EQUIVARIANT ALGEBRAIC TOPOLOGY

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Abstract. This paper develops the introductory theory of equivariant algebraic topology. We first define G-CW complexes and prove some basic homotopy-theoretic results - Whitehead's theorem, cellular and CW approximation, and the Freudenthal suspension theorem. We then define ordinary (Bredon) homology and cohomology theories and give an application to Smith theory. Our treatment of this material closely follows that of [3] and [4].

Contents

1. G-CW complexes

One of the first difficulties encountered in equivariant algebraic topology lies in formulating the correct definition of a G-CW complex. Two questions immediately arise:

(1) Should the cells to be attached remain discs, or are more general spaces needed?
(2) What type of intrinsic action should cells possess?

Clearly some modification to the usual definition is needed, as attaching regular discs D^n equipped with trivial G-action only constructs spaces where points have isotropy group G. To construct spaces whose points have a variety of isotropy groups, it is natural to consider attaching discs crossed with different orbits G/H. This slight generalization turns out to be sufficient for many situations, as defining G-CW complexes in this way will recover the usual theorems - Whitehead's theorem, cellular approximation, and CW approximation. For technical (point-set) reasons subgroup for us will mean closed subgroup, and all spaces will be compactly generated and weak Hausdorff.

Definition 1.1. A pair of G-spaces $(X,\ A)$ is a relative G-CW complex if X is the colimit of G-spaces X_n with inclusions $i_n : X_n \to X_{n+1}$, such that X_0 is the disjoint union of A and orbits G/H, and X_n is obtained from X_{n-1} by attaching

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equivariant n-cells $G/H \times D^n$ via G-maps $G/H \times S^{n-1}$, as in the following pushout diagram:

$$\begin{array}{ccc} \coprod_{\alpha \in I} G/H_\alpha \times S^{n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\alpha \in I} G/H_\alpha \times D^n & \longrightarrow & X_n. \end{array}$$

Here D^n and S^{n-1} have the trivial G-action. Letting $A = \emptyset$, we specialize to a G-CW complex X. A G-CW complex $(Y,\ B)$ is a subcomplex of $(X,\ A)$ if Y is a G-subspace of X, B is a closed G-subspace of A, and $Y_n = Y \cap X_n$ in the CW decomposition.

Remark 1.2. Quotients of G-CW complexes behave as expected, but there is a slight subtlety with respect to products that results from having to define a cell structure on the product of two orbits $G/H \times G/K$. Algebraically we have $G/H \times G/K \cong G \times G/H \times K$. If G is a compact Lie group, then this isomorphism is a homeomorphism, as the map $f : G \times G \to G/H \times G/K$ defined by $f(g, g') = (gH, g'K)$ is proper and continuous, hence a quotient map in the category of compactly generated spaces. Thus the product of two G-CW complexes is a $(G \times G)$-CW complex. In the special case that G is discrete, the product of orbits is a disjoint union of orbits, so the product of two G-CW complexes is a G-CW complex.

Remark 1.3. Observe that G-maps $\phi : G/H \times S^n \to X$ correspond bijectively to maps $\phi' : S^n \to X^H$ by letting $\phi'(x) = \phi(H, x)$ or conversely $\phi(gH, x) = g\phi'(x)$. One can often reduce the equivariant theory to the non-equivariant case by means of this observation.

The following discussion will illustrate some of the strengths and shortcomings of our chosen definition. Let G be a discrete group. We seek conditions under which an ordinary CW complex X equipped with an action by G can be exhibited as a G-CW complex with the same skeleta. Such an action should at the least respect the cell structure: if E is a n-cell of X, then for every $g \in G$, gE should again be a n-cell of X. Also, since we only consider attaching discs with trivial G-action, we must demand that the given action on a cell be trivial if that action sends the cell into itself. Note that in practice this condition can often be satisfied by a repeated subdivision of X. Any action that satisfies these two conditions is termed *cellular.*

Proposition 1.4. *Let X be a CW complex with a cellular action of G. Then X is a G-CW complex with the same skeleta.*

