EXPANDER GRAPHS AND PROPERTY (T)

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ABSTRACT. Families of expander graphs are sparse graphs such that the number of vertices in each graph grows yet each graph remains difficult to disconnect. Expander graphs are of great importance in theoretical computer science. In this paper we study the connection between the Cheeger constant, a measure of the connectivity of the graph, and the smallest nonzero eigenvalue of the graph Laplacian. We show for expander graphs these two numbers are strictly bounded away from zero. Given a finitely generated locally compact group satisfying Kazhdan's property (T), we construct expanders from the Cayley graphs of finite index normal subgroups with finite generating sets. We follow Alexander Lubotzky's treatment in [7].

CONTENTS

1. INTRODUCTION

In this expository paper we explicitly construct expander families. Expander graphs solve the most basic problem in the design of networks; designing a robust network to connect a large number of disjoint sets of users. Expander graphs strike the ideal balance between the number of connections between nodes and the reliability of the network as the number of nodes grows.

The method of construction we will employ follows A. Lubotzky's onstruction in chapter 4 of [7]. In the construction we exploit the connections between graphs and representations of locally compact topological groups. In order to translate the problem of constructing expander graphs to an algebraic one we will look at the smallest nonzero eigenvalue of the graph Laplacian. To translate back to the world of graphs we will construct a graph by taking the Cayley graph of a group satisfying property (T) where the edges represent elements of a finite generating set.

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It is desirable to explicitly construct expander graphs because of their numerous applications in the field of theoretical computer science. We briefly summarize M. Klawe's excellent review of these applications which can be found in the introduction of [6]. Expander graphs are used in the construction of sparse graphs with dense long paths, the design of fault-tolerant microelectronic chips, and in an algorithm that reduces the number of calls to a random number generator made by almost any Monte-Carlo algorithm while still running in polynomial time. Throughout this paper we will primarily be concerned with regular graphs because these are the most useful in applications.

The author owes a great intellectual debt to A. Lubotzky's treatment of expander graphs and property (T) in chapters 3 and 4 of [7]. The author has tried to present the basic theory of expander graphs in a self-contained paper while providing motivation and more detailed explanations.

2. Basics of Expander Graphs

The most intuitive definition of expanders is purely combinatorial. Eventually when we explicitly construct expanders we will have to translate to the language of algebra. The most natural object of study is the smallest nonzero eigenvalue of the graph Laplacian, λ_1. The first definition does not provide us with any obvious way of connecting these two properties of a graph. We introduce a number associated with every graph, the Cheeger constant, which allows us to compare the expansion properties of different graphs. A family of graphs is an expander graph family if the Cheeger constant is bounded away from zero. Later we will prove the Cheeger constant is bounded away from zero for an expander family if and only if λ_1 is bounded away from zero.

Definition 2.1. Given a k-regular graph $X = (V, E)$ with $|V| = n$ vertices we call it an (n, k, c) expander if there exists $c > 0$ such that

$$|\partial A| \geq c\left(1 - \frac{|A|}{n}\right)|A| \tag{2.2}$$

for all subsets $A \subseteq V$ where we have denoted the boundary of A by $\partial A = \{v \in V \mid d(v, A) = 1\}$.

While this definition is only valid for finite graphs, we are primarily interested in infinite graphs. Finite graphs are uninteresting because for every finite graph we can find $c > 0$ such that the graph is an expander graph for that c. If we allow ourselves to consider infinite graphs, for instance the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generators 1 and -1 as n goes to infinity, we see that there is no c such that this graph is an expander graph for all n. In order to approximate infinite graphs we will consider families of finite graphs without a finite bound on the number of vertices in each graph.

Definition 2.3. A family of expander graphs $\{X_i\}_{i\in I}$ is a collection of graphs such that each X_i is a (n_i, k, c)-expander where k and c are fixed for all X_i and n_i goes to infinity.

Next we introduce a constant associated with a graph that measures its expansion properties. In an expander graph for any proper subset, A, of the vertices there is a vertex outside of A connected by an edge to a vertex in A. When c is large there are more vertices outside of A connected by an edge to vertices inside A. More edges must be removed from the graph in order to disconnect it into two disjoint pieces. In order to measure the difficulty required to disconnect the graph we assign a number, the Cheeger constant, to each graph. This number will be essential in the construction of expander graphs.

