THE CANTOR-SCHROEDER-BERNSTEIN PROPERTY IN CATEGORIES

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Abstract. A category is said to have the Cantor-Schroeder-Bernstein property if, whenever there are monic maps $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, there is an isomorphism $h\colon C \to D$. I will explore which categories have this and related properties; most well-known categories lack this property, although usually particular subcategories can be found which satisfy it.

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1. The Cantor-Schroeder-Bernstein Theorem

The theorem sometimes known as the Cantor-Bernstein Theorem, and sometimes known as the Schroeder-Bernstein theorem, and henceforth here known as the Cantor-Schroeder-Bernstein Theorem, gives an order structure on the category of sets.

Theorem 1.1 (Cantor-Schroeder-Bernstein Theorem)**.** *Given sets A and B and injective functions $f\colon A \to B$ and $g\colon B \to A$, there is a bijective function $h\colon A \to B$.*

Proof. Partition the disjoint union $A \sqcup B$ into sequences as follows: given $a \in A$ or $b \in B$, construct a two-sided sequence by iterating the functions f and g to the right, and taking preimages f^{-1} and g^{-1} to the left as long as they are defined—because f and g are injective, when the preimage is nonempty, it is a singleton set.

$$... \to f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to ...$$

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Going to the right, the sequence will either eventually cycle, or continue forever without repeating. To the left, it can either cycle, continue without repeating, stop at an element of A, or stop at an element of B. Thus, there are four distinct classes of sequences:

(1) Sequences which repeat after some number of terms, and thus repeat forever in both directions.
(2) Sequences which continue infinitely in both directions.
(3) Sequences which continue infinitely to the right, and stop at an element of A on the left.
(4) Sequences which continue infinitely to the right, and stop at an element of B on the left.

Now, define $h\colon A \to B$ at $a \in A$ by the kind of sequence that a is in:

$$h(a) = \begin{cases} f(a) : a \text{ is in a sequence of type (1), (2), or (3)} \\ g^{-1}(a) : a \text{ is in a sequence of type (4)} \end{cases}$$

Within any given sequence, h is a bijection between elements of A and B. Each sequence is uniquely determined by any element of the sequence, and every element of $A \sqcup B$ is in exactly one sequence, so they provide a partition of $A \sqcup B$, so a bijection within each of the sequences is a bijection from A to B. Note that neither f nor g were necessarily bijections themselves, but the existence of both of them allowed the construction of a bijection. □

If a relation on sets is defined as $A \leq B$ if and only if there is an injection $f\colon A \to B$, then $A \leq B$ and $B \leq A$ together imply that $A \cong B$. Up to isomorphism, this means that there is a poset underlying the category of sets. Not all categories have such posets, but the existence of one implies a balance between the complexity of the objects and the complexity of the maps.

On one end of this spectrum is the category of sets, where the objects are each determined up to isomorphism by a single cardinal number, while the maps are all set functions. The maps are fairly numerous, but the objects are relatively simple. On the other end of the spectrum are actual posets, where the objects could be assigned arbitrarily complicated structure, but there is at most one map between any two objects. Many of the categories encountered on a regular basis fall outside of this spectrum, although usually an underlying poset can be found by restricting to a sufficiently small subcategory.

2. Basic Definitions and The Finite Case

Because the Cantor-Schroeder-Bernstein Theorem implies a fairly rigorous structure on the category of sets, it is worthwhile to examine in what other categories an analogous property might hold.

Definition 2.1. A category $\mathcal{C}$ is said to have the **CSB property** if whenever there is a pair of monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, there is an isomorphism $h\colon C \to D$.

An alternative statement of the Cantor-Schroeder-Bernstein theorem is that the category of sets has the CSB property. It is not the only category with that property:

Example 2.2. **WellOrd**, the category of well-ordered sets with order-preserving maps, has the CSB property. If there are injections between two well-ordered sets X and Y, then $|X| = |Y|$; an order-preserving injection between well-ordered sets of equal size is a bijection, so is an isomorphism.

Most of the categories considered going forward will be either algebraic or topological, so it is worth noting that two basic examples of categories in these areas do not have the CSB property.

Example 2.3. The category **Grp** of all groups and group homomorphisms does not have the CSB property. If F_2 is the free group with 2 generators $\{a, b\}$ and F_3 is the free group with 3 generators $\{x, y, z\}$, then define $i\colon F_2 \to F_3$ by $i(a) = x$ and $i(b) = y$, and define $f\colon F_3 \to F_2$ by $f(x) = a^2$, $f(y) = ab$, and $f(z) = b^2$. Both i and f are injective homomorphisms, so they are monomorphisms in Grp, but F_2 and F_3 are non-isomorphic, by virtue of being free groups with different numbers of generators.