Proof. G acting cellularly on X ensures that the ordinary n-skeleton X_n is a G-subspace of X. It follows that X is the colimit in the category of G-spaces of the X_n with inclusion maps $i_{n-1} : X_{n-1} \to X_n$. The proof will be complete once we verify that X_n is obtained from X_{n-1} by attaching equivariant n-cells. To this end, let I be the discrete set indexing the n-cells of X and let $\psi : I \times D^n \to X_n$ be the associated map of n-cells into X which restricts to the attaching map on $I \times S^{n-1}$. Since G acts cellularly on X it acts on I, and we may decompose I into a disjoint union of orbits I_α, $\alpha \in J$. Since G is discrete, for each α we have a homeomorphism $I_\alpha \cong G/H_\alpha$ for $H_\alpha = \{h | hi_\alpha = i_\alpha\}$ the isotropy group of some $i_\alpha \in I_\alpha$. Define G-maps $\phi_\alpha : G/H_\alpha \times D^n \to X_{n-1}$ by $\phi_\alpha(gi_\alpha, x) = g\psi(i_\alpha, x)$; note that we need the triviality condition on a cellular action to ensure $h\phi_\alpha(i_\alpha, x) = \phi_\alpha(i_\alpha, x)$. These maps restrict on $G/H_\alpha \times S^{n-1}$ to give attaching G-maps ϕ'_α. It remains to be seen that the following diagram is a pushout in the category of G-spaces:

$$\begin{array}{ccc} \coprod_{\alpha\in J} G/H_\alpha \times S^{n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\alpha\in J} G/H_\alpha \times D^n & \longrightarrow & X_n. \end{array}$$

But this is clear by the universal property: X_{n-1} and $G/H_\alpha \times D^n$ are identified with closed G-subspaces of X_n, so we can glue together given G-maps $X_{n-1} \to Y$ and $G/H_\alpha \times D^n \to Y$ to obtain a map $X_n \to Y$ that makes the required diagram commute. □

The converse to this proposition also holds; namely, given a G-CW complex we can exhibit it as an ordinary CW complex.

Proposition 1.5. *Let X be a G-CW complex and H a subgroup of G. Then X as an H-space is an H-CW complex with the same skeleta.*

Proof. G remains discrete, so as in the previous proposition any G-orbit G/K may be decomposed into the disjoint union of H-orbits. The same proof then carries through. □

Returning to the second condition of a cellular action (triviality on self maps of cells), we observe that there are occasions where this condition is too restrictive. Namely, if one has a cellular action on a triangulated manifold, then one would want the dual cell decomposition to admit the same cellular action, without having to perform an arbitrary subdivision. That this fails to hold in general implies the general failure of Poincaré duality, and recovering this property is one of the motivations behind defining the more general notion of a G-CW complex alluded to earlier. For more details see [3], chapter X.

2. Basic homotopy theory

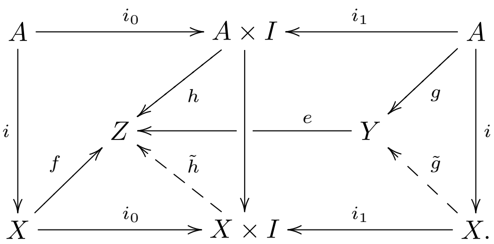
It is our aim in this section to establish equivariant analogues of some fundamental results in homotopy theory - Whitehead's theorem, cellular and CW approximation, and the Freudenthal suspension theorem. We will use the notation $[X, Y]_G$ for homotopy classes of G-maps from X to Y, preferring to omit the subscript if the group G is clear from context.

For us an ordinary pair (X, A) is n-connected if $\pi_0(A) \to \pi_0(X)$ is a surjection and $\pi_i(X, A, a) = 0$ for $1 \le i \le n$ and each $a \in A$, and a map $f : X \to Y$ is a n-equivalence if (M_f, X) is n-connected. Remark 1.3 suggests how to translate the notions of weak equivalence and dimension to the equivariant setting. Let ν be a function from conjugacy classes of subgroups of G to $\mathbb{N} \cup \{\infty\}$.

Definition 2.1. A map $e : Y \to Z$ is a ν-equivalence if $e^H : Y^H \to Z^H$ is a $\nu(H)$-equivalence for all H. If ν is ∞-valued on all H then e is a weak equivalence. A relative G-CW complex X has dimension ν if its cells of orbit type G/H all have dimension $\le \nu(H)$.

The proof of the usual homotopy extension and lifting property is typical of how one proves equivariant theorems by recourse to established nonequivariant results.

Theorem 2.2 (HELP)**.** *Let A be a subcomplex of a G-CW complex X of dimension ν and let $e : Y \to Z$ be a ν-equivalence. Suppose given maps $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram:*



Then there exist maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commute.

