THE BLACK-SCHOLES MODEL AND EXTENSIONS

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ABSTRACT. This paper will derive the Black-Scholes pricing model of a European option by calculating the expected value of the option. We will assume that the stock price is log-normally distributed and that the universe is risk-neutral. Then, using Ito's Lemma, we will justify the use of the risk-neutral rate in these initial calculations. Finally, we will prove put-call parity in order to price European put options, and extend the concepts of the Black-Scholes formula to value an option with pricing barriers.

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1. INTRODUCTION

The Black-Scholes formula developed by Fischer Black and Myron Scholes in 1973 was revolutionary in its impact on the financial industry. Today, many of the techniques and pricing models used in finance are rooted in the ideas and methods presented by these two men. This paper will serve as an exposition of the formula with extensions to more exotic options with barriers and will also highlight two different methods for solving the options pricing problem.

We will first derive the formula by determining the expected value of the option, a different method than the one originally employed by Black and Scholes. This method, although it is somewhat less rigorous, gives the same result, namely that the price of a European call option is given by

$$C = S_0 N\left(\frac{rT + \frac{\nu^2 T}{2} + \ln\frac{S_0}{K}}{\nu\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT - \frac{\nu^2 T}{2} + \ln\frac{S_0}{K}}{\nu\sqrt{T}}\right),$$

where S_0 is the initial price of the stock, $N(x)$ represents the cumulative distribution function of a standard normal variable, r is the risk-free interest rate, K is the strike price of the option, T is the amount of time until the option expires, and ν is the annual volatility of the stock price.

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The derivation of this formula requires some non-intuitive assumptions. As a result, we will define some basic terminology about risk, and then we will invoke Ito's Lemma to derive the Black-Scholes equation, named so because it was used by Black and Scholes in their original derivation. The basic idea here is that, by hedging away all risk in our portfolio, it becomes perfectly reasonable to assume that people are risk-neutral. This is a very necessary step, though, for most people are naturally risk-averse. This section will essentially follow the methods employed by Black and Scholes and, along with the derivation for barrier options, will highlight the basic method that they used and a different approach to the problem than that of expected value.

We will introduce the concept of no-arbitrage, also known as the no-free-lunch principle, in order to develop the idea of put-call parity. This method of solving for the European put option price is much simpler than repeating the original derivation and provides insight into basic ideas in financial mathematics.

Our last task will be to extend the basic principles of the Black-Scholes equation (not the formula above) to price barrier options, which are options whose validities are contingent upon hitting some pre-determined stock price. Intuitively, because they have an extra imposition, barrier options should be worth less than a regular option. In order to price them, we will use the same technique employed by Black and Scholes in which they transformed the Black-Scholes equation into the heat equation. The key difference will be in the boundary conditions, a fact that emphasizes the versatility of this technique in the pricing of more exotic options.

2. Derivation

We begin with a review of some basic terminology in probability theory.

Definition 2.1. The *cumulative distribution function*, F, of the random variable X is defined for all real numbers b, by

$$F(b) = \mathbf{P}\{X \leq b\}$$

We say X admits a *probability density function* or *density* f if

$$\mathbf{P}\{X \leq b\} = F(b) = \int_{-\infty}^{b} f(x)\,\mathrm{d}x$$

for some nonnegative function f.

Definition 2.2. X is a *normal random variable* with parameters μ and $\sigma^2 > 0$ if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

Thus, the cumulative distribution function of a standard normal random variable, i.e. one with mean 0 and variance 1, is given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}}\,\mathrm{d}y$$

Definition 2.3. If X is a continuous random variable having a probability density function $f(x)$ then the *expected value* of X is given by

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

Note that the expected value will always be given by a bold-faced $\mathbf{E}$, while a normal E merely represents some variable or parameter.

Definition 2.4. The random variable X is *log-normally distributed* if for some normally distributed variable Y, $X = e^Y$, that is, $\ln X$ is normally distributed.

Now that we have made a few basic definitions, we will delve into some ideas that will be necessary for the derivation.

Definition 2.5. A *call option* is a contract between two parties in which the holder of the option has the right (not the obligation) to buy an asset at a certain time in the future for a specific price, called the strike price.

Definition 2.6. A *put option* is a contract between two parties in which the holder of the option has the right (not the obligation) to sell an asset at a certain time in the future for a specific price, also called the strike price.

A European option is a simply an option that can be exercised only at the expiry of the option, which is specified in the contract.

Definition 2.7. A *universe* is a class that contains as its elements all the entities that one wishes to consider for a given situation.

