IRREDUCIBLE REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

NEEL PATEL

Abstract. The goal of this paper is to study the irreducible representations of semisimple Lie algebras. We will begin by considering two cases of such algebras: $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. First, we discover the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. The process used in doing so will guide us through our development of the irreducible representations of $\mathfrak{sl}_3(\mathbb{C})$. We will note several key similarities in the processes used to produce irreducible representations of both these examples. Using these two examples, we will begin to study the irreducible representations of semisimple Lie algebras in general.

Contents

1. Irreducible Representations of $\mathfrak{sl}_2(\mathbb{C})$

A simple Lie algebra is a non-abelian Lie algebra with the Lie algebra itself and 0 as its only ideals. A semisimple Lie algebra is a direct sum of simple Lie algebras. We begin our study of semisimple Lie algebras with one of the more basic examples: $\mathfrak{sl}_2(\mathbb{C})$ which is the set of 2×2 matrices with trace zero. We construct the following basis for this Lie algebra:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now, let V be an irreducible and finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Then, we can decompose V by analyzing the action of H on V. This action gives us a decomposition of V into a direct sum $V = \oplus V_\lambda$. The λ are eigenvalues living in the complex numbers such that for any $v \in V_\lambda$, $H \cdot v = \lambda \cdot v$. So, we have decomposed V into subspaces V_λ and H maps V_λ into V_λ. We know desire to find out how A and B act on V and specifically on each V_λ.

Date: AUGUST 31, 2010.

To do so, we first calculate the commutator relations between H, A, and B:

$$[A, B] = H \qquad [H, A] = 2A \qquad [H, B] = -2B$$

Now, we consider the action of A on $v \in V_\lambda$:

$$\begin{aligned} H(A \cdot v) &= A(H \cdot v) + H(A \cdot v) - A(H \cdot v) \\ &= A(H \cdot v) + [H, A] \cdot v \\ &= A(\lambda \cdot v) + 2A \cdot v = (\lambda + 2) \cdot v \end{aligned}$$

Thus, we see that $A \cdot v$ is also an eigenvector for H with eigenvalue $\lambda + 2$. Thus, A maps V to $V_{\lambda+2}$.

Similarly, $H(B \cdot v) = (\lambda - 2) \cdot v$. So, $B \cdot v$ is an eigenvector for H with eigenvalue $\lambda - 2$. B maps V_λ to $V_{\lambda-2}$. Recall that we assumed V to be irreducible. Thus, V is invariant under action by $\mathfrak{sl}_2(\mathbb{C})$. So, this imposes the condition that the eigenvalues, λ, in the decomposition of V are equivalent to one another modulo 2. We rewrite the irreducible representation as $V = \oplus_{n \in \mathbb{Z}} V_{\lambda_0 + 2n}$. V is finite-dimensional, and hence, there our sequence of eigenvalues must also be finite. Suppose N is the final element in the sequence of eigenvalues: $\lambda_0 + 2n$.

Since our string of subspaces ends with V_N, we can effective write $V_{N+2} = 0$. Thus, $A \cdot v = 0$ for any $v \in V$. Let us pick some nonzero $v \in V_N$ and consider the repeated action of B on this vector v. So, $B^n(v)$ will lie in the space V_{N-2n}. We now provide an important theorem in the study of representations of $\mathfrak{sl}_2(\mathbb{C})$ that have consequences in the proof beyond the statement of the theorem. We begin first with a lemma. We will use the result of this lemma in both the theorem follwing it and in later discussion.

Lemma 1.1. *$A \cdot B^n(v) = n(N - n + 1) \cdot B^{n-1}(v)$ where $v \in V_N$.*

Proof. We use induction. Consider the base case $A(B(v)) = [A, B](v) + B(A(v)) = H \cdot v + B(0) = N \cdot v$. Now, let us assume that $A \cdot B^k(v) = k(N - k + 1) \cdot B^{k-1}(v)$ is true for $k = n - 1$, so $A \cdot B^{n-1}(v) = (n - 1)(N - n) \cdot B^{n-2}(v)$.

$$\begin{aligned} A \cdot B^n(v) &= [A, B](B^{n-1}(v)) + B(A(B^{n-1}(v))) \\ &= H(B^{n-1}(v)) + B((n - 1)(N - n + 1) \cdot B^{n-2}(v)) \\ &= (N - 2n + 2) \cdot B^{n-1}(v) + (n - 1)(N - n + 1) \cdot B^{n-1}(v) \\ &= n(n - N + 1) \cdot B^{n-1}(v) \end{aligned}$$

Thus, we are done by induction. □

Theorem 1.2. *V is spanned by $\{v, B(v), B^2(v), \ldots, B^n(v), \ldots\}$.*

Proof. Let $U = span(\{v, B(v), B^2(v), \ldots\})$. V is irreducible, and hence, the only invariant subspaces of V are V and 0. U is obviously not the 0 subspace and $U \subset V$. Thus, all we need to show is that U is invariant under the action of $\mathfrak{sl}_2(\mathbb{C})$. (Once we recognize that U is invariant under the action of $\mathfrak{sl}_2(\mathbb{C})$, then we see that $U = V$. In order to prove invariance, we study the action of the three generator matrices, A, B, and H on our space U. B preserves this space trivially since B takes an element in $B^n(v)$ to an element in $B^{n+1}(v)$.

