) \cdot 3) = \frac{1}{24} \cdot 24 = 1.$$

Again, we don’t need to compute $\langle \chi_Y, \chi_W \rangle$. We already know the decomposition: $Y = U \oplus U' \oplus V \oplus V'$.

**Edges:** Call this permutation representation $Z$. By following the same process as described above, we can find that $\chi_Z$ has the values 12, 2, 0, 0, 0 on the respective conjugacy classes, and therefore $Z$ decomposes as $Z = U \oplus V'^2 \oplus V' \oplus W$.

### 4. Representations of $S_n$

Recall that $S_n$ has exactly as many irreducible representations as it does conjugacy classes, and each conjugacy class is a cycle-type equivalence class. Now, there is a one-to-one correspondence between the set of cycle-types and the ways to write $n$ as the sum of positive integers. For example, $S_4$ has the following five cycle-types:
Permutation of the Form | Can Also be Written As | Corresponds to the Sum
---|---|---
\(\iota\) | (1)(2)(3)(4) | 1 + 1 + 1 + 1
(12) | (12)(3)(4) | 2 + 1 + 1
(123) | (123)(4) | 3 + 1
(12)(34) | (12)(34) | 2 + 2
(1234) | (1234) | 4

A fundamental tool for studying the representations of \(S_n\) is the Young Diagram, which is an array of boxes: for a given partition \(n = \lambda_1 + \lambda_2 + \cdots + \lambda_k\) (which we denote by \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\)) where \(\lambda_i \geq \lambda_{i+1}\) for every \(1 \leq i \leq k\), we draw a row of \(\lambda_1\) boxes. Beneath it, we draw a row of \(\lambda_2\) boxes, and so on so that the last row contains \(\lambda_k\) boxes and each row is as long as or shorter than the one above it. Thus, the Young Diagrams corresponding to the partitions of 4 are:

\[
\begin{array}{ccc}
\lambda_1 = 3 & 1 & 2 & 3 \\
\lambda_2 = 2 & 4 & 5 \\
\lambda_3 = 1 & 6
\end{array}
\]

Define the following two sets, which are, in fact, subgroups of \(S_n\):

\[
P_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves the set of numbers in each row}\}\]
\[
Q_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves the set of numbers in each column}\}\]

For our particular numbered diagram above, we have:

\[
P_\lambda = \{\iota, (12), (23), (13), (123), (132), (45), (12)(45), (12)(45), (23)(45), (13)(45), (123)(45), (132)(45)\}
\]
\[
Q_\lambda = \{\iota, (14), (16), (46), (146), (25), (14)(25), (16)(25), (46)(25), (146)(25), (164)(25)\}
\]

Before we proceed, we need:

**Definition 4.1.** Let \(F\) be any field, and let \(G\) be a group. The group algebra of \(G\) over \(F\) is the vector space over \(F\) generated by the basis \(\{e_x \mid x \in G\}\) with
multiplication defined by
\[
\left( \sum_{x \in G} c_x x \right) \left( \sum_{y \in G} d_y y \right) = \sum_{x,y \in G} c_x d_y x y,
\]
where the \(c_x\)'s and \(d_y\)'s are scalars in \(F\). Note that if we let \(F = \mathbb{C}\), then \(\mathbb{C}G\) becomes a permutation representation under the group action defined by \(g \cdot e_x = e_{gx}\).

Now define the following three elements of the group algebra \(\mathbb{C}S_n\):
\[
a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma, \\
b_\lambda = \sum_{\tau \in Q_\lambda} \text{sgn}(\tau) e_\tau, \text{ and} \\
c_\lambda = a_\lambda b_\lambda.
\]
It turns out that the subspace \(\mathbb{C}S_n \cdot c_\lambda\) is an irreducible representation of \(S_n\), and distinct partitions of \(\lambda\) correspond to distinct irreducible representations. It is important to know that although the number of conjugacy classes of any finite group will always equal the number of its irreducible representations, very rarely do we see an explicit bijection between these sets as we do here.

**Example 4.2.** To see this in action, let’s find all irreducible representations of \(S_3\) again. There are three partitions of 3 and therefore three corresponding Young Diagrams. We will number each one in the manner described above:

\[
\lambda = (3) \quad \mu = (2,1) \quad \nu = (1,1,1)
\]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \quad \begin{array}{cc}
1 & 2 \\
3 & \\
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