The concept of a universe allows us to specify and isolate certain conditions that must hold for a theorem or idea to be true. In the universe in which we will be dealing, we will assume that the risk-free interest rate, r, is always available. This allows us to make reasonable simplifications in our argument that will help us reach our final result.

Construction 2.8. The *forward price of a stock* is the current price of the stock, S_0, plus an expected return which will exactly offset the cost of holding the stock over a period of time t. Thus, as the only cost of holding the stock in our case is the risk-free interest lost, the forward price is

$$S_0 e^{rt}$$

where r is the risk-free interest rate.

Definition 2.9. A universe is *risk-neutral* if for all assets A and time periods t, the value of the asset $C(A, 0)$ at $t = 0$ is the expected value of the asset at time t discounted to its present value using the risk-free rate.

$$C(A, 0) = e^{-rt}\mathbf{E}[C(A, t)]$$

where r is the continuously compounded risk-free interest rate.

Lemma 2.10. *Let S_0 be the initial value of the stock price, S_t be the price at time t, and denote by ν annual volatility in the percent change in the stock price, i.e. the standard deviation of the percent change in the price over one year. Finally, assume S_t is a log-normally distributed random variable, i.e. $\ln \frac{S_t}{S_0}$ is normally distributed with mean μ and variance σ, and let the mean of the log-normal distribution be located at the forward price of the stock. Then, $\mu = \mu(t)$, $\sigma = \sigma(t)$, and*

$$\sigma = \nu^2 t \tag{2.11}$$

$$\mu = (r - \frac{\nu^2}{2})t \tag{2.12}$$

It may not be initially clear why this is an important lemma, but it is actually quite useful in the derivation of the Black-Scholes formula. It allows us to compute the drift, μ, of the stock in terms of the risk-free interest rate, annual volatility, and time and, thus, becomes very useful in simplifying complex expressions and obtaining a nice result.

Proof. Equation 2.11 follows from induction. It is clear that after one year, $\ln \frac{S_1}{S_0}$ has variance $(\nu^2)1$. If we assume that after $t-1$ years, $\ln \frac{S_{t-1}}{S_0}$ has variance $(\nu^2)(t-1)$, then after t years

$$\begin{aligned}\ln \frac{S_t}{S_0} &= \ln \frac{S_{t-1}S_t}{S_0 S_{t-1}} \\ &= \ln \frac{S_{t-1}}{S_0} + \ln \frac{S_t}{S_{t-1}}\end{aligned}$$

and has variance $(\nu^2)(t-1) + \nu^2 = \nu^2 t$.

Equation 2.12 results from the following:

$$\begin{aligned}F(a) &= \mathbf{P}\{S_t \leq a\} \\ &= \mathbf{P}\{S_0 e^{x_t} \leq a\} \\ &= \mathbf{P}\{x_t \leq \ln \frac{a}{S_0}\} \\ &= \frac{1}{\sqrt{2\sigma\pi}} \int_{-\infty}^{\ln \frac{a}{S_0}} e^{\frac{-(x_t-\mu)^2}{2\sigma}}\, \mathrm{d}x\end{aligned}$$

Differentiating with respect to a yields the density function for S_t, given by

$$f(x) = \frac{1}{\sqrt{2\sigma\pi}x} e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\sigma}}$$

By assumption, $\mathbf{E}[S_t] = S_0 e^{rt}$, so

$$\begin{aligned}\mathbf{E}[S_t] &= \int_0^{\infty} \frac{1}{\sqrt{2\sigma\pi}x} x e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\sigma}}\, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\sigma\pi}} \int_0^{\infty} e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\sigma}}\, \mathrm{d}x\end{aligned}$$

Let $z = \frac{\ln \frac{x}{S_0} - \mu}{\sqrt{\sigma}}$, then $\mathrm{d}z = \frac{\mathrm{d}x}{x\sqrt{\sigma}}$ with $x = S_0 e^{z\sqrt{\sigma}+\mu}$, so that

$$\begin{aligned}\mathbf{E}[S_t] &= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} e^{z\sqrt{\sigma}+\mu}\, \mathrm{d}z \\ &= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2} + z\sqrt{\sigma}+\mu}\, \mathrm{d}z \\ &= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-\sqrt{\sigma})^2}{2} + \mu + \frac{\sigma}{2}}\, \mathrm{d}z \\ &= \frac{S_0 e^{\mu + \frac{\sigma}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-\sqrt{\sigma})^2}{2}}\, \mathrm{d}z\end{aligned}$$