Now consider the action of H on U. Consider the vector $B^n(v)$. This vector lives inside of V_{N-2n}. Thus, $H \cdot B^n(v) = (N-2n)B^n(v)$ as per our earlier calculation. So U is invariant under the action of H.
Proving that U is invariant under A is a little more tricky. We use the previous lemma. A takes arbitrary basis vector $Y^k(v)$ to a scalar multiple of $Y^{k-1}(v)$. Thus, A preserves U as well. □

We also found from the above theorem that each V_λ is a one-dimensional subspace of V. This follows from the fact that V is spanned by one vector from each subspace V_λ and the subspaces V_λ's are independent of one another.
The finite-dimensionality of V implies that we have both an upper and lower bound on the sequence of eigenvalues. Thus, if N is the last term in the sequence and M is the smallest positive integer such that B annihilates our v then we get that:
$0 = A \cdot B^M(v) = M(N-M+1) \cdot B^{M-1}(v)$
Well, we get that $N-M+1=0$ since the other terms on the right hand side are not necessarily always 0. We get that $N = M-1$. Thus, N is a non-negative integer since M is a positive integer. We also know that the eigenvalues of H are symmetric about 0, i.e. if λ is an eigenvalue, then $-\lambda$ is an eigenvalue:

Theorem 1.3. *The eigenvalues of H are symmetric about* 0.

To summarize the above results, for each integer N, there exitsts a unique irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ that is $N+1$ dimensional and H acting on this representation has eigenvalues $-N, -N+2, \ldots, N-2, N$ and with each eigenvalue occuring with multiplicity one. Let us call the $n+1$-dimensional irreducible representations $V^{(n)}$. Now that we have the general properties of an irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, let us construct a concrete structure for them.
We begin with $V^{(0)}$. Well, let us first consider $V = \mathbb{C}$, the trivial representation. We this is exactly the one dimensional irreducible representation $V^{(0)}$. We now look at $V = \mathbb{C}^2$. Well, this space has basis x, y. $H \cdot x = x$ and $H \cdot y = -y$ so we have that $V = \mathbb{C} \cdot x \oplus \mathbb{C} \cdot y$. The first term in the decomposition is the eigenspace with eigenvalue 1 and the second term is the space with eigenvalue -1. Thus, $\mathbb{C}^2 = V^{(1)}$.

Now let us fix $V = \mathbb{C}^2$. Consider $V^{(n)} = Sym^n V$. This has basis $\{x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n\}$. We want to show that $V^{(n)}$ is the n-th irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.

$$H \cdot (x^{n-i}y^i) = (n-i) \cdot H(x) \cdot x^{n-i-1}y^i + i \cdot H(y) \cdot x^{n-i}y^{i-1} = (n-2i) \cdot x^{n-i}y^i$$

Thus, considering the action of H over all possible i, we see that the eigenvalues of H are $n, n-2, \ldots, -n+2, -n$. Each eigenvalue occurs exactly once. Hence, $V^{(n)} = Sym^n\mathbb{C}^2$ is our n-th irreducible representation. We have thus reached the following conclusion:

Theorem 1.4. *The unique n-th dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is the n-th symmetric power of $\mathbb{C}^2$.*

2. $\mathfrak{sl}_2(\mathbb{C})$ vs. $\mathfrak{sl}_3(\mathbb{C})$ and the Decomposition of $\mathfrak{sl}_3(\mathbb{C})$

Our study of the irreducible representations of $\mathfrak{sl}_3(\mathbb{C})$ will proceed through analogy to the study of $\mathfrak{sl}_2(\mathbb{C})$. When we started our study of $\mathfrak{sl}_2(\mathbb{C})$ we outlined a set of basis matrices for it. We chose H in this case to be the generator for the set of diagonal matrices in $\mathfrak{sl}_2(\mathbb{C})$. In the case of $\mathfrak{sl}_3(\mathbb{C})$, we chose the subspace of diaganol matrices. This is a two-dimensional subspace, which we shall call $\mathfrak{h}$. The matrices in this subspace commute, and thus, are simultaneously diagonalizable.

Now, recall that in the case of $\mathfrak{sl}_2(\mathbb{C})$, we considered the arbitrary representation V and decomposed it into a direct sum of subspaces $V = \oplus V_\lambda$ such that each V_λ was an eigenspace of H. Now, in the case of $\mathfrak{sl}_3(\mathbb{C})$, we are dealing with a subspace rather than a single matrix H. This subspace, $\mathfrak{h}$, has elements that always commute, and thus, we can once again decompose a representation V into a similar decomposition. However, because it is a subspace, our eigenvalues are not constants, but rather linear functionals. So, for any $v \in V_\lambda$, where λ is a linear functional on $\mathfrak{h}$, we have the following:

$H \cdot v = \lambda(H) \cdot v$ with $H \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$, so $\lambda(H)$ is a scalar.

So, in sum, given finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$, we get a decomposition of $V = \oplus V_\lambda$ with $\lambda \in \mathfrak{h}^*$ as the eigenvalue for its eigenspace V_λ. Since V is finite-dimensional, we have a finite number of λ's.

Recall that in the case of $\mathfrak{sl}_2\mathbb{C}$, we wanted to determine how A and B acted on V after we found its decomposition under the action of H. Well, in the case of $\mathfrak{sl}_3\mathbb{C}$, we run into a bit of trouble because $\mathfrak{sl}_3\mathbb{C}$ is dimension eight and $\mathfrak{h}$ is dimension two. In $\mathfrak{sl}_2\mathbb{C}$, we used the commutator relations to determine the actions of A and B. Well, the commutator relations are essentially the adjoint action of H on A and B. Thus, essentially what we actually did in that example was to decompose $\mathfrak{sl}_2\mathbb{C}$ into eigenspaces, and the matrices A and B were eigenvectors for the adjoint action of H with eigenvalues 2 and -2 respectively.