Proof. We construct $\tilde{g}$ and $\tilde{h}$ by first inducting on the dimension of skeleta and then working cell by cell. Thus we may suppose X is obtained from A by attaching a cell $G/H \times D^n$. Letting $(X, A) = (G/H \times D^n, G/H \times S^{n-1})$, we see that the assertion is equivalent to a corresponding nonequivariant statement with $e : Y^H \to Z^H$ and $(X, A) = (D^n, S^{n-1})$; our hypotheses are such that we can apply ordinary HELP [2]. □

Theorem 2.3 (Whitehead)**.** *Let $e : Y \to Z$ be a ν-equivalence and X be a G-CW complex. Then $e_* : [X, Y] \to [X, Z]$ is a bijection if X has dimension less than ν and a surjection if X has dimension ν.*

Proof. For surjectivity, consider the pair $(X, \emptyset)$. Given f a map in $[X, Z]$, the lift $\tilde{g}$ given by HELP is the desired element of $[X, Y]$. For injectivity, consider the pair $(X \times I, X \times \partial I)$; note that as I has the trivial action, $X \times I$ is a G-CW complex of one dimension higher. Given $e_*\phi$ and $e_*\phi'$ G-homotopic maps from X to Z, let f be the homotopy, g be ϕ and ϕ', and h be given by $h(x, s, t) = f(x, s)$. Then ϕ and ϕ' are G-homotopic via $\tilde{g}$. □

Corollary 2.4. *If $e : Y \to Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence. In particular, a weak equivalence of G-CW complexes is a G-homotopy equivalence.*

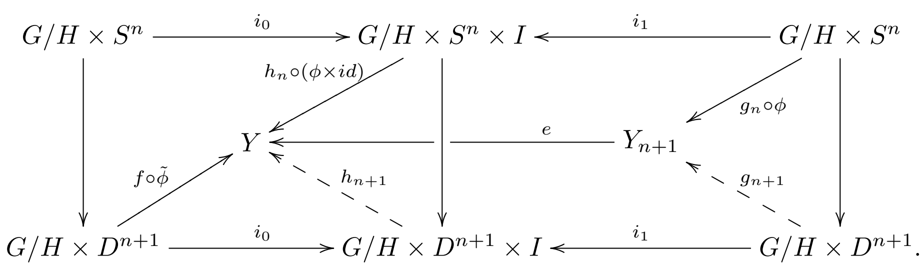
Proof. The assertion is a formal (category-theoretic) consequence of the bijection established by Whitehead's theorem. □

A map f between G-CW complexes (X, A) and (Y, B) is cellular if $f(X_n) \subset Y_n$ for all n. We have the following cellular approximation result.

Theorem 2.5 (Cellular Approximation)**.** *Let (X, A) and (Y, B) be relative G-CW complexes, (X', A') be a subcomplex of (X, A), and $f : (X, A) \to (Y, B)$ be a G-map whose restriction to (X', A') is cellular. Then f is G-homotopic rel $X' \cup A$ to a cellular G-map $g : (X, A) \to (Y, B)$.*

Recall the idea of the proof in the nonequivariant setting (as given for example in [2]): one proceeds inductively over skeleta, supposing given a cellular map $g : X_n \to Y_n$ and a homotopy $h_n : X_n \times I \to Y_n$ such that $(h_n)_0 = f|_{X_n}$, and extending this to a cellular map and homotopy one dimension higher. The extension problem is solved by an application of HELP once one establishes the n-connectivity of the pair (X, X_n). The analogous condition in the equivariant context is n-connectivity of (X^H, X_n^H) for all n and subgroups H of G. This will be proven as Lemma 2.7. Assuming the lemma, the theorem follows as in the nonequivariant case; for the sake of completion we recapitulate the proof.

Proof. To start the induction, note that 0-connectivity of (X^H, X_0^H) implies the existence of a G-homotopy $h_0 : X_0 \times I \to Y$ rel $X' \cup A$ from $f|_{X_0}$ to a map $g_0 : X_0 \to Y_0$. Now suppose given $g_n : X_n \to Y_n$ and $h_n : X_n \times I \to Y$ a G-homotopy rel $X' \cup A$ from $f|_{X_n}$ to g_n. For an attaching map $\phi : G/H \times S^n \to X_n$ of a cell $\tilde{\phi} : G/H \times D^{n+1} \to X$, we have the following diagram:



Here $e : Y_{n+1} \to Y$ is the natural inclusion, which by assumption is a $(n+1)$-equivalence, so an application of HELP gives the indicated maps. Considering all such attaching maps of $(n+1)$-cells yields the desired maps h_{n+1} and g_{n+1}, completing the induction. □

It remains to prove the stated connectivity result. This is an application of homotopy excision, the statement of which we first record (as given in [5], pp. 133).

Theorem 2.6 (Blakers-Massey)**.** *Let X be the union of open subspaces A and B with non-empty intersection $C = A \cap B$. Suppose that*

$$\pi_i(A, C, *) = 0,\ 0 < i < m,\ m \geq 1$$

$$\pi_i(B, C, *) = 0,\ 0 < i < n,\ n \geq 1$$

for each basepoint $* \in C$. Then the inclusion $(B, C) \to (X, A)$ induces a injection on π_i for $1 \leq i < m + n - 2$ and a surjection on π_i for $1 \leq i \leq m + n - 2$.