Example 2.4. The category **Top** of topological spaces and continuous maps does not have the CSB property. A simple case showing this is the closed and open unit intervals, $[0, 1]$ and $(0, 1)$. In one direction, there is the inclusion $i\colon (0, 1) \to [0, 1]$, and in the other direction, there is the map $f\colon [0, 1] \to (0, 1)$ defined by $f(x) = 1/4 + x/2$, but the spaces are not homeomorphic, since [0,1] is compact while (0,1) is not.

This example also demonstrates that the category **Pos** of partially-ordered sets and order-preserving maps does not have the CSB property; both of these spaces are totally-ordered sets, and the continuous functions given preserve order, meaning that $x \leq y$ implies $f(x) \leq f(y)$.

While groups in general do not have the CSB property, finite groups do; given $f\colon G \rightarrowtail H$ and $g\colon H \rightarrowtail G$, then $|G| \leq |H|$ and $|H| \leq |G|$, so $|G| = |H|$; then f, as an injective function between finite sets of equal size, is a bijection. Since f is a bijective homomorphism, it is an isomorphism.

A similar argument works for any structures where bijective morphisms are isomorphisms; however, the CSB property is satisfied by an even larger class of finite objects.

Theorem 2.5. *If a category $\mathcal{C}$ has a faithful functor $F\colon \mathcal{C} \to$ **FinSet** to the category of finite sets, then $\mathcal{C}$ has the CSB property.*

Proof. Consider objects $C, D \in \mathcal{C}$ and monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$. Then we have that $F(gf)$ is a function $FC \to FC$; because FC is a finite set, there are only finitely many functions from FC to itself, so for some $k < n \in \mathbb{Z}$, $(F(gf))^n = (F(gf))^k$. Then by functoriality of F, $F((gf)^n) = F((gf)^k)$. Because F is faithful, $(gf)^n = (gf)^k$; gf, as the composition of monomorphisms, is a monomorphism, so $(gf)^{n-k} = 1_C$. Thus, g is a split epimorphism, so g is an isomorphism. □

This actually proves something stronger than the theorem states; if there are monomorphisms in both directions between objects, then not only is there an isomorphism, but those monomorphisms themselves are isomorphisms. This is similar, and we will show equivalent, to the concept of Dedekind finiteness.

Definition 2.6. A category has the **strong CSB property** if whenever there are monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, both f and g are isomorphisms.

Definition 2.7 (Stout [7])**.** An object C in a category $\mathcal{C}$ is called **Dedekind finite** if all monic endomorphisms $f\colon C \rightarrowtail C$ are isomorphisms.

Proposition 2.8. *A category $\mathcal{C}$ has the strong CSB property if and only if all objects in the category are Dedekind finite.*

Proof. ($\Longrightarrow$) Given a monic endomorphism $f\colon C \rightarrowtail C$ in a category with the strong CSB property, then the strong CSB property can be applied by taking both objects to be C, and both monomorphisms to be f; then because the category has the strong CSB property, f is an isomorphism, so then C is Dedekind finite. C was an arbitrary object in a category with the strong CSB property, so all objects in such a category are Dedekind finite.

($\Longleftarrow$) Since C and D are Dedekind finite, given monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, both gf and fg are isomorphisms. Let k be the inverse of gf; then $gfk = 1_C$ making g a split epimorphism. It is already a monomorphism so it is an isomorphism; the same argument applies to g. □

This is an important distinction between categories that have the CSB property and those that have the strong CSB property; the former is just a statement about the structure of the category as a whole, while the latter is really a statement about the objects in the category, which in turn implies a structure for the category as a whole.

It is worth noting that the original question was motivated by the Cantor-Schroeder-Bernstein Theorem in the category of sets. In that category, all monic maps are split monic (having a left inverse). All the definitions given thus far can be reformulated using split monomorphisms instead of monomorphisms.

Definition 2.9. A category $\mathcal{C}$ is said to have the **split CSB property** if whenever there is a pair of split monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, there is an isomorphism $h\colon C \to D$.

Definition 2.10. A category has the **strong split CSB property** if whenever there are split monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, both f and g are isomorphisms.

Definition 2.11. An object C in a category $\mathcal{C}$ is called **split Dedekind finite** if all split monic endomorphisms $f\colon C \rightarrowtail C$ are isomorphisms.