So, in the case of $\mathfrak{sl}_3\mathbb{C}$ we do that same. Decompose $\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\lambda)$, where each $\mathfrak{g}_\lambda$ is an eigenspace with eigenvalue $\lambda \in \mathfrak{h}^*$ under the adjoint action of $\mathfrak{h}$. Concretely, this means that if $H \in \mathfrak{h}$ and $A \in \mathfrak{g}_\lambda$ then:

$$[H, A] = ad(H)(A) = \lambda(H) \cdot A$$

Now, we essentially explicitly calculate the decomposition of $\mathfrak{sl}_3\mathbb{C}$. Consider the remaining six basis elements of $\mathfrak{sl}_3\mathbb{C}$, which are $E_{i,j}$ where $i \neq j$. Let $H \in \mathfrak{h}$ be:

$$H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}$$

We know define the linear functional $L_i : \mathfrak{h} \to \mathbb{R}$ by $L_i(H) = h_i$. The reason for developing this linear functional will be clear in the following claim.

Claim 2.1. *There are six subspaces $\mathfrak{g}_\lambda$ in the decomposition of $\mathfrak{g} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\lambda)$. The λ's are of the form $L_i - L_j$ for $i \neq j$.*

Proof. Consider H of the form above and a matrix $G \in \mathfrak{g} - \mathfrak{h}$, i.e. G is a non-diaganol matrix in $\mathfrak{g}$. Say G has entries g_{ij}. Now, we act on G by the matrix H.

$ad(H)(G) = [H, G] = HG - GH = \{(h_i - h_j)g_{ij}\}$.
This means that the entries of G are of the form $(h_i - h_j)g_{ij}$. Now, if we want $ad(H)(G) = \lambda(H)(G)$ where $\lambda(H)$ is a scalar, then one can easily check that our eigenvectors are E_{ij}, the matrices with 1 in the $ij - th$ entry and 0 elsewhere with eigenvalues $h_i - h_j = (L_i - L_j)(H)$. The claim is obvious from here. □

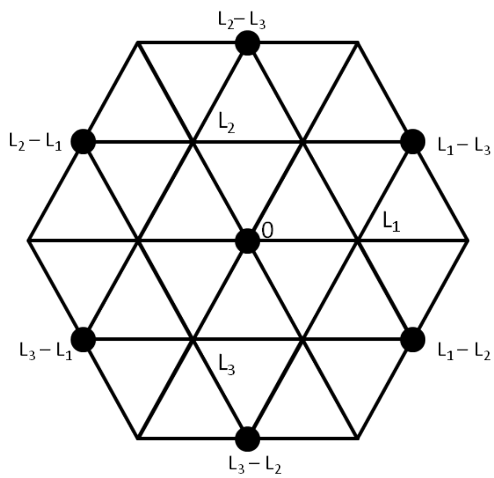
So, we can rewrite the $\mathfrak{g}_\lambda$'s as $\mathfrak{g}_{L_i - L_j}$ for $i \neq j$ and $i, j = 1, 2, 3$. We now introduce the idea of a root lattice. This requires some definitions.

Definition 2.2. The eigenvalues of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ are called the *roots* of $\mathfrak{g}$. The subspaces corresponding to the roots are called *root spaces*.

Note that we never consider 0 to be a root. Now we define root lattice:

Definition 2.3. The *root lattice* is the lattice generated by the roots.

Without further ado, here is the root lattice for $\mathfrak{sl}_3\mathbb{C}$.



In this diagram, the solid vertices represent the root spaces of $\mathfrak{g}$, with the 0 vertex representing $\mathfrak{h}$. Note that $\mathfrak{h}$ acts on the root spaces by mapping each one to itself. In notation form, $ad(\mathfrak{h}) : \mathfrak{g}_\lambda \to \mathfrak{g}_\lambda$. Now, we want to find out what the adjoint action of $\mathfrak{g}_\lambda$ does to the other root lattices.Let $H \in \mathfrak{h}$, $A \in \mathfrak{g}_\lambda$, and $B \in \mathfrak{g}_\gamma$.

$$\begin{aligned}[H, [A, B]] &= [A, [H, B]] + [[H, A], B] \\ &= [A, \gamma(H) \cdot B] + [\lambda(H) \cdot A, B] \\ &= (\lambda(H) + \gamma(H)) \cdot [A, B]\end{aligned}$$

Thus, $\lambda + \gamma \in \mathfrak{h}^*$ is also a root for the adjoint action of $\mathfrak{h}$ with root space generated by $[A, B]$. We can also write $ad(\mathfrak{g}_\lambda) : \mathfrak{g}_\gamma \to \mathfrak{g}_{\lambda+\gamma}$. Thus, you may notice that the action of certain root spaces on other root spaces will annihilate that space. For instance, the action of $\mathfrak{g}_{L_2 - L_3}$ will annihilate $\mathfrak{g}_{L_2 - L_1}$, $\mathfrak{g}_{L_1 - L_3}$, and itself.

Now that we have decomposed $\mathfrak{g}$ under the adjoint action, let us move on to decomposing an irreducible representation V of $\mathfrak{sl}_3\mathbb{C}$. Recall earlier, we stated that we can decompose any finite-dimensional representation into eigenspaces of the action of $\mathfrak{h}$ on V. Let $V = \oplus V_\lambda$ be our decomposition. Now, just as we did in the case of $\mathfrak{sl}_2\mathbb{C}$, we want to find out how all of the root spaces act on each V_λ. Before we do so, let us introduce some formal definitions.