Letting $x = z - \sqrt{\sigma}$, we see that

$$\mathbf{E}[S_t] = \frac{S_0 e^{\mu + \frac{\sigma}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\,\mathrm{d}x \tag{2.13}$$

$$= S_0 e^{\mu + \frac{\sigma}{2}} \tag{2.14}$$

except the last integral is equal to $\sqrt{2\pi}$. Thus, since $\sigma = \nu^2 t$, $S_0 e^{\mu + \frac{\nu^2 t}{2}} = S_0 e^{rt}$ and $\mu = (r - \frac{\nu^2}{2})t$. $\square$

Theorem 2.15. *(Black-Scholes) In a risk-neutral universe with an initial stock price S_0 and a log-normally distributed stock price S_t, as in Lemma 2.6, at time t, the value C of a European call option at time $t = 0$ with strike K, and expiration time T, and r being the continuously compounded risk-free rate is*

$$C = S_0 N\left(\frac{rT + \frac{\nu^2 T}{2} + \ln\frac{S_0}{K}}{\nu\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT - \frac{\nu^2 T}{2} + \ln\frac{S_0}{K}}{\nu\sqrt{T}}\right) \tag{2.16}$$

where N is the cumulative distribution function of the standard normal variable.

Proof. We have $C(S,T) = \max(S_T - K, 0)$
By assumption,

$$\begin{aligned}
C(S,0) &= e^{-rT}\mathbf{E}[C(S,T)] \\
&= e^{-rT}\mathbf{E}[\max(S_T - K, 0)] \\
&= e^{-rT}\int_K^{\infty} \frac{1}{\sqrt{2\pi T}\nu x}(x-K) e^{\frac{-(\ln\frac{x}{S_0} - \mu)^2}{2\nu^2 T}}\,\mathrm{d}x \\
&= e^{-rT}\int_K^{\infty} \frac{1}{\sqrt{2\pi T}\nu} e^{\frac{-(\ln\frac{x}{S_0} - \mu)^2}{2\nu^2 T}}\,\mathrm{d}x - e^{-rT}\int_K^{\infty} \frac{1}{\sqrt{2\pi T}\nu x} K e^{\frac{-(\ln\frac{x}{S_0} - \mu)^2}{2\nu^2 T}}\,\mathrm{d}x
\end{aligned}$$

One can see that the first integral is in fact the same one encountered in Lemma 2.10, hence the first term will simplify to

$$e^{-rT} S_0 e^{\mu + \frac{\nu^2 T}{2}} \int_A^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\,\mathrm{d}y$$

with

$$A = \frac{\ln\frac{K}{S_0} - \mu - \nu^2 T}{\nu\sqrt{T}}$$

To see this, note that this is the same integral as in equation 2.9 except for the lower limit has been changed to A.

By using equation 2.13 for the value of μ and recognizing that this integral represents the cumulative distribution function for the standard normal variable, we see that this is in fact

$$\begin{aligned}
S_0\left(1 - N\left(\frac{\ln\frac{K}{S_0} - rT - \frac{\nu^2 T}{2}}{\nu\sqrt{T}}\right)\right) &= S_0 N\left(-\frac{\ln\frac{K}{S_0} - rT - \frac{\nu^2 T}{2}}{\nu\sqrt{T}}\right) \\
&= S_0 N\left(\frac{\ln\frac{S_0}{K} + rT + \frac{\nu^2 T}{2}}{\nu\sqrt{T}}\right)
\end{aligned}$$

which gives the first term of equation 2.16.

Now we examine the second term. Let $z = \frac{\ln \frac{x}{S_0} - \mu}{\nu\sqrt{T}}$, then $\mathrm{d}z = \frac{\mathrm{d}x}{x\nu\sqrt{T}}$ and

$$
\begin{aligned}
-e^{-rT}\int_K^\infty \frac{1}{\sqrt{2\pi T}\nu x} K e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\nu^2 T}}\,\mathrm{d}x &= -e^{-rT}\int_{A+\nu\sqrt{T}}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}\,\mathrm{d}z \\
&= -e^{-rT}K\left(1 - N\left(A + \nu\sqrt{T}\right)\right) \\
&= -Ke^{-rT}N\left(-A - \nu\sqrt{T}\right) \\
&= -Ke^{-rT}N\left(\frac{\ln \frac{S_0}{K} + rT - \frac{\nu^2 T}{2}}{\nu\sqrt{T}}\right)
\end{aligned}
$$

which is the second term of equation 2.16 and completes the proof.