In particular, if (B, C) is n-connected and (X, A) is 0-connected, then (X, A) is n-connected.

Lemma 2.7. *For each G-CW complex X and subgroup H of G, the pair (X^H, X_n^H) is n-connected.*

Proof. It suffices to show that (X_{n+1}^H, X_n^H) is n-connected for all n; this clearly implies (X_{n+k}^H, X_n^H) is n-connected for all $k \geq 1$, and compactness of the sphere allows us to pass to the colimit X^H. We have the following pushout diagram:

$$\begin{array}{ccc} \coprod_{\alpha \in I} G/H_\alpha^H \times S^n & \longrightarrow & X_n^H \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in I} G/H_\alpha^H \times D^{n+1} & \longrightarrow & X_{n+1}^H. \end{array}$$

The left inclusion is an n-equivalence since $S^n \to D^{n+1}$ is an n-equivalence and π_i commutes with products. By excision, (X^H_{n+1}, X^H_n) is n-connected for $n \geq 1$. To check 0-connectivity of (X^H, X^H_0), note that any point of X^H_1 lies in some $G/K^H \times D^1$, and thus is connected via a path to some point in $G/K^H \times S^0 \subset X^H_0$. □

Corollary 2.8. *Let X and Y be G-CW complexes. Then any G-map $f : X \to Y$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

Proof. The first statement is clear. For the second, given such a homotopy $h : X \times I \to Y$ which is cellular on the subcomplex $X \times \partial I$, cellular approximation yields the desired cellular homotopy. □

Now that cellular approximation is in hand we may prove CW approximation.

Theorem 2.9 (CW Approximation)**.** *For any G-space X, there is a G-CW complex ΓX and a weak equivalence $\gamma : \Gamma X \to X$. For a G-map $f : X \to Y$ and a CW approximation $\gamma' : \Gamma Y \to Y$, there is a G-map $\Gamma f : \Gamma X \to \Gamma Y$, unique up to G-homotopy, such that $\gamma' \circ \Gamma f \simeq f \circ \gamma$. In particular, ΓX is unique up to G-homotopy equivalence.*

Proof. We shall obtain ΓX as the colimit of a sequence of G-CW complexes Y_i, $i \geq 0$, and γ will be induced by maps $\gamma_i : Y_i \to X$ such that $\gamma_{i+1}|_{Y_i} = \gamma_i$. In imitation of the proof of ordinary CW approximation, γ_0 will be surjective on all homotopy groups, and the kernels of maps between successively higher homotopy groups will be killed off by explicitly attaching those homotopies to the G-CW complex. We choose representative maps $f : S^n \to X^H$ for each element of $\pi_n(X^H, x)$ and construct Y_0 as the disjoint union of spaces $G/H \times S^n$ indexed by the maps f. Then γ_0 is defined on each component of Y_0 by its index f. Inductively, suppose we have Y_n and γ_n such that the induced map on $\pi_i(Y^H_n, y)$ is surjective for all i and bijective for $i < n$. Choose representative maps (f, g) for each pair of elements of $\pi_n(Y^H_n, y)$ that have equal image under $(\gamma_n)_*$; by cellular approximation we may suppose the images of f and g lie in the n-skeleton of Y_n. Construct Y_{n+1} by attaching $G/H_+ \wedge S^n \wedge I_+$ along each (f, g), and define γ_{n+1} on the new cells using a homotopy from $(\gamma_n)_*(f)$ to $(\gamma_n)_*(g)$. Note that $G/H_+ \wedge S^n \wedge I_+$ is indeed a G-CW complex as collapsing the line through the basepoint forms a quotient complex. It follows Y_{n+1} is a G-CW complex which contains Y_n as a subcomplex. Clearly γ_{n+1} still induces a surjection on all homotopy groups, and as we have not modified the n-skeleton when passing to Y_{n+1}, γ_{n+1} still induces a bijection up to dimension $n - 1$. By construction, γ_{n+1} also induces a bijection in dimension n, and the induction is complete. Since π_n commutes with colimits, passage to the colimit gives a weak equivalence $\gamma : \Gamma X \to X$. The second statement is a corollary of Whitehead's theorem (Theorem 2.3). □

We conclude this section with a result whose proof is not quite as immediate - an equivariant version of the Freudenthal suspension theorem. Here all spaces in sight are based. The classical statement reads as follows (as given in [3], pp. 116).