Definition 2.4. The eigenvalues of the action of $\mathfrak{h}$ on V are called the *weights* of our representation, V. The associated eigenvectors to those eigenvalues are called *weightvectors* and the subspaces they generate are called *weightspaces*.

Now, let us determine the action of each root space on V_λ. Let us take arbitrary root space $\mathfrak{g}_\gamma$ and $B \in \mathfrak{g}_\gamma$ and $H \in \mathfrak{h}$. Let $v \in V_\lambda$.

$$\begin{aligned} H(B(v)) &= B(H(v)) + [H, B](v) = B(\lambda(H) \cdot v) + \gamma(H) \cdot B(v) \\ &= (\lambda(H) + \gamma(H)) \cdot B(v) \end{aligned}$$

Thus, $B(v)$ is also a weight vector for the action of $\mathfrak{h}$ on V and it has weight $\lambda + \gamma$. So we can say, in notational form, that $\mathfrak{g}_\gamma : V_\lambda \to V_{\lambda+\gamma}$. This is a similar result to the case of $\mathfrak{sl}_2(C)$ where the action of an element in a root space on a vector, v in V, produced another weight vector for the action of H on V. Recall that in $\mathfrak{sl}_2(C)$, our weights differs by $mod 2$. In this case, since all roots of $\mathfrak{sl}_3(C)$ are of the form $L_i - L_j$ for $i \neq j$ and $i, j = 1, 2, 3$. Thus, we have just shown the following:

Theorem 2.5. *The weights of a representation V of $\mathfrak{sl}_3(C)$ are congruent to one another modulo the root lattice. In other words, the difference between any two weights is a linear combination of the roots: $L_i - L_j$s.*

3. Roots Spaces and the Highest Weight Vector

We continue our analogy with $\mathfrak{sl}_2(C)$. In that case, we looked at the weights of the representation V and picked the maximal eigenvalue in the positive direction, which we picked to mean the last term in the sequence $\lambda + 2n$ where λ might have been our initial weight. This weight was considered maximal or extremal in some sense. In the case of $\mathfrak{sl}_3(C)$, we have no real concept of maximal yet since our weights differ by modulo the root lattice. So in order to fix this problem, we need to pick some linear functional $L : \triangle_R \to \mathbb{R}$, where $\triangle_R$ is the notation for the root lattice. After picking this linear function, we extend it to $L : \mathfrak{h}^* \to \mathbb{C}$. We want the hyperplane $L = 0$ to not contain any of the points in our root lattice. Now we can talk about what we mean by a maximal or extremal weight. It is the weight, say λ, such that $Re(L(\lambda))$ is maximal.

Now, in the case of $\mathfrak{sl}_2(C)$, we took this weight space associated with the maximal weight, which was in the kernel of the action of A, and acted on it repreatedly by B to get the rest of the weight spaces. However, for $\mathfrak{sl}_3(C)$, one might ask how exactly does one decide how we pick the vector and root spaces such that the vector is in the kernel of some of the root spaces and the other root spaces can act repeatedly on that vector. Well, this is why we picked a linear functional. The idea is that we will choose a vector that is from our maximal weight space and is a weight vector for $\mathfrak{h}$ and then act on it by the root spaces $\mathfrak{g}_\lambda$'s such that $L(\lambda) > 0$. Well, if our weight space, say V_γ, was associated with the maximal weight and if the action of $\mathfrak{g}_\lambda$ takes that weight space to $V_{\gamma+\lambda}$. Well, since L is a linear functional, then $Re(L(\gamma + \lambda)) = Re(L(\gamma) + L(\lambda)) > Re(L(\gamma))$. But, $L(\gamma)$ is already maximal, forcing $\mathfrak{g}_\lambda$ to annihilate V_γ. So exactly half of our root spaces will annihilate our vector. We call these our positive root spaces. The other root spaces are the negative root spaces.

Let us pick linear functional L such that $L(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3$

with $a+b+c=0$ and $a>b>c$. Thus, our positive root spaces are $\mathfrak{g}_{L_1-L_2}, \mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$. The other root spaces are our negative root spaces. Note that in the study of $\mathfrak{sl}_2\mathbb{C}$, our positive root space was generated by A and our negative root space was generated by B. In that case, we picked such a vector and acted on it repeatedly by B to get a set of vectors that spanned all of our representation. We do the same here. Let us define such a vector:

Definition 3.1. The *highest weight vector* is a vector in our representation such that it is a weight vector for the subspace $\mathfrak{h}$ and is annihilated by the action of the positive root spaces.

Definition 3.2. The *highest weight* is the associated weight to the highest weight vector.

We now present a theorem analogous to Theorem 1.2 used in the $\mathfrak{sl}_2\mathbb{C}$ case.

Theorem 3.3. *Given highest weight vector v, relative to our linear functional defined above, of an irreducible representation V of $\mathfrak{sl}_3\mathbb{C}$, V is spanned by the images of v under the repeated action of the generators of the negative root spaces.*

Before beginning the proof, let us introduce a definition first.

Definition 3.4. Given a set X, a *word* of length n is an expression with the form $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$.