Proposition 2.12. *A category $\mathcal{C}$ has the strong split CSB property if and only if all the objects in the category are split Dedekind finite.*

Proof. ($\Longrightarrow$) Given a split monic endomorphism $f\colon C \rightarrowtail C$ in a category with the strong split CSB property, then the strong split CSB property can be applied by taking both objects to be C, and both split monomorphisms to be f; then because the category has the strong split CSB property, f is an isomorphism, so C is split Dedekind finite. C is an arbitrary object in a category with the strong split CSB property, so all objects in such a category are split Dedekind finite.

($\Longleftarrow$) Given split Dedekind finite objects C and D and split monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$, both fg and gf are split monomorphisms and hence isomorphisms, so both f and g are epimorphisms. Then, being both split monomorphisms and epimorphisms, both f and g are isomorphisms. As a result, a category where all objects are split Dedekind finite has the strong split CSB property. □

Our vocabulary is now complete; no further terms will be necessary to describe the various degrees of the CSB property that can be satisfied by a category.

Because all monic set functions are split monic, the Dedekind finite sets are the same as the split Dedekind finite sets. If the countable axiom of choice is assumed, then the Dedekind finite sets are precisely the sets of finite cardinality. However, it turns out that if countable choice is rejected, there are models of set theory in which sets exist that have infinite cardinality, but which are still Dedekind finite.

Proposition 2.13 (Herrlich [2])**.** *A set A is Dedekind finite if and only if there is no injection $f\colon \mathbb{N} \rightarrowtail A$.*

Proof. ($\Longrightarrow$) If there is an injection $f\colon \mathbb{N} \rightarrowtail A$, then define a non-surjective injection $h\colon A \rightarrowtail A$ as follows:

$$h(x) = \begin{cases} f(n+1): & x = f(n) \\ x: & \text{else} \end{cases} \tag{2.14}$$

($\Longleftarrow$) If A isn't Dedekind finite, then take a non-surjective injection $g\colon A \rightarrowtail A$, and define an injection $f\colon \mathbb{N} \rightarrowtail A$ by taking some $x \in A \setminus g(A)$, then define recursively:

$f(0) = x$

$f(n+1) = g(f(n))$

This is injective: assuming on the contrary that $f(n) = f(k)$ gives that $g^n(x) = g^k(x)$. Then, since g is injective, $g^{n-k}(x) = x$, but x was chosen to be outside the image of A under g, so we have a contradiction. Thus f is an injection from $\mathbb{N}$ to A. □

Infinite sets which are Dedekind finite have a number of strange properties; they can admit surjections onto sets of arbitrarily large cardinality, and their power sets don't have to be Dedekind finite (and if the power set is Dedekind finite, the power set of the power set must not be Dedekind finite). Those which cannot surject onto $\mathbb{N}$, or equivalently, which have Dedekind finite power sets, are called **Weakly Dedekind Finite**. The earlier result on faithful functors to finite sets can be expanded through weakly Dedekind finite sets.

Proposition 2.15. *If a category $\mathcal{C}$ has a faithful functor $F\colon \mathcal{C} \to$ **Set** with the set FC weakly Dedekind finite for all objects $C \in \mathcal{C}$, then $\mathcal{C}$ has the strong CSB property.*

Proof. Take objects $C, D \in \mathcal{C}$, and monomorphisms $f\colon C \rightarrowtail D$ and $g\colon D \rightarrowtail C$. Then $F(gf)\colon FC \to FC$ is a function from a weakly Dedekind finite set to itself. Considering the iteration $(F(gf))^n(a)$ for $a \in FC$. If this sequence continues infinitely without repeating, then an injection $h\colon \mathbb{N} \rightarrowtail FC$ can be defined by $h(n) = (F(gf))^n(a)$. Thus, repeated applications of $F(gf)$ takes any element of FC ultimately into a cycle. The size of that cycle is bounded, for if it weren't, a surjection $\phi\colon FC \twoheadrightarrow \mathbb{N}$ could be defined by taking an element of FC that lands in a cycle of the n^{th} largest size to n. As well, the time it takes for an element to enter its cycle is also bounded, by the same reasoning—a surjection onto $\mathbb{N}$ could be defined by taking elements that take the n^{th} longest to enter their cycle to n. Let k be the least number such that every element of FC is in its cycle by $(F(gf))^k$, and let m be the least common multiple of all the cycle lengths. Then $(F(gf))^k = (F(gf))^{k+m}$, so by functoriality of F, $F((gf)^k) = F((gf)^{k+m})$, and since F is faithful, $(gf)^k = (gf)^{k+m}$. Both g and f are monic, so $1_C = (gf)^m$. This means that g is split epic. Being monic and split epic, g is an isomorphism. The same argument applies to f, so $\mathcal{C}$ has the strong CSB property. □

We don't need to restrict to weakly Dedekind finite sets if all we are trying to establish is the strong split CSB property.