Theorem 2.10. *Let Y be an n-connected space, $n \geq 1$, and let X be a finite CW complex. Then the suspension map $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is surjective if dim $X \leq 2n + 1$ and bijective if dim $X \leq 2n$.*

In providing an equivariant generalization of this theorem, it is appropriate to consider spheres with a variety of G-actions. Let G be a compact Lie group and let V be a finite-dimensional real representation of G. As a space $V = \mathbb{R}^n$, and the G-action on V is specified by a map of Lie groups $\rho : G \to O(V)$. Let $S^V = V \cup \{\infty\}$ be the one-point compactification of V with basepoint ∞, on which the G-action is trivial. Note that we recover S^n from the trivial n-dimensional representation of G. For a based G-space X, define $\Sigma^V X = X \wedge S^V$ and $\Omega^V X = [S^V, X]$. Here $\Omega^V X$ has G-action $(g \cdot f)(x) = gf(g^{-1}x)$. We have the adjunction $[\Sigma^V X, Y] \cong [X, \Omega^V Y]$. This gives the following commutative diagram:

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\Sigma^V} & [\Sigma^V X, \Sigma^V Y] \\ & \searrow^{\eta_*} & \downarrow \cong \\ & & [X, \Omega^V \Sigma^V Y]. \end{array}$$

Here η is the unit of the adjunction. Whitehead's theorem then reduces the problem of when Σ^V is an isomorphism to a question about the connectivity of the map η. Let $c(X)$ denote the connectivity of a space X, where $c(X) = -1$ if X is not path-connected.

Theorem 2.11 (Freudenthal Suspension). *The map $\eta : Y \to \Omega^V \Sigma^V Y$ is a ν-equivalence if ν satisfies the following two conditions:*

(1) $\nu(H) \leq 2c(Y^H) + 1$ for all subgroups H with dim $V^H > 0$.
(2) $\nu(H) \leq c(Y^K)$ for all pairs of subgroups $K \subset H$ with dim $V^K >$ dim V^H.

Proof. Consider the following diagram,

$$\begin{array}{ccc} \pi_n(Y^H) & \xrightarrow[(\eta^H)_*]{} & \pi_n((\Omega^V \Sigma^V Y)^H) \\ \downarrow \cong & & \downarrow \cong \\ [S^n, Y]_H & \xrightarrow[\eta_*]{} & [S^n, \Omega^V \Sigma^V Y]_H \\ & \searrow^{\Sigma^V} & \downarrow \cong \\ & & [S^{n+V}, \Sigma^V Y]_H. \end{array}$$

We see that η^H is a $\nu(H)$-equivalence if Σ^V is bijective for $n < \nu(H)$ and surjective as well for $n = \nu(H)$. The following diagram indicates how to reduce this problem to one where the classical Freudenthal suspension theorem can be applied:

$$\begin{array}{ccc} [S^n, Y]_H & \xrightarrow{\Sigma^V} & [S^{n+V}, \Sigma^V Y]_H \\ \cong \downarrow R_1 & & \downarrow R_2 \\ [S^n, Y^H] & \xrightarrow{\Sigma^{V^H}} & [S^{n+V^H}, \Sigma^{V^H} Y^H]. \end{array}$$

Here R_1 and R_2 are the restrictions to H-fixed point sets. Our hypothesis $\nu(H) \leq 2c(Y^H) + 1$ implies that Σ^{V^H} is a $\nu(H)$-equivalence by the classical Freudenthal suspension theorem. To lift this result to Σ^V it suffices to show that R_2 is injective for $n \leq \nu(H)$. Thus suppose that $g : S^{n+V} \to \Sigma^V Y$ is a H-map such that $R_2(g) =$

0. Let $i : S^{n+V^H} \to S^{n+V}$ be the inclusion and Ci the mapping cone. Then a nullhomotopy of $R_2(g)$ gives a map $h : Ci \to \Sigma^V Y$, where $h = g$ on the base. To obtain a nullhomotopy of g we want to extend h to the cone CS^{n+V}. Recall the following fact from obstruction theory ([5], pp. 205):

- Given a map $h : A \to Y$ and a space X obtained from A by attaching cells along maps ϕ_α, there exists an extension $\tilde{h} : X \to Y$ of h iff the maps $h\phi_\alpha$ are nullhomotopic.