Proof. In the case of our linear functional L above, our negative root space generators are $E_{2,1}, E_{3,2}$, and $E_{3,1}$. Let us note before we start that $[E_{1,2}, E_{2,3}] = E_{1,3}$ and $[E_{2,1}, E_{3,2}] = E_{3,1}$. This allows us to complete the proof of the theorem by only considering the action of $E_{1,2}$, $E_{2,3}$, $E_{2,1}$ and $E_{3,2}$.
We proceed by induction. Say U is the set generated by the repeated actions of $E_{2,1}, E_{3,2}$, and $E_{3,1}$. Let U_i be spanned by the sets of vectors that are equivalent to a word of length i acting on the highest weight vector v, where the word is an expression comprised of powers of $E_{2,1}$ and $E_{3,2}$. For example, one basis vector for U_3 would be $E_{2,1}E_{3,2}E_{2,1}(v)$. Clearly, U is the union of all of the U_i. Now, we want to show that U is invariant under the action of the positive root spaces. To do this we show that $E_{1,2}$ and $E_{2,3}$ map U_{i+1} to U_i.
So, suppose the above claim is true for $i = n-1$. Take an arbitrary vector $v_n \in U_n$. Then, $v_n = A(v_{n-1})$ where A is $E_{2,1}$ or $E_{3,2}$ and v_{n-1} lives in U_{n-1}. Let us consider the case where $A = E_{2,1}$ first. In this calculation, I use a fact that I will discuss at greater length later. The fact is that $[E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j} = H_{i,j}$ is in $\mathfrak{h}$. (Let λ be its associated eigenvalue.)

$$\begin{aligned}
E_{1,2}(v_n) &= E_{1,2}(E_{2,1}(v_{n-1})) \\
&= E_{2,1}(E_{1,2}(v_{n-1})) + [E_{1,2}, E_{2,1}](v_{n-1}) \\
&= E_{2,1}(v_{n-2}) + \lambda([E_{1,2}, E_{2,1}])(v_{n-1}),\ where\ v_{n-2} \in U_{n-2}
\end{aligned}$$

Well, $E_{2,1}(v_{n-2})$ is in U_{n-1} and the second term is just a scalar multiple of a vector in U_{n-1} so $E_{1,2}$ does indeed map U_n to U_{n-1}. We can do the same type of calculation for $E_{2,3}$, and hence, for $E_{1,3}$. The negative root spaces trivially preserve U since they take U_n to U_{n+1} by definition. □

Say our highest weight vector lives in V_λ. Well, this theorem tells us that our weight space V_λ is one dimensional, and hence, our highest weight vector generates

V_λ and is the unique eigenvector for $\mathfrak{h}$ with weight λ. Next, we observe that since the weight spaces $V_{\lambda+k(L_2-L_1}$ and $V_{\lambda+k(L_3-L_2}$ are generated by $E_{2,1}^k(v)$ and $E_{3,2}^k(v)$ respectively, then those weight spaces must also be one-dimensional. Finally, this theorem also tells us that the weights of V, our irreducible representation, occur in a $\frac{1}{3}$-plane with a corner at λ. One can see this by noticing the direction in which the actions of the negative root spaces take the weight space V_λ. The proof of the above theorem also leads to the following theorem:

Theorem 3.5. *Let's say we have a representation V of $\mathfrak{sl}_3\mathbb{C}$ and a highest weight vector $v \in V$. The subrepresentation W generated by the images of v under repeated action of the negative root spaces is irreducible.*

Proof. We already know that W is a subrepresentation by the proof above. We now have to show it is irreducible. Well, suppose W was not irreducible. Then, W would be a direct sum of two representations. Say $W = W_1 + W_2$. Say λ is the weight of W for v. Then $W_\lambda = W_{1_\lambda} + W_{2_\lambda}$. But, W_λ is one-dimensional, and hence, either W_{1_λ} or W_{2_λ} is zero, and so, the other is all of W. □

We now have the following corollary to this theorem.

Corollary 3.6. *Every irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ has a highest weight vector that is unique up to scalar multiplication.*

Proof. Suppose an irreducible representation, U had two distinct highest weight vectors, v and w. Then, they would produce two distinct subrepresentations by the above theorem, say V and W respectively. So at least one subrepresentations would not be equal to the entire representation. This is a contradiction since our representation was irreducible. □

4. Subalgebras Isomorphic to $\mathfrak{sl}_2\mathbb{C}$

Recall now that we stated that the weights of our representations live in a $\frac{1}{3}$-plane. Well, one of the boundaries is created by the action of $E_{2,1}$ on the highest weight vector v. We will now examine this boundary. Say our highest weight vector lives in the space V_λ. Then, our boundary consists of the weights $\lambda, \lambda + L_2 - L_1, \ldots, \lambda + k(L_2 - L_1), \ldots$