Proposition 2.16. *If a category $\mathcal{C}$ has a faithful functor $F: \mathcal{C} \to \mathbf{Set}$ with the set FC Dedekind finite for all objects $C \in \mathcal{C}$, then $\mathcal{C}$ has the strong split CSB property.*

Proof. Take split monomorphisms $f: C \rightarrowtail D$ and $g: D \rightarrowtail C$; then $F(gf): FC \rightarrowtail FC$ is a function from a Dedekind finite set to itself. Because functors preserve split monomorphisms, $F(gf)$ is a split monomorphism, so it is a bijection. Then, if k is the left inverse of gf, $F(k)$ is the left inverse of $F(gf)$. As the left inverse of a bijection, $F(k)$ is the two-sided inverse, so $F(gf) \circ F(k) = 1_{FC}$, and by functoriality of F, $F(gfk) = F(1_C)$. Because F is faithful, $gfk = 1_C$, so gf is an isomorphism, meaning that g is an epimorphism. Since it was already a split monomorphism, g is an isomorphism. The same argument applies to f, so $\mathcal{C}$ has the strong split CSB property. □

3. CSB Sometimes Holds in Algebra

Algebraic structures need not have the CSB property—earlier it was shown that groups do not, but even far more restricted cases fail to meet the criteria.

Example 3.1. The category **Ab** of abelian groups and homomorphisms does not have the CSB property. Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{2^i}$ and $H = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{4^i}$. Define $f: G \to H$ by taking the $\mathbb{Z}_{2^i}$ component to the $\mathbb{Z}_{4^i}$ component via multiplication by 2, and define $g: H \to G$ by taking the $\mathbb{Z}_{4^i}$ component by identity to the $\mathbb{Z}_{2^{2i}}$ component. Both of these maps are injective homomorphisms, so they are monomorphisms, and G and H aren't isomorphic: there is an element $x \in G$ with $x + x = 0$ but no $y \in G$ such that $y + y = x$, but there is no such element in H.

The groups above are countable, so even that is not enough for abelian groups to have the CSB property. What is enough is some amount of finiteness—in particular, being finitely-generated.

Theorem 3.2. *The category of finitely-generated abelian groups and homomorphisms has the CSB property.*

Proof. Let G and H be finitely-generated abelian groups; by the fundamental theorem of finitely-generated abelian groups, they are both the product of a finite abelian group and some number of copies of $\mathbb{Z}$. Denote the torsion part of G by G_0 and the torsion part of H by H_0; then $G \cong G_0 \times \mathbb{Z}^g$ and $H \cong H_0 \times \mathbb{Z}^h$ for some $g, h \in \mathbb{N}$. Take injective homomorphisms $\phi: G \rightarrowtail H$ and $\psi: H \rightarrowtail G$. If $x \in G_0$,

$$0_H = \phi(0_G) = \phi(x^{|G_0|}) = \phi(x)^{|G_0|}$$

so ϕ maps G_0 into H_0. If $y \in H_0$, we get

$$0_G = \psi(0_H) = \psi(y^{|H_0|}) = \psi(y)^{|H_0|}$$

so ψ maps H_0 into G_0. Then, because there are injections between the finite groups G_0 and H_0, $G_0 \cong H_0$ by Theorem 2.5. Now, let π_G and π_H be the projections $\pi_G \colon G \to \mathbb{Z}^g$, $\pi_H : H \to \mathbb{Z}^h$. Both the restriction $(\pi_H \circ \phi)|_{\mathbb{Z}^g}$ and the restriction $(\pi_G \circ \psi)|_{\mathbb{Z}^h}$ are injections: for if $\pi_H(\phi(x)) = \pi_H(\phi(y))$, it means that $\phi(x)^{|H_0|} = \phi(y)^{|H_0|}$, so then $\phi(x^{|H_0|}) = \phi(y^{|H_0|})$. Because ϕ is injective, $x^{|H_0|} = y^{|H_0|}$, and since x and y are elements of a free abelian group, $x = y$. The same argument applies to $(\pi_G \circ \psi)|_{\mathbb{Z}^h}$. Then $(\pi_H \circ \phi)|_{\mathbb{Z}^g} : \mathbb{Z}^g \rightarrowtail \mathbb{Z}^h$ and $(\pi_G \circ \psi)|_{\mathbb{Z}^h} : \mathbb{Z}^h \rightarrowtail \mathbb{Z}^g$ are both injections between free abelian groups; the first implies $g \leq h$, and the second implies $h \leq g$, so together we get that $g = h$. Then

$$G \cong G_0 \times \mathbb{Z}^g = G_0 \times \mathbb{Z}^h \cong H_0 \times \mathbb{Z}^h \cong H$$