□

3. Ito's Lemma

Some readers may have a problem with the above derivation. It certainly leads to the correct result, but we have made several assumptions that are not necessarily justified. First of all, it is not clear that a stock price should or will be log-normally distributed. This presents a real problem for the formula, but it is not something that we will deal with in this paper. Another problem in our assumptions is that of risk-neutrality, which is clearly not true. Most people are not risk-neutral, i.e. for some risky asset A and time period t, they will value the asset at $C(A, 0) < e^{-rt}\mathbf{E}[C(A, t)]$. In a sense, they must be compensated for the risk that they are bringing upon themselves. In this section, we will present the ideas behind this assumption and attempt to justify why it is, in fact, perfectly acceptable to make.

Definition 3.1. A stochastic process, W_t, for $t \geq 0$, is a *Brownian Motion* if $W_0 = 0$, and for all t and s, with $s < t$,

$$W_t - W_s$$

is continous, has a normal distribution with variance $t - s$, and the distribution of $W_t - W_s$ is independent of the behavior W_r for $r \leq s$.

Definition 3.2. The family X of random variables X_t *satisfies the stochastic differential equation* (SDE),

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t \tag{3.3}$$

if for any t,

$$X_{t+h} - X_t - h\mu(t, X_t) - \sigma(t, X_t)(W_{t+h} - W_t)$$

is a random variable with mean and variance which are $\mathrm{o}(h)$ and W_t is a Brownian motion.

Definition 3.4. A stochastic process S_t is said to follow a *Geometric Brownian Motion* if it satisfies the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \tag{3.5}$$

with μ and σ constants and W_t a Brownian motion.

Definition 3.6. An *Ito Process*, X_t, is a process that satisfies the stochastic differential equation

$$\mathrm{d}X_t = \mu(t)X_t\,\mathrm{d}t + \sigma(t)X_t\,\mathrm{d}W_t$$

Theorem 3.7. *(Ito's Lemma) Let X_t be an Ito process satisfying equation 3.3, and let $f(x,t)$ be a twice-differentiable function; then $f(X_t,t)$ is an Ito process, and*

$$\mathrm{d}(f(X_t,t)) = \frac{\partial f}{\partial t}(X_t,t)\,\mathrm{d}t + \frac{\partial f}{\partial X_t}\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,\mathrm{d}X_t^2 \tag{3.8}$$

where $\mathrm{d}X_t^2$ *is defined by*

$$\mathrm{d}t^2 = 0 \tag{3.9}$$

$$\mathrm{d}t\,\mathrm{d}W_t = 0 \tag{3.10}$$

$$\mathrm{d}W_t^2 = \mathrm{d}t \tag{3.11}$$

Remark 3.12. Proving this theorem is beyond our means at this time, but we would like to say something about equations 3.9-3.11 since they will be essential in deriving the Black-Scholes equation. Equations 3.9 and 3.10 might seem plausible based on the fact that $\mathrm{d}t$ is infintessimal, i.e. it seems reasonable that $\mathrm{d}t^2 = 0$ and $\mathrm{d}t\,\mathrm{d}W_t = 0$. However, equation 3.11 requires more justification. In order to provide a non-rigorous justification, it will be useful to examine random walk on Z. Imagine that a man takes a step of length 1 or -1 with equal probability at time t for $t > 0$, where t is a natural number. Let W_t be the sum of the steps from time $t = 1$ to t, then $\mathbf{E}[W_t] = 0$. Now consider W_t^2. $W_t^2 = t$ since each step now has length 1 and there are t steps. Thus, it seems reasonable that $|W_t| = O(\sqrt{t})$. So in Δt steps, $W_t = O(\sqrt{\Delta t})$. As $\Delta t \to 0$, $W_t^2 \approx \Delta t$, since the time increment and steps have become arbitrarily small. Equation 3.11 follows. This is certainly not a proof, but hopefully the reader will accept this for now.

Now let's get back to risk-neutrality. The standard model for changes in stock prices is geometric Brownian motion since stock prices are presumed to be log-normally distributed. Thus, using Ito's Lemma and equation 3.5, we will now justify the use of the risk-free interest rate in the derivation.

Theorem 3.13. *Given a European call option $C(S,t)$, with expiry T and strike price K, on a stock with price S that follows a geometric Brownian motion, and with r being the continuously compounding risk-free interest rate, then*

$$\frac{\partial C}{\partial t}(S,t) + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) = rC \tag{3.14}$$

This equation is known as the Black-Scholes equation (not to be confused with the Black-Scholes formula derived earlier, although it was originally used to derive the formula), and once shown, we will explain why this equation (and proof) justifies the use of the risk-free rate.