Definition 2.4. For a graph $X = (V, E)$ and A a subset of the vertices define the Cheeger ratio to be

$$\hat{h}_A(X) = \frac{|E(A, A^c)|}{\min\{|A|, |A^c|\}} \tag{2.5}$$

where $E(A, A^c)$ denotes the edges between A and A^c. The Cheeger constant is defined to be

$$h(X) = \inf_{A \subseteq V} \hat{h}_A(X).$$

The Cheeger ratio will be useful when we wish to show an upper bound on the Cheeger constant because for any subset of the vertices, the inequality $\hat{h}_A(X) \geq h(X)$ holds. We can use the definitions to show constructing a family of (n, k, c)-expander graphs for a given $c > 0$, fixed k, and n going to infinity is equivalent to constructing an infinite family of k-regular graphs with a Cheeger constant strictly greater than zero.

Proposition 2.6. *Let X be a k-regular graph with n vertices.*

i) If X is an (n, k, c)-expander then $h(X) \geq \frac{c}{2}$.
ii) If X is an (n,k,c) expander then X is an $(n, k, \frac{h(X)}{k})$-expander.

Proof of (i). Pick a subset, A, of the vertices such that the Cheeger ratio calculated for the set A is equal to the Cheeger constant.

$$\frac{|E(A, B)|}{\min\{|A|, |A^c|\}} = \inf_{A \subseteq V} \frac{|E(A, B)|}{\min\{|A|, |A^c|\}} = h(X)$$

Notice the inequality $|\partial A| \leq |E(A, A^c)|$ must hold. In addition $\max\{|A|, |A^c|\} \geq n/2$ because $A \cup A^c = V$. Next a simple calculation shows

$$\frac{c}{2} \leq \frac{n}{(n - |A|)} \frac{|\partial A|}{2|A|} = \frac{n}{2} \frac{|\partial A|}{|A||A^c|} \leq \frac{n}{2} \frac{|E(A, A^c)|}{|A||A^c|} \leq \frac{|E(A, A^c)|}{\min\{|A|, |A^c|\}} = h(X).$$

□

Proof of (ii). Let A be a subset of the vertices. Without loss of generality assume $|A| \leq n/2$. If this is not true then $|A^c| \leq n/2$ and we apply the method of the proof to A^c. By definition we can bound the Cheeger constant by the ratio

$$h(X) \leq \frac{|E(A, A^c)|}{\min\{|A|, |A^c|\}} \leq \frac{k|\partial A|}{|A|}.$$

We need to show that if we substitute $h(X)/k$ for the constant in the definition of an expander graph than the inequality still holds.

$$\frac{h(X)}{k} \left(1 - \frac{|A|}{n}\right) |A| \leq |\partial A| \left(1 - \frac{|A|}{n}\right) \leq |\partial A|$$

□

2.1. **Existence of Expander Graphs.** The goal of the paper, explicit construction of expander families, is very difficult to achieve. However, the existence of expander families can be shown with combinatorial methods. In fact one can prove most k-regular graphs are expander graphs. The proof can be found in section 1.2 of [7]. We have omitted the proof because the argument shares no similarities with the construction we will present. It is interesting to note the problem of constructing expanders is stated in purely combinatorial language yet there is no known combinatorial method to construct them. Instead we must translate this problem into an algebraic one in order to use group theory to present the solution to the problem.

3. Graph Laplacian

In addition to [7], we give credit to [4, p. 472] for this explanation of the graph Laplacian. We would like to define the graph Laplacian in a similar manner to the Laplacian defined in elementary vector calculus for a real valued function. Namely for a real valued function f the Laplacian of f is equal to the divergence of the gradient of f. We can define analogous notions of the gradient and divergence for functions that assign a number to every vertex of a graph.

The derivative measures rate and direction of change. Just as in Euclidean space we arbitrarily define an orientation to make simplifications, for graphs we can make the notation simpler by defining an orientation on the edges of the graph. The particular orientation chosen will have no effect on the results presented in this section. Given a graph $X = (V, E)$, for every edge e, we denote the terminal vertex of the edge with respect to the orientation by e^+ and the initial vertex of the orientation by e^-. We can represent this orientation with the matrix $D_{e,v} = \begin{cases} +1 & \text{if } v = e^+ \\ -1 & \text{if } v = e^- \\ 0 & \text{else .} \end{cases}$

In vector calculus we think of the gradient as measuring the change in the function as we move along the coordinate axes. Likewise we should expect the combinatorial gradient to measure the change in our function as we move along each edge from a given vertex.

Definition 3.1. Given a real-valued function f on the vertices of the graph, $X(V, E)$, we define the operator $d : \mathbb{R}(V) \to \mathbb{R}(E)$ by $df(e) = f(e^+) - f(e^-)$. We use $\mathbb{R}(V)$ to denote the set of real-valued functions on the vertices of the graph and $\mathbb{R}(E)$ to denote the set of real-valued functions on the edges of the graph.