In the first diagram, 1, 2 and 3 are all sitting in one row, so any reshuffling of these numbers will preserve the row. The only permutation that will preserve the columns, however, is the identity. Therefore, \(P_\lambda = S_3\) and \(Q_\lambda = \{\iota\}\). This gives us
\[
a_\lambda = e_\iota + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}, \\
b_\lambda = e_\iota, \\
c_\lambda = (e_\iota + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)})e_\iota \\
= e_\iota + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}.
\]
The associated irreducible representation, then, is \(\mathbb{C}S_3 \cdot c_\lambda = \mathbb{C} \cdot c_\lambda = \langle c_\lambda \rangle\) because multiplying by any element in the basis of \(\mathbb{C}S_3\) will simply rearrange the addends of \(c_\lambda\) but not change the sum. Notice that the subspace generated by \(c_\lambda\) is one-dimensional. Furthermore, because \(\sigma \cdot r c_\lambda = r c_\lambda\) for any \(\sigma\) in \(S_3\) and for any scalar \(r\), the action of every \(\sigma\) leaves every vector in \(\langle c_\lambda \rangle\) fixed, which means \(\langle c_\lambda \rangle\) is the trivial representation.
In the second diagram, \( P_\mu = \{ \iota, (12) \} \) and \( Q_\mu = \{ \iota, (13) \} \). Therefore, we have
\[
\begin{align*}
a_\mu &= e_\iota + e_{(12)}, \\
b_\mu &= e_\iota - e_{(13)}, \\
c_\mu &= (e_\iota + e_{(12)})(e_\iota - e_{(13)}) \\
&= e_\iota - e_{(13)} + e_{(12)} - e_{(132)}.
\end{align*}
\]

According to our algorithm, the associated irreducible representation is \( \mathbb{C}S_3 \cdot c_\mu \). To find out what this subspace is, we multiply \( c_\mu \) by the basis elements of \( \mathbb{C}S_3 \):
\[
\begin{align*}
e_\iota(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_\iota - e_{(13)} + e_{(12)} - e_{(132)}, \\
e_{(12)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(12)} - e_{(132)} + e_{(13)} - e_{(132)}, \\
e_{(13)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(13)} - e_\iota + e_{(123)} - e_{(23)}, \\
e_{(23)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(23)} - e_{(132)} + e_{(132)} - e_{(12)}, \\
e_{(123)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(123)} - e_{(23)} + e_{(13)} - e_{(i)}, \\
e_{(132)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(132)} - e_{(12)} + e_{(23)} - e_{(132)}.
\end{align*}
\]

This set is spanned by the first and third vectors, showing that \( \mathbb{C}S_3 \cdot c_\mu \) is the subspace \( \langle e_\iota - e_{(13)} + e_{(12)} - e_{(132)}, e_{(13)} - e_\iota + e_{(123)} - e_{(23)}, e_\iota - e_{(13)} + e_{(12)} - e_{(132)} \rangle \). This must be the standard representation, which is the only two-dimensional representation of \( S_3 \).

In the third diagram, any permutation in \( S_3 \) will preserve the column, but only the identity will fix the rows. Therefore, we have \( P_\nu = \{ \iota \} \) and \( Q_\nu = S_3 \). This gives us
\[
\begin{align*}
a_\nu &= e_\iota, \\
b_\nu &= e_\iota - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}, \\
c_\nu &= e_\iota(e_\iota - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}) \\
&= e_\iota - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}.
\end{align*}
\]

Again, the associated irreducible representation is \( \mathbb{C}S_3 \cdot c_\nu = \mathbb{C} \cdot c_\nu = \langle c_\nu \rangle \) because multiplying by any element in the basis of \( \mathbb{C}S_3 \) will rearrange the addends of \( c_\nu \) and negate their signs. This subspace is also one-dimensional. Furthermore, for any \( \sigma \) in \( S_3 \) and for any scalar \( r \), \( \sigma \cdot r c_\nu \) is \( r c_\nu \) if \( \sigma \) is even and \( -r c_\nu \) if \( \sigma \) is odd. Thus, \( \langle c_\nu \rangle \) is the alternating representation.

Not only do we have a straightforward way of constructing every irreducible representation of \( S_n \), but we also have an explicit formula for computing the character of each one: For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of \( n \), let \( C_i \) denote any conjugacy class of \( S_n \). Let \( j \) be an index that runs from 1 through \( n \). If we write any element in \( C_i \) as the product of disjoint cycles, then define \( i_j \) to be the number of cycles of length \( j \) in this product. Introduce \( k \) independent variables \( x_1, x_2, \ldots, x_k \). Define the \( j \)th power sum to be \( P_j(x) = x_1^j + x_2^j + \cdots + x_k^j \). Define the discriminant of \( \{ x_1, x_2, \ldots, x_k \} \) to be
\[
\Delta(x) = \begin{vmatrix}
1 & x_k & \cdots & x_k^{k-1} \\
1 & x_1 & \cdots & x_1^{k-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_{k-1} & \cdots & x_{k-1}^{k-1}
\end{vmatrix}
\]
Let \( \ell_s = \lambda_s + k - s \) for every \( 1 \leq s \leq k \). Finally, if \( f(x) \) is some polynomial function of \( x_1, x_2, \ldots, x_k \), let \( [f(x)]_{(\ell_1, \ell_2, \ldots, \ell_k)} \) denote the coefficient on the term \( x_1^{\ell_1} x_2^{\ell_2} \cdots x_k^{\ell_k} \). Then:
Theorem 4.3. (The Frobenius Formula) The character of the irreducible representation of $S_n$ associated with $\lambda$ is given by

$$\chi_\lambda(C_i) = \left[ \Delta(x) \cdot \prod_j P_j(x)^{i_j} \right]_{(i_1,i_2,...,i_k)}.$$ 