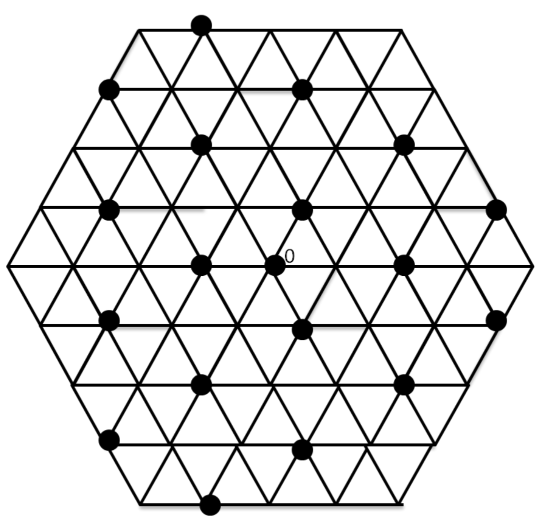
Because our representation is finite-dimensional, this sequence must terminate at some point, say $k = n$. Well, we know that $E_{1,2}$ annihilates the space V_λ and $E_{2,1}$ annihilates the space $V_{\lambda+n(L_2-L_1)}$. Just as in the $\mathfrak{sl}_2\mathbb{C}$ case, we could also calculate the exact expression for the value of n. However, there is a useful trick we can utilize instead. Now, let me talk a little bit about the fact I stated earlier regarding the commutator of $E_{i,j}$ and $E_{j,i}$. Let us call the commutator $[E_{j,i}, E_{i,j}] = E_{i,i} - E_{j,j} = H_{i,j}$, which is clearly in $\mathfrak{h}$ since it is a diagonal matrix with trace 0. So, for instance, $[E_{1,2}, E_{2,1}] = H_{1,2}$. These three matrices, as one can check, actually form a subalgebra. We call this subalgebra $\mathfrak{sl}_{1,2}$ which is actually isomorphic to $\mathfrak{sl}_2\mathbb{C}$. Well, in the case of $\mathfrak{sl}_2\mathbb{C}$, we saw that the eigenvalues of an irreducible representation were symmetric about the origin and integers. We now have to draw an analogy to this case. We consider the weight spaces that are invariant under $\mathfrak{sl}_{1,2}$, which are the spaces with weights of form $\lambda + k(L_2 - L_1)$. Let us call the direct sum of these weight spaces $V_{1,2}$. Well, since $V_{1,2}$ is invariant under the action of our subalgebra $\mathfrak{sl}_{1,2}$, then it is a subrepresentation of V and a representation of

$\mathfrak{sl}_2\mathbb{C}$. The origin in this case is the line, l, in the lattice determined by the equation $< H_{1,2}, l >= 0$. Our weights are symmetric about this origin. We can now do the same for the other boundary which is created by applying the matrix $E_{3,2}$ to the highest weight vector repeatedly.

Now that we have two borders figured out, we have essentially used the initial highest weight vector to the best of our ability. We now want to find another highest weight vector. Well, let us consider the vector $w = E_{2,1}^{n-1}(v)$. Well, $E_{2,1}$ annihilates this vector, as does $E_{2,3}$ and $E_{1,3}$. The last two annihilate w since they take w above, so to speak, the border created by the action of $E_{2,1}$ on v. This vector lives in the space $V_{\lambda+(n-1)(L_2-L_1)}$. Well, since we want our positive roots to annihilate our highest weight vector, we must choose another linear functional, say L' such that $L'(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3$ with $a + b + c = 0$ and $b > a > c$. If we had chosen this linear functional to start with, our highest weight vector would be w in our first analysis instead of v. So, now we play the same game with w instead of v and create another boundary by the repeated action of $E_{3,1}$ on w. Once again, the matrices $E_{1,3}$, $E_{3,1}$ and $[E_{1,3}, E_{3,1}] = H_{1,3}$ form a basis for a subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{C}$ and we have that the weights $\lambda+(n-1)(L_2-L_1), \lambda+(n-1)(L_2-L_1)+(L_3-L_1), \ldots, \lambda+(n-1)(L_2-L_1)+k(L_3-L_1), \ldots$ are symmetric about the origin, which in this case is the line defined by $< H_{1,3}, l >= 0$. We can continue this until we form a hexagon shaped border, with the hexagon symmetric about each lines defined by $< H_{i,j}, l >= 0$ for $i \neq j$ and $i, j = 1, 2, 3$. We also know that the weights on the border are congruent to one another modulo the root lattice.

We also know that we can consider all of the weight spaces generated by combinations of actions of our various basis matrices for $\mathfrak{g} - \mathfrak{h}$. It is clear that all of these weights are points inside of the hexagon that are congruent to the border points modulo the root lattice. Thus, we have the following theorem and diagram.

Theorem 4.1. *Let V be an irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. Then, there is a linear functional λ such that all of the weights that are congruent to λ modulo the root lattice and lie inside of the hexagon formed by the images of λ under the reflections about the lines $< H_{i,j}, l >= 0$.*



As seen in the picture above, the weights, indicated by the solid dots, are symmetric about the lines $< H_{i,j}, l >= 0$. ($< H_{i,j}, l >= 0$ is the line that runs through the origin and k where k is not equal to i or j. For example, $< H_{1,2}, l >= 0$ runs through the origin and L_3.) The weights are also equivalent to one another modulo the root lattice. As shown, the weights do form a hexagon shape.

5. Irreducible Representations of $\mathfrak{sl}_3\mathbb{C}$

We will begin by noting, without proof, that if a representation V has highest weight vector v with highest weight λ_v and another representation W has highest weight vector w with highest weight λ_w, then $v \otimes w$ is a highest weight vector with highest weight $\lambda_v + \lambda_w$. Before I go further let me note that we shall call(given it exists) $V_{s,t}$ the irreducible representation with highest weight $sL_1 - tL_3$.