Proof. By equation 3.8,

$$\begin{aligned} \mathrm{d}C &= \frac{\partial C}{\partial t}(S,t)\,\mathrm{d}t + \frac{\partial C}{\partial S}(S,t)\,\mathrm{d}S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S,t)\,\mathrm{d}S^2 \\ &= \left(\frac{\partial C}{\partial t}(S,t) + \mu S\frac{\partial C}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)\right)\mathrm{d}t + \sigma S\frac{\partial C}{\partial S}(S,t)\,\mathrm{d}W_t \end{aligned}$$

since

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}W_t$$

and

$$\begin{aligned} \mathrm{d}S^2 &= \mu^2 S^2\,\mathrm{d}t^2 + \mu\sigma S^2\,\mathrm{d}t\,\mathrm{d}W_t + \sigma^2 S^2\,\mathrm{d}W_t^2 \\ &= \sigma^2 S^2\,\mathrm{d}t \end{aligned}$$

by equations 3.9, 3.10, and 3.11.

Now consider a portfolio consisting of the call option and α stocks. Then the cost of the portfolio is C+αS. By the same argument as above, we see that

$$\begin{aligned} \mathrm{d}(C+\alpha S) &= \left(\frac{\partial C}{\partial t}(S,t) + \mu S\frac{\partial C}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t) + \alpha\mu S\right)\mathrm{d}t \\ &+ \sigma S\left(\frac{\partial C}{\partial S}(S,t) + \alpha\right)\mathrm{d}W_t \end{aligned}$$

Now we let $\alpha = -\frac{\partial C}{\partial S}(S,t)$ to hedge away all risk in our portfolio.

$$\mathrm{d}(C+\alpha S) = \left(\frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)\right)\mathrm{d}t$$

As one can see, the random component $\mathrm{d}W_t$, is now gone. The portfolio has no risk or randomness. This is an important result, for, since it is risk-free, it must grow over time at the risk-free rate r. Thus, $\frac{\mathrm{d}}{\mathrm{d}t}(C+\alpha S) = r(C+\alpha S) = r\left(C - S\frac{\partial C}{\partial S}\right)$ and

$$r\left(C - S\frac{\partial C}{\partial S}\right) = \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)$$

Rearranging gives us the Black-Scholes equation:

$$\frac{\partial C}{\partial t}(S,t) + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t) = rC$$

□

By hedging away all randomness, we make sure that the portfolio has no risk, and that allows us to use the assumption of risk-neutrality. If we did not do this, then it seems very natural that the price of an option must take on the perceived risk with which the investor views the stock. That being said, it is also unlikely that there would be a unique price, for it is highly improbable that everyone would agree on the risk level.

4. Put-Call Parity

Definition 4.1. A *portfolio* is a collection of assets held by an institution or individual.

Definition 4.2. A portfolio is said to be an *arbitrage portfolio* if today it has zero value, and in the future it has positive value.

We will be dealing with the assumption of arbitrage-free pricing. Arbitrage clearly exists in the real world. One example is simply different prices for different assets in different markets. One might be able to buy something in one market and sell it for a higher price in a different market and make a profit. However, instances like these are usually eliminated quickly. People notice when there are arbitrage opportunities, and they pounce on them, thus adjusting prices so that these opportunities vanish.

Another instance of arbitrage can occur when dealing with two stocks. Suppose stock A and B are worth the same at time $t = 0$, but at time T, A is worth twice as much as before and B is still at its initial value. Then, one can create a portfolio that is long A and short B. Clearly, the portfolio has positive value at time T but zero value at time $t = 0$.

No-arbitrage means no free lunch, that a person can't make a riskless profit when he starts out with some portfolio with no value. Such an assumption makes things simpler and provides for a certain amount of necessary order in the world of financial mathematics. Without it, we would not be able to come up with a unique price for options.

Example 4.3. Suppose 1 dollar exchanges for 5 yen, 1 pound exchanges for 10 yen, and 1 dollar exchanges for 1 pound. Clearly there is an arbitrage opportunity here, for one could take a dollar, exchange it for a pound, exchange the pound for 10 yen, and exchange the yen for 2 dollars.