G and H were arbitrary finitely-generated abelian groups. The only additional assumption made was that there were injections $\phi \colon G \rightarrowtail H$ and $\psi \colon H \rightarrowtail G$. The conclusion is that $G \cong H$, so the category of finitely-generated abelian groups and homomorphisms has the CSB property. □

Abelian groups still only have one operation. It might be hoped that by having both addition and multiplication, the CSB property could be achieved for all objects of that type, but that is not the case. Not only do rings and even commutative rings fail to have the CSB property in general, but so do fields.

Example 3.3. The category **Field** of fields and field homomorphisms does not have the CSB property. Consider the field of complex numbers $\mathbb{C}$ and the field of rational functions with complex coefficients $\mathbb{C}(x)$. The algebraic closure of $\mathbb{C}(x)$ is an algebraically closed field the order of the continuum, so it is isomorphic to $\mathbb{C}$ [6]. There are then canonical inclusions $\mathbb{C} \hookrightarrow \mathbb{C}(x)$ and $\mathbb{C}(x) \hookrightarrow \mathbb{C}$, the first including into constant functions, the second the inclusion into the algebraic closure. However, $\mathbb{C}$ is algebraically closed while $\mathbb{C}(x)$ is not, so they are not isomorphic.

Fields are one of the most strictly defined structures in algebra, but even they don't have the CSB property. However, the theory of uncountable algebraically closed fields has a good deal more rigidity; algebraically closed fields of order κ and characteristic p are all isomorphic [6]. This result allows a broader category to claim the CSB property:

Proposition 3.4. *The category of uncountable algebraically closed fields and field homomorphisms has the CSB property.*

Proof. If we have injections $i \colon F \rightarrowtail K$ and $j \colon K \rightarrowtail F$, then F and K have the same characteristic, because injective homomorphisms preserve characteristic. Field homomorphisms are set functions, so the Cantor-Schroeder-Bernstein theorem implies equality of order. Therefore, the result stated above implies that $F \cong K$. □

It is a bit disappointing to look at example after example of algebraic object, and then be forced to restrict drastically in order to achieve the CSB property. To end the section on a more positive note, we have the one case where no restriction is necessary.

Proposition 3.5. *The category of vector spaces over a fixed field K and linear transformations between them has the CSB property.*

Proof. By the rank-nullity theorem, an injective map $A \colon V \rightarrowtail W$ implies $\dim V \leq \dim W$, and an injective map $B \colon W \rightarrowtail V$ implies $\dim W \leq \dim V$. Thus, $\dim V = \dim W$, so $V \cong W$. □

4. Dedekind Finiteness in Algebra

By the proof of Theorem 2.4, whenever the underlying set of an algebraic object is finite, that object is Dedekind finite, assuming morphisms are uniquely determined by their underlying set functions. However, the added structure of algebraic objects means that there can be Dedekind finite objects built on sets which aren't Dedekind finite. Because Dedekind finiteness implies the strong CBS property, and the strong CSB property implies the CSB property, the categories of Dedekind finite objects are going to be subcategories, usually proper, of those objects with the CSB property.

One particularly useful thing about Dedekind finiteness, as opposed to the CSB property, is that, as a property of objects instead of categories, it is easier to come up with general theorems like the following:

Theorem 4.1. *In a concrete category $\mathcal{C}$ where bijective endomorphisms are isomorphisms, objects satisfying the descending chain condition on subobjects are Dedekind finite. An object C is said to have the descending chain condition on subobjects if all chains of the form $C \supseteq C_1 \supseteq C_2 \supseteq ...$ where all the set inclusions are injective homomorphisms, eventually stabilize.*

Proof. Take an object C satisfying the descending chain condition on subobjects; then for an injective endomorphism $f \colon C \rightarrowtail C$, there is a descending chain

$$C \supseteq f(C) \supseteq f^2(C) \supseteq \cdots \supseteq f^n(C) \supseteq \cdots$$