Now, we will begin by quickly looking at representations of the form $Sym^n V$ and $Sym^n V^*$ where $V = \mathbb{C}^3$. Let us note that $Sym^n V$ and $Sym^n V^*$ are $V_{n,0}$ and $V_{0,n}$ respectively. Let us consider how this can be seen by observation. $Sym^1 V = \mathbb{C}^3$ has three weights: L_1, L_2, L_3 corresponding to their respective weight vectors e_1, e_2, e_3 which are the standard basis vectors. $Sym^1 V^* = V^*$ has weights $-L_1, -L_2, -L_3$ with respective weight vectors e_1^*, e_2^*, e_3^*. Now, the weights of the symmetric power of V are pairwise sums of the weights of V. For example, $Sym^2 V$ has weights $2L_1, 2L_2, 2L_3, L_1 + L_2, L_1 + L_3, L_2 + L_3$ and the weights occur with multiplicity one. So, $Sym^2 V$ is $V_{2,0}$. In drawing the weight diagrams, we discover, and one can check, that the weights of $Sym^n V$ forms a triangle of weights with vertices at nL_1, nL_2, nL_3. The weights of $Sym^n V^*$ form a triangle with vertices at $-nL_1, -nL_2, -nL_3$. Thus, $Sym^n V$ and $Sym^n V^*$ are $V_{n,0}$ and $V_{0,n}$ respectively.

We shall now finalize our study of the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$ by proving the following:

Theorem 5.1. *Given any pair of natural numbers s and t, there exists a unique, finite-dimensional irreducible representation, $V_{s,t}$, that has highest weight $sL_1 - tL_3$.*

Proof. The existence portion of this theorem follows from the fact that the highest weight of $Sym^s V \otimes Sym^t V^*$ will be $sL_1 - tL_3$, and so, there is an irreducible subrepresentation contained in it with highest weight $sL_1 - tL_3$. This irreducible subrepresentation is generated by the repeated action of the negative root spaces on the highest weight vector.
We move onto proving the uniqueness portion of the theorem. Say V and V' are irreducible representations. Consider highest weight vectors v and v' in V and V' both with highest weight λ. We can say that $V \oplus V'$ has highest weight vector $v + v'$ with highest weight λ. Thus, we can generate an irreducible subrepresentation of $V \oplus V'$ with highest weight λ. Let us call this subrepresentation W. The projection maps from W to V and W to V' are nonzero maps obviously. Thus, by Schur's Lemma, since the three representations in question are all irreducible, the projection maps must be isomorphisms. Hence, $V \cong V'$. □

In the proof above, we mentioned the subrepresentation of $Sym^s V \otimes Sym^t V^*$ that has highest weight $sL_1 - tL_3$. We shall delve into this idea just a little further. Consider the contraction map $c_{s,t} : Sym^s V \otimes Sym^t V^* \to Sym^{s-1} V \otimes Sym^{t-1} V^*$ defined by $c_{s,t}((v_1 \cdot \ldots \cdot v_s) \otimes (v_1^* \cdot \ldots \cdot v_t^*)) = \sum < v_i, v_j^* > ((v_1 \cdot \ldots \cdot v_i \cdot \ldots \cdot v_s) \otimes (v_1 \cdot \ldots \cdot v_j^* \cdot \ldots \cdot v_t^*))$. We have the following proposition:

Proposition 5.2. *The kernel of the contraction map $c_{s,t}$ is the irreducible representation $V_{s,t}$.*

6. General Theory of Semisimple Lie Algebra Representations

We will examine the general theory by describing the several steps used to find the irreducible representations of an arbitrary semisimple Lie algebra, $\mathfrak{g}$.

1. In this step, we analyze the semisimple Lie algebra's action on itself via the adjoint representation. Just as in the cases we examined, the first step is to find a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that acts diagonally on $\mathfrak{g}$.

Definition 6.1. A *Cartan subalgebra* is the maximal abelian subalgebra of $\mathfrak{g}$.

Now, the question arises, how can we tell that we have chosen the proper subalgebra? To answer this, let us first decompose $\mathfrak{g}$ into a direct sum $\mathfrak{h} \oplus (\oplus \mathfrak{g}_\lambda)$ where each $\mathfrak{g}_\lambda$ represents an eigenspace of $\mathfrak{g}$ under the action of $\mathfrak{h}$ with eigenvalue λ. Recall that the eigenvalues are called *roots* and the eigenspaces are called *rootspaces*. The eigenvalues are linear functionals on $\mathfrak{h}$. Our Cartan subalgebra is the eigenspace with eigenvalue 0. If any of the other eigenspaces also possessed an eigenvalue of 0 then our original choice of $\mathfrak{h}$ was not maximal. This is how we can tell if we chose $\mathfrak{h}$ correctly. Our roots,λ, and root spaces, $\mathfrak{g}_\lambda$, also have some interesting properties. First, our root spaces will be one-dimensional. Second, our roots will be symmetric, i.e. if λ is a root, then $-\lambda$ is also a root. Finally, the sets of roots, which we shall call R, generates a lattice, which we shall call $\triangle_R$ that has rank equal to the dimension of $\mathfrak{h}$.

2. Now that we have knowledge of the roots and root spaces of our Lie algebra, let us consider an arbitrary finite-dimensional, irreducible representation of $\mathfrak{g}$. Let us call this irreducible representation V. The action of $\mathfrak{h}$ on V admits a decomposition $V = \oplus V_\lambda$. The action has eigenvalues λ, which is a linear functional on $\mathfrak{h}$. Recall that the eigenvalues are called weights and the corresponding eigenspaces are called weight spaces. Now, we want to know how the rest of our Lie algebra acts on V. Well, just as in the cases examined, $\mathfrak{g}_\lambda : V_\gamma \to V_{\lambda+\gamma}$. We can also see that since V is irreducible and the direct sum $W = \oplus_{\gamma \in \triangle_R} V_{\lambda+\gamma}$ (given some weight λ) is invariant under the action of $\mathfrak{g}$, we must have that $V = W$. In other words, all of the weights are congruent to one another modulo the root lattice.