Theorem 4.4. *If, in an arbitrage-free world, portfolios A and B are such that at time T, A is worth at least as much as B, then at any time $t < T$, A will be worth at least as much as B.*

Proof. Let portfolio C consist of being long portfolio A and short portfolio B. Then at time T, the value of C is $A(T) - B(T)$ which is greater than or equal to 0. As C is an arbitrage-free portfolio, the value of C at any time $t < T$ is $A(t) - B(t)$ which also must be greater than or equal to 0 since, if it weren't, there is clearly an arbitrage opportunity, which is a contradiction. Thus, the value of A at any time $t < T$ must also be greater than or equal to that of B. □

Theorem 4.5. *(Put-Call Parity) Let $C(t)$ be the value of a European call option on an asset S with strike price K and expiration T. Let $P(t)$ be the value of a European put option on the same asset S with the same strike price and expiration. Finally, let S have a final value at expiration of S_T, and let $B(t,T)$ represent the value of a risk-free bond at time t with final value 1 at expiration time T. If these assumptions hold and there is no arbitrage, then*

$$C(t) + KB(t,T) = P(t) + S_t \tag{4.6}$$

Proof. Consider first a portfolio X that consists of one put option and one share of S. At time T, portfolio X has value

$$X_v = \begin{cases} K, & \text{if } S_t \leq K \text{ as the option will be worth } K - S_T \text{ and the share } S_T \\ S_T, & \text{if } S_T \geq K \text{ as the option will be worth 0 and the share } S_T. \end{cases}$$

Now consider a portfolio Y that consists of one call option and K bonds that pay 1 at time T with certainty. Then, at time T, portfolio Y has value

$$Y_v = \begin{cases} K, & \text{if } S_t \leq K \text{ as the option will be worth 0 and the bonds } K \\ S_T, & \text{if } S_T \geq K \text{ as the option will be worth } S_T - K \text{ and the bonds } K. \end{cases}$$

We can see that whatever value S takes at time T, portfolios X and Y have the same value. Thus, from Theorem 4.3, at any time $t < T$, the portfolios must also have the same value. It follows then that

$$C(t) + KB(t,T) = P(t) + S(t)$$

□

Now, it is straightforward to obtain the price of a European put option with strike price K and expiration time T. Recall from equation 2.16 that

$$C(0) = S_0 N\left(\frac{rT + \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT - \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right)$$

Also note that, if we assume the interest rate r is constant, which we have implicitly done in this paper, $B(0,T) = e^{-rT}$. And so,

$$\begin{aligned} P(0) &= C(0) - S_0 + Ke^{-rT} \\ &= S_0 N\left(\frac{rT + \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right) - S_0 + Ke^{-rT} - Ke^{-rT} N\left(\frac{rT - \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right) \\ &= -S_0\left(1 - N\left(\frac{rT + \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right)\right) + Ke^{-rT}\left(1 - N\left(\frac{rT - \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right)\right) \\ &= -S_0 N\left(-\frac{rT + \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right) + Ke^{-rT} N\left(-\frac{rT - \frac{\nu^2 T}{2} + \ln \frac{S_0}{K}}{\nu\sqrt{T}}\right) \end{aligned}$$

And we are done.

5. Barrier Options

Now we will examine a specific type of option called a barrier option. Unlike a normal vanilla option, the barrier option is contingent upon hitting some stock price, called the barrier, at any time before its expiry. There are two types of barrier options, knock-out options and knock-in options. A knock-out option becomes worthless if at any time before the expiry, the stock price reaches the barrier, while a knock-in option only provides a pay-off once the stock price crosses the barrier. In this paper, we will address down-and-out call options and, as a consequence, down-and-in call options. There is a very convenient relationship between the two that will be apparent once the pay-offs of the two are defined.

Definition 5.1. The payoff of a *down-and-out call option* with strike price K and barrier at B is

$$\max(S_T - K, 0)$$

unless at any time $t < T$, S_t passes below B.

One thing that should be pointed out is that it makes no sense for the barrier to be above the initial stock price. If this were the case, the option is immediately worthless. Thus, the option is called "down-and-out" because if it goes down past the barrier, the option "knocks out" and becomes worthless. Even if the final price of the stock, S_T, were well above K, the option's pay-off is 0 if the stock price ever hits the barrier. If it does not reach the barrier at any time before the expiry, then the payoff is identical to a normal European call option.

Definition 5.2. The payoff of a *down-and-in call option* with strike price K and barrier at B is

$$\max(S_T - K, 0)$$

if and only if at some time $t < T$, S_t passes below B.

S_t must pass below the barrier for some $t < T$ for the option to pay-off anything. As with the down-and-out call option, however, hitting the barrier in no way guarantees a positive payoff, since S_T must still be above K. Otherwise, the payoff is 0.