Because C satisfies the descending chain condition, this chain eventually stabilizes, so there is some $k \in \mathbb{N}$ such that $k \leq n \Rightarrow f^k(C) = f^n(C)$. Since f is injective, then f is bijective on the set $f^k(C) \subseteq C$. Applying f to C over k iterations keeps $f^k(C)$ fixed, while sending C into $f^k(C)$; injectivity of f makes this impossible unless $f^k(C) = C$, so f was surjective all along, and as a bijective endomorphism, by the hypothesis on $\mathcal{C}$, f is an isomorphism, meaning that C is Dedekind finite. □

The descending chain condition implied Dedekind finiteness, but the converse is not true; a simple example of this is the rational numbers:

Example 4.2. The additive group of rational number $\mathbb{Q}$ is Dedekind finite, but does not satisfy the descending chain condition on subgroups. Endomorphisms of $\mathbb{Q}$ are all of the form $\phi(x) = a \cdot x$ for some $a \in \mathbb{Q}$. The only one of these that is not surjective is when $a = 0$, but this case is also not injective, so $\mathbb{Q}$ is Dedekind finite. However, because $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$, and as will be discussed below, $\mathbb{Z}$ does not satisfy the descending chain condition on subgroups, $\mathbb{Q}$ does not satisfy the descending chain condition on subgroups.

The infinite cyclic group $\mathbb{Z}$ is a key example of a group that isn't Dedekind finite; in particular, it is a member of the category of finitely generated abelian groups, which has the CSB property. That category has the CSB property without having the strong CSB property. Meanwhile, $\mathbb{Q}$ is the first case that we've seen in which a Dedekind finite object has infinitely many injective endomorphisms.

Example 4.3. Despite not being a Dedekind finite group, when considered as a ring, $\mathbb{Z}$ is Dedekind finite, because the added requirement for homomorphisms that $\phi(1) = 1$ implies that the only ring endomorphism on $\mathbb{Z}$ is the identity.

This doesn't mean that all rings are Dedekind finite—example 3.3 shows that $\mathbb{C}$ has a non-surjective injective endomorphism.

Once again, the section will end with vector spaces, where everything works as we would hope.

Theorem 4.4. *The Dedekind finite vector spaces are precisely the finite-dimensional vector spaces.*

Proof. That finite-dimensional vector spaces are Dedekind finite follows from the fact that they satisfy the descending chain condition on subspaces. Then, any infinite dimensional vector space isn't Dedekind finite, simply by taking a countable subset $\{v_1, v_2, ...\}$ of the basis, and constructing a linear map T by $T(v_i) = v_{i+1}$, and for any other basis element v, $T(v) = v$. T is then an injective linear map, but v_1 has no preimage, so T is not surjective. □

A restricted case of Theorem 4.1, along with a much broader discussion of Dedekind finiteness in module categories, can be found in [4]. For a particular examination of Dedekind finite abelian groups, see [1].

5. Split Dedekind Finiteness in Algebra

In abelian categories, the splitting lemma says that a short exact sequence is split if and only if the middle term is a direct sum. Starting with a split monomorphism $f\colon A \rightarrowtail A$, a short exact sequence can be formed:

$$0 \longrightarrow A \xrightarrow{f} A \xrightarrow{\pi} A/f(A) \longrightarrow 0$$

Then, because f is split, this sequence is split, so $A \cong A \oplus A/f(A)$. As a result, we get the following result:

Proposition 5.1. *An object A in an abelian category is split Dedekind finite if and only if $A \cong A \oplus C$ implies $C \cong 0$.*

Proof. If split monic endomorphisms are all isomorphisms, then in particular the inclusion endomorphism: $A \hookrightarrow A \oplus C$ is an isomorphism; however, if C is not trivial, then $0_A + c \in A \oplus C$ is not in the image of f, a contradiction, so C must be the 0 object in the category. In the other direction, the diagram above shows that any split monic endomorphism f of A gives a decomposition $A \cong A \oplus A/f(A)$; if $A/f(A) \cong 0$, then f must be an isomorphism. □

This proposition makes it easy to find objects which aren't split Dedekind finite; for any non-trivial object C, then $\bigoplus_{i=0}^{\infty} C$ is nnt split Dedekind finite, by the shift morphism taking the i^{th} component to the $(i+1)^{\text{st}}$. It also hints at the following characterization of fields, given that a direct sum of fields isn't a field:

Theorem 5.2. *All fields are split Dedekind finite.*

Proof. Given a field F and a split monomorphism $\phi\colon F \rightarrowtail F$, then ϕ must have a left inverse ϕ', which is a surjective field homomorphism; all surjective field homomorphisms are isomorphisms, and ϕ, as the right inverse of an isomorphism, is the two-sided inverse of the isomorphism, and thus an isomorphism itself. □

Fields are split Dedekind finite because all split monic field morphisms are isomorphisms. This is equivalent to saying that fields are never direct sums of other fields. This property can be defined in the general setting of abelian categories.