Proof. See Section 4.3 of [3]. \qed

Example 4.4. Let’s compute $\chi_{(2,1)}$ for the conjugacy class of transpositions in $S_3$. Here $k = 2$, so we introduce two independent variables $x_1$ and $x_2$. Each transposition can be written as a product of a 2-cycle and a 1-cycle, so we have $i_1 = 1, i_2 = 1, i_3 = 0$. This means the polynomial inside the brackets of the formula above is $(x_2 - x_1)(x_1^2 + x_2^2)(x_1^2 + x_2^2) = x_2^4 - x_1^2$. Therefore, the coefficient on the term $x_1^2x_2$ is 0. Notice that this agrees with the value for $\chi_V(12)$ (where $V$ is the standard representation) that we computed when we were constructing the character table of $S_3$.

5. Schur Functors

In this section, we will see how, given a representation of a group (finite or not), we can use the irreducible representations of $S_n$ to discover new representations of that group.

Definition 5.1. If $V$ and $V'$ are representations of groups $H$ and $H'$, respectively, then the external tensor product $V \boxtimes V'$ is the representation $V \otimes V'$ of $H \times H'$ where the group action is given by $(h,h') \cdot (v \otimes v') = hv \otimes h'v'$ for every $(h,h')$ in $H \times H'$ and every $v \otimes v'$ in $V \otimes V'$.

Definition 5.2. If $\rho : GL_n \rightarrow GL_k(W)$ is a representation of $GL_n$, then $\rho$ sends matrices to matrices:

$$\rho \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} f_{11}(a_{11},\ldots,a_{nn}) & \cdots & f_{1n}(a_{11},\ldots,a_{nn}) \\ \vdots & \ddots & \vdots \\ f_{n1}(a_{11},\ldots,a_{nn}) & \cdots & f_{nn}(a_{11},\ldots,a_{nn}) \end{pmatrix}.$$ 

If the $f_{ij}$’s in the image matrix are polynomial functions, we say that $W$ is a polynomial representation.

The group $GL(V)$ has a natural representation: $V$ equipped with the group action given by $T \cdot v = T(v)$ for every $T$ in $GL(V)$. Just as with finite groups, any tensor power of $V$ is also a representation of $GL(V)$. The group action here is

$$T \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = T(v_1) \otimes T(v_2) \otimes \cdots \otimes T(v_n)$$ 

$V^\otimes n$ is also naturally a representation of $S_n$ under the group action given by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$ 

Notice that these actions commute. We can therefore compose them to obtain a new group action on $V^\otimes n$ of $GL(V) \times S_n$, making $V^\otimes n$ into a representation of $GL(V) \times S_n$.

Definition 5.3. As shown in Section 4, any partition $\lambda$ of $n$ corresponds to a Young symmetrizer $c_\lambda$. The image of the action of $c_\lambda$ on $V^\otimes n$, which we denote by $S_\lambda(V)$, is another representation of $GL(V)$. We call the functor (between the
category of representations of $G$ and itself) that sends a representation $V$ to $S_{\lambda}(V)$ the Schur Functor associated with the partition $\lambda$ and denote it by $F_{\lambda}$.

**Theorem 5.4.** $V \otimes^n$ can be decomposed as

$$V \otimes^n \cong \bigoplus_{\lambda \vdash n} S_{\lambda}(V) \otimes^{m_{\lambda}}$$

where each $S_{\lambda}(V)$ is an irreducible representation of $GL(V)$ and $m_{\lambda}$ is the dimension of $V_{\lambda}$, the irreducible representation of $S_n$ associated with the partition $\lambda$ of $n$.

Another way of saying this is that $V \otimes^n$ can be decomposed as $V \otimes^n = \bigoplus_i V_i$ where each $i$ corresponds to a partition of $n$ and each $V_i$ has the form $M_i \Box W_i$, and $M_i$ and $W_i$ are irreducible representations of $S_n$ and $GL(V)$, respectively. Furthermore, if the dimension of $V$ is greater than or equal to $n$, then every irreducible representation of $S_n$ occurs exactly once in the decomposition, and every polynomial irreducible representation of $GL(V)$ occurs in the decomposition of exactly one $V \otimes^n$ as $n$ varies from 1 to $\infty$. For a given $n$, we would like to find out exactly what the $W_i$’s are in the decomposition of $V \otimes^n$. Let $m_{\lambda} = \dim(M_i)$. If we ignore the $CS_n$-module structure on $S_n$, we have

$$V \otimes^n = \bigoplus_i C^{m_i} \otimes W_i = \bigoplus_i (W_i)^{m_i}.$$ 

It turns out that we can recover each $W_i$ by applying the Schur Functor associated with the partition $i$ to $V \otimes^n$.

By starting with just one representation of $GL(V)$ (namely, $V$), we can recover all irreducible representations of $GL(V)$. The same method can be used to find many, if not all, of the irreducible representations of other groups besides $GL(V)$. There is also an explicit formula for computing the characters of these resulting representations called the Weyl Character Formula. It would be an interesting topic for further study.

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**References**