Because there is an extra condition on the barrier option, intuitively it seems that it must be worth less than a normal European option. In fact, one can see that, in a portfolio with exactly one down-and-out call option and one down-and-in option with the same strike price and barrier and for the same stock, exactly one of the barrier options will pay $\max(S_T - K, 0)$, which is the payoff off of a normal European call option. Thus, there is a very nice relationship between these two barrier options, namely that

$$C_I + C_O = C_v \tag{5.3}$$

where C_I stands for the price of the down-and-in call option, C_O for the price of the down-and-out call option, and C_v for the price of the normal European call option. We will now derive the price of the down-and-out call option and, afterward, it will be very simple to determine the price of the down-and-in call option.

Our first goal in pricing barrier options is to reduce the Black-Scholes equation to the heat equation, along with the extra condition that if the barrier is reached at any time during the option's life, the option becomes worthless. The heat equation is a well-known equation in physics and mathematics, and there are several ways to solve it. We will use a slight trick that presupposes that a solution exists, but other methods can be seen in [5].

The barrier option is essentially a normal option with an extra constraint. Thus, it satisfies the Black-Scholes equation (3.13) with the additional condition that

$$C(B, t) = 0$$

where B is the barrier of the option.

$$\frac{\partial C}{\partial t}(S,t) + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) = rC$$
$$C(S,T) = \max(S - K, 0)$$

$$C(B,t) = 0$$

Now we will use several change of variables in order to reduce this to the heat equation. Let $S = Be^x$, then

$$\begin{aligned}\frac{\partial C}{\partial x} &= \frac{\partial C}{\partial S}\frac{\partial S}{\partial x} \\ &= \frac{\partial C}{\partial S}S\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 C}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial C}{\partial x} \\ &= \frac{\partial}{\partial x}\left(S\frac{\partial C}{\partial S}\right) \\ &= \frac{\partial S}{\partial x}\frac{\partial C}{\partial S} + S\frac{\partial}{\partial x}\left(\frac{\partial C}{\partial S}\right) \\ &= S\frac{\partial C}{\partial S} + S\frac{\partial^2 C}{\partial S \partial x} \\ &= S\frac{\partial C}{\partial S} + S^2\frac{\partial^2 C}{\partial S}\end{aligned}$$

since

$$\frac{\partial C}{\partial x} = S\frac{\partial C}{\partial S}$$

Now we arrive at,

$$\frac{\partial C}{\partial t} + (r - \frac{1}{2}\sigma^2)\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2} = rC$$

And the boundary conditions become

$$C(x,T) = \max(Be^x - K, 0) \text{ and } C(0,t) = 0$$

Now let $t = T - \tau/\frac{1}{2}\sigma^2$. Then,

$$-\frac{1}{2}\sigma^2\frac{\partial C}{\partial \tau} + (r - \frac{1}{2}\sigma^2)\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2} = rC \qquad \text{since } \frac{\partial t}{\partial \tau} = -1/\frac{1}{2}\sigma^2$$

and the boundary conditions become

$$C(x,0) = \max(Be^x - K, 0) \text{ and } C(0,\tau) = 0$$

Finally, let $C = Be^{\alpha x + \beta \tau}u(x,\tau)$. Then,

$$-\frac{1}{2}\sigma^2\left(\beta u + \frac{\partial u}{\partial \tau}\right) + (r - \frac{1}{2}\sigma^2)\left(\alpha u + \frac{\partial u}{\partial x}\right) + \frac{1}{2}\sigma^2\left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) = ru$$

We want to eliminate the u term and the $\frac{\partial u}{\partial x}$ term so we can set up 2 equations to find values of α and β such that this will occur.

$$-\frac{1}{2}\sigma^2\beta + (r - \frac{1}{2}\sigma^2)\alpha + \frac{1}{2}\sigma^2\alpha^2 - r = 0$$

$$r - \frac{1}{2}\sigma^2 + \sigma^2\alpha = 0$$

So

$$\alpha = -\frac{r}{\sigma^2} + \frac{1}{2} \qquad \text{and} \qquad \beta = -\frac{r^2}{\sigma^4} - \frac{1}{4} - \frac{r}{\sigma^2}$$

With these values for α and β, we arrive at the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

for $0 < x < \infty$, $\tau > 0$ and with boundary conditions

$$u(x,0) = U(x) = \max(e^{x-\alpha x} - \frac{K}{B}e^{-\alpha x}, 0), \qquad x > 0$$

and

$$u(0,\tau) = 0$$

This problem has now become that of the flow of heat in an infinite bar. However, there is a slightly new condition in $u(0,\tau) = 0$. This essentially says that at any time t, the temperature is 0 at $x = 0$. This can be modelled by creating a two-bar semi-infinite problem with one side hot and the other cold, so that the heat flow exactly cancels out and the temperature is 0 at $x = 0$.