Definition 5.3. An object A in an abelian category $\mathcal{A}$ is **Indecomposable** if $A \cong B \oplus C$ implies that $B \cong 0$ or $C \cong 0$.

Thus, in the particular case where $B \cong A$ above, all indecomposable objects are split Dedekind finite. While objects don't have to decompose nicely into indecomposable objects, when they do, the Dedekind finite objects can be precisely determined.

Theorem 5.4. *If all objects in a category $\mathcal{A}$ can be written uniquely, up to ordering, as a countable direct sum of indecomposable objects, then an object is split Dedekind finite if and only if each of its indecomposable components appears finitely many times.*

Proof. $\Rightarrow$ Given an object, one of whose indecomposable components appears infinitely many times, construct the endomorphism on that object performing a shift operation on countably many of that component, and taking the identity on every other component. This is a non-surjective split monic endomorphism, so that object isn't split Dedekind finite.

$\Leftarrow$ Take an object A whose indecomposable components each appears finitely many times. Let R_i be the indecomposable components, and let r_i be the number of times R_i appears in A. Then, given a split monic endomorphism f of A, each copy of each R_i must be taken to a direct-sum-decomposition copy of R_i. Since there are only r_i copies, this is simply a shuffling of the components, so f is a bijection on each component. Then f is surjective on A as a whole, so f is an isomorphism, meaning that A is split Dedekind finite. □

Example 5.5. The category of abelian groups which are countable direct sums of cyclic groups all have a unique decomposition into a countable sum of cyclic groups of prime power and infinite order, which are all indecomposable. Thus, the split Dedekind finite objects in this category are precisely those which have finite index for $\mathbb{Z}$ and all $\mathbb{Z}_{p^n}$.

Remark 5.6. It is clear that the strong CSB property implies both the CSB property and the strong split CSB property, that the CSB property implies the split CSB property, and that the strong split CSB property implies the split CSB property. It is also clear from examples demonstrated thus far that each of the opposite implications does not hold. The only remaining pair of properties to consider are the CSB property and the strong split CSB property.

The above example demonstrates that the CSB property and the strong split CSB property are incomparable—neither implies the other. The split Dedekind finite objects which are countable direct sums of cyclic groups make up a category with the strong split CSB property, but not the CSB property, since it contains both $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{2^i}$ and $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{4^i}$. On the other hand, the category with one object, $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$, and all of its endomorphisms, trivially has the CSB property, because all of the objects in the category are isomorphic, but it does not have the strong split CSB property, since its only object is not split Dedekind finite.

6. Split CSB in Algebra

This section contains two pieces of speculation—a proposed counterexample, and a conjecture.

Example 6.1. Let F_2 be the free group on two elements, let F_3 be the free group on three elements, and call K the direct sum $F_2 \oplus F_3$. Then, the following two groups may be a counterexample to the claim that **Grp**, the category of groups, has the split CSB property: $G = \bigoplus_{i=1}^{\infty} F_3$, and $H = \bigoplus_{i=1}^{\infty} K$. There are split monomorphisms in both directions, from G to H by including each copy of F_3 into the corresponding copy of K, and from H to G by taking the i^{th} component to the sum of the $(2i-1)^{\text{st}}$ and $(2i)^{\text{th}}$ components, by mapping F_3 to F_3 identically, and taking the generators of F_2 to two of the generators of F_3. All of these component maps are split monomorphisms, so the full maps are split monomorphisms; the only outstanding question is actually demonstrating that G and H are non-isomorphic, which is not a trivial task, and which may not ultimately be true.

Conjecture 6.2. *All abelian categories, and in particular, the category* **Ab** *of abelian groups, have the split CSB property.*

7. CSB Rarely Holds in Topology

Example 2.3 shows that the whole category **Top** of topological spaces and continuous maps does not have the CSB property. In fact, the much more restricted category of path-connected compact Hausdorff spaces also fails to have the CSB property:

Example 7.1. Let X be a closed disc of radius $1/2$ in $\mathbb{R}^2$, and let Y be the union in $\mathbb{R}^2$ of a closed disc of radius $1/2$, a circle of radius 1, and an interval going from $(1/2, 0)$ to $(1, 0)$. Both of these are path-connected compact Hausdorff spaces. Define a monomorphism $i\colon X \hookrightarrow Y$ by including X into Y; define a monomorphism $f\colon Y \rightarrowtail X$ by $f(y) = y/2$. These are continuous monomorphisms in both directions, but X and Y do not even have the same fundamental group, so they cannot be homeomorphic.