CS^{n+V} is obtained from Ci by attaching cells of the form $H/K \times D^{m+1}$, $K \subset H$, $\dim V^K > \dim V^H$, $m \leq n + \dim V^K$ (accounting for the other orbit types in S^{n+V}). By hypothesis, $n \leq \nu(H) \leq c(Y^K)$, so $\pi_m((\Sigma^V Y)^K) = 0$ for $m \leq n + \dim V^K$. By the fact, we have the desired extension $\tilde{h} : CS^{n+V} \to \Sigma^V Y$. □

Note that the second condition is indeed necessary. For example, let $G = \mathbb{Z}/2$, $n \geq 3$, and V be the real one-dimensional sign representation of G. One has the following computation ([3], pp. 119):

$$[S^n, S^n] = \mathbb{Z} \qquad [S^{n+V}, S^{n+V}] = \mathbb{Z}^2$$
$$[S^{n+V}, \Sigma^{n+V} G_+] = \mathbb{Z}^2 \qquad [S^{n+2V}, \Sigma^{n+2V} G_+] = \mathbb{Z}.$$

Dropping the second condition from the theorem would require the maps $\Sigma_V : [S^n, S^n] \to [S^{n+V}, S^{n+V}]$ and $\Sigma_V : [S^{n+V}, \Sigma^{n+V} G_+] \to [S^{n+2V}, \Sigma^{n+2V} G_+]$ to be isomorphisms, a contradiction.

3. Ordinary homology and cohomology

We begin with the equivariant Eilenberg-Steenrod axioms for a homology theory on G-CW pairs. Since orbits are thought of as points in the definition of a G-CW complex, we see that the dimension axiom should be expanded to now specify initial values on all orbits G/H (for instance, this is necessary if our homology theory is to be unique). We thus must conceive of homology taking coefficients in some sort of generalized abelian group. Let $\mathcal{G}$ denote the category of orbit G-spaces with G-maps between them, and let $h\mathcal{G}$ denote the associated homotopy category. Define a covariant coefficient system N to be a functor from $h\mathcal{G}$ to $\mathcal{A}b$ and a contravariant coefficient system M to be a functor from $h\mathcal{G}^{\text{op}}$ to $\mathcal{A}b$.

Definition 3.1. Let N be a covariant coefficient system. An ordinary, or Bredon, equivariant homology theory with coefficients in N consists of functors $H_n^G(X, A; N)$ for each integer n from the homotopy category of G-CW pairs to the category of abelian groups together with natural transformations $\partial : H_n^G(X, A; N) \to H_{n-1}^G(A; N)$, which satisfy the following axioms. Here $H_n^G(X; N)$ is shorthand for $H_n^G(X, \emptyset; N)$.

- Dimension: For each orbit space G/H, $H_0^G(G/H; N) = N(G/H)$ and $H_n^G(G/H; N) = 0$ for all other integers.
- Exactness: The following sequence is exact:

$$\ldots \longrightarrow H_n^G(A; N) \longrightarrow H_n^G(X; N) \longrightarrow H_n^G(X, A; N) \xrightarrow{\partial} H_{n-1}^G(A; N) \longrightarrow \ldots.$$

- Excision: If X is the union of subcomplexes A and B, then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism $H_*^G(A, A \cap B; N) \cong H_*^G(X, B; N)$.

- Additivity: If (X,A) is the disjoint union of a set of pairs (X_i,A_i), then the inclusions $(X_i,A_i) \to (X,A)$ induce an isomorphism $\sum_i H_*^G(X_i,A_i;N) \cong H_*^G(X,A;N)$.

Axioms for cohomology are dual; the notation is $H_G^*(X,A;M)$, where M is a contravariant coefficient system. Given a homology theory on G-CW pairs we obtain uniquely a homology theory on pairs of G-spaces by mandating in addition that a weak equivalence between spaces induce an isomorphism on homology; we have the expected results on CW approximation of pairs and excisive triads allowing us to define $H_*^G(X,A;N) = H_*^G(\Gamma X, \Gamma A; N)$.

We turn towards the construction of cellular homology and cohomology. Let X be a G-CW complex. One naturally obtains a functor from $h\mathcal{G}^{\mathrm{op}}$ to $h\,\mathrm{Top}$, the homotopy category of topological spaces, by sending G/H to X^H. Then given a G-map $f : G/H \to G/K$, $f(eH) = gK$, we have a map $F : X^K \to X^H$ defined by $F(x) = gx$. In general we define $\underline{H}_n(X)$ to be the composition of this functor with H_n. $\underline{H}_n(X,Y)$ is defined similarly. Now define a chain complex of contravariant coefficient systems $\underline{C}_n(X) = \underline{H}_n(X_n, X_{n-1}; \mathbb{Z})$. The boundary map $d : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$ is given objectwise by the connecting homomorphism of the triple $(X_n^H, X_{n-1}^H, X_{n-2}^H)$. A similar construction works for the reduced chain complex $\tilde{\underline{C}}_*(X)$.