The heat equation is invariant under reflection so $u(x,\tau)$ and $u(-x,\tau)$ are both solutions for it. Now, we will solve for all x, instead of just $x > 0$. Now,

$$u(x,0) = \begin{cases} U(x), & x > 0 \\ -U(-x), & x < 0 \end{cases}$$

or

$$u(x,0) = \begin{cases} \max(e^{x-\alpha x} - \frac{K}{B}e^{-\alpha x}, 0), & x > 0 \\ -\max(e^{\alpha x - x} - \frac{K}{B}e^{\alpha x}, 0), & x < 0 \end{cases}$$

This guarantees that $u(0,\tau) = 0$. Rather than integrating this, there is a slight trick that we can use to find the price of the down-and-out option. (For a more traditional approach, see [5].) Consider a normal European call option with the same expiry and exercise price but no barrier. Let its value be $C_v(S,T;K)$ and let $U_v(x,\tau)$ be the corresponding solution for the heat equation. When $S < K$, $C_v(S,T;K) = 0$. Thus, since $S = Be^x$, when $x < \ln\frac{K}{B}$ then $U_v(x,\tau) = 0$. As we assumed that the strike price K was higher than the barrier B, then $\ln\frac{K}{B} > 0$. Thus, if we set $U(x) = 0$ for $x < 0$, then we extend $U(x)$ for all x and $U(x)$ is now equal to $U_v(x)$. We can now write

$$u(x,0) = U_v(x) - U_v(-x)$$

and this holds for all x. Thus,

$$u(x,\tau) = U_v(x,\tau) - U_v(-x,\tau)$$

since both sides of this equation satisfy all conditions and both solve the heat equation. By uniqueness of the solution, they must be equivalent.

$$C_v(S,t;K) = C_v(Be^x, t(\tau);K) = Be^{\alpha x + \beta\tau}U_v(x,\tau)$$

shows that

$$U_v(x,\tau) = e^{-\alpha x - \beta\tau}C_v(Be^x, t(\tau);K)/B$$

and

$$U_v(-x,\tau) = e^{\alpha x - \beta\tau}C_v(Be^{-x}, t(\tau);K)/B$$

Thus, the value of the down-and-out call option is

$$\begin{aligned}
C_O(S,t) &= Be^{\alpha x+\beta\tau}u(x,\tau) \\
&= Be^{\alpha x+\beta\tau}(U_v(x,\tau)-U_v(-x,\tau)) \\
&= Be^{\alpha x+\beta\tau}(e^{-\alpha x-\beta\tau}C_v(Be^x,t(\tau);K)/B - e^{\alpha x-\beta\tau}C_v(Be^{-x},t(\tau);K)/B) \\
&= C_v(Be^x,t(\tau);K) - e^{2\alpha x}C_v(Be^{-x},t(\tau);K) \\
&= C_v(S,t;K) - \left(\frac{S}{B}\right)^{2\alpha} C_v(B^2/S,t;K)
\end{aligned}$$

or

$$\begin{aligned}
C_O = S_0 N\left(\frac{rT+\frac{\nu^2 T}{2}+\ln\frac{S_0}{K}}{\nu\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{rT-\frac{\nu^2 T}{2}+\ln\frac{S_0}{K}}{\nu\sqrt{T}}\right) \\
- B\left(\frac{S_0}{B}^{-2r\sigma^{-2}}\right)N\left(\frac{rT+\frac{\nu^2 T}{2}+\ln\frac{B^2}{S_0K}}{\nu\sqrt{T}}\right) \\
-\left(\frac{S_0}{B}^{1-2r\sigma^{-2}}\right)Ke^{-rT}N\left(\frac{rT-\frac{\nu^2 T}{2}+\ln\frac{B^2}{S_0K}}{\nu\sqrt{T}}\right)
\end{aligned}$$

And by equation 5.3, it is clear that the price of the down-and-in call option is

$$\begin{aligned}
C_I = B\left(\frac{S_0}{B}^{-2r\sigma^{-2}}\right)N\left(\frac{rT+\frac{\nu^2 T}{2}+\ln\frac{B^2}{S_0K}}{\nu\sqrt{T}}\right) \\
+\left(\frac{S_0}{B}^{1-2r\sigma^{-2}}\right)Ke^{-rT}N\left(\frac{rT-\frac{\nu^2 T}{2}+\ln\frac{B^2}{S_0K}}{\nu\sqrt{T}}\right)
\end{aligned}$$

Pricing for other types of barrier options is done similarly. For further study, see [4].

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