8. Compact Manifolds are Dedekind Finite

It is in fact quite difficult to find examples of topological spaces which are Dedekind finite, unless you know where to look.

Definition 8.1. A **topological manifold** of dimension n is a second-countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Manifolds can be intuitively thought of as nice-looking topological spaces: euclidean space $\mathbb{R}^n$, n-sphere S^n, and the n-torus $\mathbb{T}^n$ are the best-known examples. Compact manifolds, like S^n and T^n, are those nice spaces which are in some sense finite. One such sense in which they are finite is the following:

Theorem 8.2. *Compact connected manifolds are Dedekind finite.*

Proof. Take an injective map $f\colon M \rightarrowtail M$ for some compact connected manifold M. By the invariance of domain theorem [8], f is an open map. Because f is an injective map from a compact space to a Hausdorff space, it is a homeomorphism onto its image, so in particular, its image is compact. Thus, the image $f(M)$ is both an open and a closed subset of M. Since it is nonempty, $f(M) = M$, and so because f is a homeomorphism onto its image, f is a homeomorphism. □

Corollary 8.3. *Compact manifolds with finitely many connected components are Dedekind finite.*

Proof. Each connected component, as a connected clopen set, must be taken to another connected component, since f is an open mapping. Then, since each of the component maps are homeomorphisms, f is a homeomorphism. □

Manifolds are not always just discussed as topological objects. They are often introduced because it is possible to give them differentiable structure. The question naturally arises, then, whether compact connected differentiable manifolds are Dedekind finite with respect to smooth maps.

Example 8.4. The circle S^1 is not Dedekind finite when considered as a differentiable manifold. This is because there exist smooth homeomorphisms of S^1 whose derivatives are zero at some point [3]. As such, these homeomorphisms cannot have smooth inverses, so they are not diffeomorphisms. Because they are smooth homeomorphisms, they are smooth injections.

However, one property can still be salvaged:

Theorem 8.5. *The category of compact connected differentiable manifolds is split Dedekind finite*

Proof. Because a compact connected differentiable manifold is Dedekind finite when considered as a topological space, given a split monic smooth map $f\colon M \rightarrowtail M$, f must be a homeomorphism. Then, f has a continuous inverse, and a smooth left inverse. As set functions, these must be identical, meaning that the continuous inverse of f is smooth, so f is a diffeomorphism. □

9. Dedekind Finite Objects and Categorical Constructions

The fundamental example of a category with the strong CSB property, **FinSet**, is closed under all finite limits and colimits. It would be nice if all such categories were closed under finite limits and colimits.

Example 9.1. The equalizer of a pair of maps between Dedekind finite objects need not be a Dedekind finite object. Take $X = Y = S^1$ considered as a topological space, and consider the equalizer E of the maps id, the identity on the circle, and f, which is the identity on the closed right half of the circle, and which contracts the open left half of the circle slightly towards the north pole. The equalizer of these maps is the inclusion of the closed interval $[0, 1]$ onto the right half of the circle X, but the closed interval isn't Dedekind finite. It has the map multiplying by $1/2$, which is a non-surjective injection from $[0, 1]$ to itself.

Example 9.2. The coequalizer of a pair of maps between Dedekind finite objects need not be a Dedekind finite object. Let $\mathcal{C}$ be the category with one object and just the identity on that object. Let $\mathcal{D}$ be the category with two objects and exactly one non-identity map, going from one object to the other. These categories are both Dedekind finite in the category of small categories, since they each only have one monic endomorphism, the identity. There are two functors from $\mathcal{C}$ to $\mathcal{D}$. The coequalizer of these two functors is the monoid $\mathbb{N}$, which is not Dedekind finite, by the map taking the generator to the generator squared.

Because neither equalizers nor coequalizers of Dedekind finite objects must be Dedekind finite, it is hopeless to expect finite limits or colimits in general of Dedekind finite objects to be Dedekind finite. However, there is one very important open question in the study of Dedekind finiteness.

Conjecture 9.3. *A finite product or coproduct of Dedekind finite objects in any category must itself be Dedekind finite.*

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