In analogy to the classical construction of homology we wish to tensor this chain complex on the right by a coefficient system. The correct notion of a tensor product of coefficient systems (or of categories in general) is that of a coend. Given a contravariant coefficient system M and covariant N, define the Abelian group $M \otimes N$ to be $\sum M(G/H) \otimes N(G/H)/\approx$, where $(f^*m,n) \approx (m, f_*n)$ for a map $f : G/H \to G/K$ and elements $m \in M(G/K)$ and $n \in N(G/H)$. Note that this construction yields the familiar adjunction $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B,C))$, where A is contravariant, C is covariant, and B is a "$\mathbb{Z} - \mathbb{Z}$ bimodule" functor $h\mathcal{G} \times h\mathcal{G}^{\mathrm{op}} \to \mathcal{A}b$. Recovering this property suggests that we indeed have the right generalization of tensor product. Now define the chain complex of abelian groups $C_n^G(X;N) = \underline{C}_n(X) \otimes N$ with boundary map $\partial = d \otimes 1$. The homology of this chain complex is $H_*^G(X;N)$.

To define cohomology with coefficients in a contravariant coefficient system M, define the cochain complex $C_G^n(X;M) = \mathrm{Hom}(\underline{C}_n(X), M)$ with coboundary map $\delta = \mathrm{Hom}(d, \mathrm{id})$. Note $\mathrm{Hom}(M', M)$ forms an abelian group, with addition defined objectwise, so we have a cochain complex of abelian groups. The cohomology of this cochain complex is $H_G^*(X;M)$.

It remains to verify the axioms. For simplicity we refer only to cohomology. Let the relative chain complex $C_*(X,A)$ equal $\tilde{\underline{C}}_*(X/A)$. That cohomology is a functor on the homotopy category of G-CW pairs, with homotopic maps inducing an isomorphism on cohomology, follows from the objectwise construction in terms of nonequivariant homology. The exactness axiom will follow once we show that $\underline{C}_*(X)$ is a projective object in the abelian category of coefficient systems - that is, $\mathrm{Hom}(\underline{C}_*(X), -)$ preserves exact sequences. The additivity and excision axioms of nonequivariant cohomology imply that the chain complex $\underline{C}_*(X)$ is a direct sum of coefficient systems of the form $\underline{\tilde{H}}_n(G/K_+ \wedge S^n) \cong \underline{H}_0(G/K)$. Let F denote the free abelian group functor on sets. We have $\underline{H}_0(G/K)(G/H) = F[G/H, G/K]_G$. A Yoneda lemma-style argument then shows $\mathrm{Hom}(\underline{H}_0(G/K), M) \cong M(G/K)$ via the assignment $\phi \mapsto \phi(1_{G/K})$. Similarly $\underline{H}_0(G/K) \otimes N \cong N(G/K)$ (every element of the sum is equivalent to one in $\underline{H}_0(G/K)(G/K) \otimes N(G/K)$). This gives the dimension and exactness axioms. Additivity and excision are clear, so we have indeed constructed an equivariant cohomology theory. Finally, note that uniqueness of cohomology follows much as it does in the nonequivariant case, the key point being still the axiomatic determination of the cohomology of the wedge X_n/X_{n-1}.

4. Smith theory

We apply Bredon cohomology to prove a classical result of P. A. Smith relating the cohomology of a space with that of its fixed points. All coefficient systems will be contravariant. Let G be a finite p-group and let X be a finite dimensional G-CW complex such that $H^*(X;\mathbb{F}_p)$ is a finite dimensional vector space. Unless otherwise specified, cohomology will be taken to have coefficients in $\mathbb{F}_p$.

Theorem 4.1. *If X is a mod p cohomology n-sphere, then X^G is empty or is a mod p cohomology m-sphere for some $m \leq n$. If p is odd, then $n - m$ is even and X^G is non-empty if n is even.*

Proof. Note that by induction on the order of G we may as well suppose that $G = \mathbb{Z}_p$, since $X^G = (X^H)^{G/H}$ for H a non-trivial normal subgroup of G. Let $FX = X/X^G$. Define coefficient systems L, M, and N so that on objects,

$$\begin{array}{ll} L(G) = \mathbb{F}_p & L(*) = 0 \\ M(G) = \mathbb{F}_p[G] & M(*) = \mathbb{F}_p \\ N(G) = 0 & N(*) = \mathbb{F}_p. \end{array}$$