3. Consider root space $\mathfrak{g}_\lambda$. As noted earlier, there exists a root space $\mathfrak{g}_{-\lambda}$. Together, along with their commutator $\mathfrak{h}_\lambda$, which is a one-dimensional subspace of $\mathfrak{h}$, they form a subalgebra $\mathfrak{sl}_\lambda$ of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2\mathbb{C}$. In notation, $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} \oplus \mathfrak{h}_\lambda = \mathfrak{sl}_\lambda \cong \mathfrak{sl}_2\mathbb{C}$.

4. We can now use what we know about $\mathfrak{sl}_2\mathbb{C}$. Let us pick three generators A_λ, $B_{-\lambda}$, and H_λ that are from $\mathfrak{g}_\lambda$, $\mathfrak{g}_{-\lambda}$, and $\mathfrak{h}_\lambda$ respectively and satisfy the commutator relation. First off, we know that the weights of any representation of $\mathfrak{sl}_2\mathbb{C}$ are symmetric about an origin. We also know that we can construct a weight lattice that is the set of all linear functionals that have integer values on each H_λ. The root lattice lies in this weight lattice and the weight diagram of any representation also lies in this weight lattice. The reason that it must have integer values on the H_λ is that

the eigenvalues of H_λ in any representation of $\mathfrak{sl}_\lambda$ must be integers since $\mathfrak{sl}_\lambda \cong \mathfrak{sl}_2\mathbb{C}$. Thus, since a representation of $\mathfrak{sl}_\lambda$ is a subrepresentation of a representation of $\mathfrak{g}_\lambda$, the weights of a representation of $\mathfrak{g}_\lambda$ must take on integers values on each H_λ as well.

Let us move onto the discussion of symmetry. Consider the hyperplane $\Pi_\lambda = \{\gamma \in \mathfrak{h}^* :< H_\lambda, \gamma >\}$ and the involution $\phi_\lambda(\gamma) = \gamma - \gamma(H_\lambda)\lambda$. So, our involution is a reflection across the line spanned by λ in the hyperplane Π_λ.

Definition 6.2. The group of these involutions are called the *Weyl Group* of $\mathfrak{g}$.

Let us use the symbol Φ to represent the Weyl group. Say $V = \oplus V_\gamma$. The subspace $V_{[\lambda]} = \oplus_{n\in\mathbb{Z}} V_{\lambda+n\cdot\gamma}$ is invariant under the involution ϕ_γ and is a subrepresentation in V for $\mathfrak{sl}_\gamma$. Our set of weights of a representation is invariant under Φ, our Weyl group, and so are the multiplicities of the weights.

4. In our study of $\mathfrak{sl}_3\mathbb{C}$, we assigned certain root spaces as negative and others as positive. Well, in the general case, we do the same. First, we pick a linear functional on the root lattice that does not pass through any of the roots. Thus, we have broke our root lattice into two halves and the negative root spaces on those on which the linear functional takes on a negative value. The positive root spaces are those on which our linear functional takes on a positive value. Recall that from here we found a *highest weight vector* that was annihilated by all of the positive root spaces. Then, we acted on this highest weight vector by the negative root spaces. Our results in the case studies is generalized in the following theorem:

Theorem 6.3. *Given a semisimple Lie algebra, $\mathfrak{g}$, any finite-dimensional representation of $\mathfrak{g}$ has a highest weight vector. The repeated action of the negative root spaces on a highest weight vector generates an irreducible subrepresentation, and every irreducible representation has, up to scalars, a unique highest weight vector.*

5. In $\mathfrak{sl}_3\mathbb{C}$, after we determined the boundary, so to speak, of our weight diagram, we were able to fill in the rest of our weight diagram. In the general case, we can do the same. The vertices of our boundary are the set of eigenvalues conjugate to the highest weight, λ, under the Weyl group. In $\mathfrak{sl}_3\mathbb{C}$, these vertices were reflections of the highest weight about the three lines of form $< H_{i,j}, l >= 0$. The border is formed by the successive applications of negative root spaces. For the weights lying inside of the border, we have the following:

Theorem 6.4. *The set of weights of a representation lie inside of the convex hull formed by the conjugates of the highest weight under the Weyl group and are congruent to the highest weight modulo the root lattice.*

The theorem above is the general case of Theorem 4.1. The next theorem we introduce, after some definitions, will be the general case of Theorem 5.1. Let us begin with a definition. (Note that the highest weight, λ is the weight such that for any positive root γ, $\lambda(H_\gamma)$ is nonnegative.)

Definition 6.5. The *Weyl chamber* is the set of points that satisfy the note above and are in the real span of our roots.

Now we can finally state our last theorem:

Theorem 6.6. *Given λ inside of the intersection of the Weyl chamber(respective to the choice of positive and negative roots) and the weight lattice, there exists a* *unique, finite-dimensional, irreducible representation of our semisimple Lie algebra that has highest weight* λ.

Thus, we end with a statement about the existence and uniqueness of irreducible representations of semisimple Lie algebras.

Acknowledgments. It is a pleasure to thank my mentor, Saravanan Thiyagarajan, for guiding me through the study of representation theory of semisimple Lie algebras and for helping me develop my understanding through both examples and theory.

References

[1] Fulton, William, and Joe Harris. Representation Theory: A First Course. New York: Springer-Verlag, 1991. Print.