Since an equivariant cohomology theory is uniquely specified by its value on orbits, $H^q_G(X;L) \cong \tilde{H}^q(FX/G)$, $H^q_G(X;M) \cong H^q(X)$, and $H^q_G(X;N) \cong H^q(X^G)$. Let I be the augmentation ideal of the group ring $\mathbb{F}_p[G]$. We have short exact sequences of coefficient systems

$$0 \longrightarrow \underline{I} \longrightarrow M \longrightarrow L \oplus N \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow M \longrightarrow \underline{I} \oplus N \longrightarrow 0.$$

Here $\underline{I}$ is notation for the coefficient system whose value on G is I and whose value on * is zero. Recall that exactness is defined objectwise. On *, exactness is obvious. For exactness on G, note that $0 \longrightarrow I \longrightarrow \mathbb{F}_p[G] \longrightarrow \mathbb{F}_p \longrightarrow 0$ is split short exact: by definition I is the kernel of the augmentation map $\epsilon : \mathbb{F}_p[G] \to \mathbb{F}_p$ given by summing the coefficients, and we have the map $g : \mathbb{F}_p \to \mathbb{F}_p[G]$ defined by $g(x) = xe$, such that $\epsilon \circ g = \mathrm{id}$. By the exactness axiom of an equivariant cohomology theory, we have long exact sequences

$$... \longrightarrow H^q_G(X;\underline{I}) \longrightarrow H^q(X) \longrightarrow \tilde{H}^q(FX/G) \oplus H^q(X^G) \longrightarrow H^{q+1}_G(X;\underline{I}) \longrightarrow ...$$

and

$$... \longrightarrow \tilde{H}^q(FX/G) \longrightarrow H^q(X) \longrightarrow H^q_G(X;\underline{I}) \oplus H^q(X^G) \longrightarrow \tilde{H}^{q+1}_G(FX/G) \longrightarrow ....$$

The purpose behind deriving these long exact sequences is that they give inequalities on the cohomological dimensions of spaces involved. Let

$$a_q = \dim \tilde{H}^q(FX/G),\ \bar{a}_q = \dim H^q_G(X;\underline{I}),\ b_q = \dim H^q(X),\ c_q = \dim H^q(X^G).$$

Then $a_q + c_q \leq b_q + \bar{a}_{q+1}$ and $\bar{a}_q + c_q \leq b_q + a_{q+1}$. These inequalities combine to yield

$$a_q + c_q + c_{q+1} + ... + c_{q+r} \leq b_q + b_{q+1} + ... + b_{q+r} + ... + a_{q+r+1}.$$

Here r is even. Letting $q = 0$ we see $\sum c_q \leq \sum b_q$. In particular, in the setup of the theorem $\sum b_q = 2$, so $\sum c_q \leq 2$. Also, letting $q = n+1$ we see $\dim H^i(X^G) = 0$ for $i \geq n+1$. Now suppose we had the Euler characteristic formula

$$\chi(X) = \chi(X^G) + p(\chi(FX/G) - 1). \tag{4.2}$$

Then $\chi(X) \equiv \chi(X^G) \bmod p$, so $\sum c_q \neq 1$ and X^G is nonempty or a mod p cohomology m-sphere for $m \leq n$. If $p > 2$, this congruence also implies $n - m$ is even and consequently that X^G is nonempty if n is even, concluding the proof.

Let us now prove (4.2). We have the following short exact sequences of coefficient systems:

$$0 \longrightarrow \underline{I}^{n+1} \longrightarrow \underline{I}^n \longrightarrow L \longrightarrow 0,\ 1 \leq n \leq p-1.$$

$\underline{I}^n$ is notation for the coefficient system whose value on G is I^n and whose value on $*$ is zero. Exactness at $*$ is obvious. To see exactness at G, note that I^n is a principal ideal generated by $(\sigma - e)^n$, where σ is a generator of $\mathbb{Z}_p$. The desired map $f : I^n \to \mathbb{F}_p$ is given by $f((\sigma - e)^n) = \sigma$ and extended by $f((\sum r_i g_i)(\sigma - e)^n) = (\sum r_i)\sigma$ and $f(x + y) = f(x) + f(y)$. This makes f a $\mathbb{F}_p$-module homomorphism with kernel I^{n+1}. From the various long exact sequences we have the equalities

$$\chi(X) = \chi(H^*_G(X;\underline{I})) + \chi(FX/G) - 1 + \chi(X^G)$$

and

$$\chi(H^*_G(X;\underline{I}^n)) = \chi(H^*_G(X;\underline{I}^{n+1})) + \chi(FX/G) - 1.$$

Using that $\underline{I}^{p-1} = L$, inductively we get (4.2).

□

To conclude, we remark that with a bit more algebra one can actually prove stronger inequalities than those given above; for details see [1].

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