We can think of this one form as a row vector where each entry corresponds to the value of f at a vertex. In matrix notation we can write this operator as $(fD)_e = f(e^+) - f(e^-)$. Likewise we can define a notion of the divergence. Let $g \in \mathbb{R}(E)$. We can think of g as a column vector where each entry corresponds to the flow out of a vertex.

Definition 3.2. The divergence of g is defined by

$$(Dg)_v = \sum_{\{e \in E | v = e^-\}} g(e) - \sum_{\{e \in E | v = e^+\}} g(e)$$

Now we are prepared to define the Laplacian of a function $f \in \mathbb{R}(V)$. As mentioned earlier the Laplacian does not depend on the orientation of the graph.

Definition 3.3. The Laplacian operator is defined by $\Delta = D^*D$ where D^* denotes the transpose of the matrix D.

A simple calculation shows we can equivalently define the graph Laplacian in a manner that is simpler to compute and does not require the choice of an orientation on the graph.

Proposition 3.4. *Let $X = (V, E)$ be a graph. Then $\Delta = S - \delta_{ij}$ where δ_{ij} is the adjacency matrix $\delta_{ij} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{else }. \end{cases}$ and S is a diagonal matrix where each diagonal entry is equal to the degree of the vertex. The notation $i \sim j$ means the vertices i and j are joined by an edge.*

Proposition 3.5. *Given two functions, $f, g \in \mathbb{R}(V)$ such that $\langle f, f\rangle$ and $\langle g, g\rangle$ are finite, the Laplacian satisfies $\langle f, \Delta g\rangle = \langle df, dg\rangle$ where we use $\langle \cdot, \cdot\rangle$ to denote the inner product, $\langle f, g\rangle = \sum_{v\in V} f(v)g(v)$. From this we conclude the Laplacian is a positive self-adjoint operator. Therefore its eigenvalues are real and nonnegative.*

Proof. We can write out the Laplacian operator in the form of a sum

$$\Delta f(v) = \deg(v)f(v) - \sum_{u\in V} \delta_{vu} f(u) = \sum_{u\in V} \delta_{vu}(f(v) - f(u)).$$

Where we have used $\deg(v)$ to denote the degree of the vertex v. Next by a computation we show the Laplacian is self-adjoint and the two definitions are equivalent

$$\begin{aligned} \langle f, (S-\delta)g\rangle &= \sum_{v\in V} f(v)\left(\deg(v)g(v) - \sum_{u\in V}\delta_{vu}g(u)\right) \\ &= \sum_{v\in V}\deg(v)f(v)g(v) - \sum_{v\in V}\sum_{u\in V}\delta_{vu}f(v)g(v) \end{aligned}$$

$$\begin{aligned} \langle df, df\rangle &= \sum_{e\in E}(df(e))\cdot(dg(e)) = \sum_{e\in E}(f(e^+) - f(e^-))(g(e^+) - g(e^-)) \\ &= \sum_{e\in E}(f(e^+)g(e^+) + f(e^-)g(e^-)) - \sum_{e\in E}(f(e^+)g(e^-) + f(e^-)g(e^+)) \\ &= \sum_{v\in V}\deg(v)f(v)g(v) - \sum_{v\in V}\sum_{u\in V}\delta_{vu}f(v)g(u) \end{aligned}$$

$$\langle df, df\rangle = \langle f, (S-\delta)g\rangle = \langle f, \Delta g\rangle.$$

□

Now we need to find a way to compute λ_1, the smallest nonzero eigenvalue of the graph Laplacian. We will use the graph theory analog of the Rayleigh Quotient method.

Proposition 3.6.

$$\lambda_1(X) = \inf_{f\in\mathbb{R}_0(V)}\left(\frac{\langle df, df\rangle}{\langle f, f\rangle}\right)$$

where $\mathbb{R}_0(V)$ denotes the space of real-valued functions on the vertices such that $\sum_{v\in V} f(v) = 0$.

Proof. The constants are orthogonal to the functions in the space $\mathbb{R}_0(V)$ because for some constant $c \in \mathbb{R}$ and $f \in \mathbb{R}_0(V)$ we can compute

$$\langle c, f\rangle = c \sum_{v \in V} f(v) = 0.$$

This combined with the fact Δ is self-adjoint imply $\lambda_1(X)$ is the smallest positive eigenvalue of Δ on $L_0^2(X)$.

In order to compute $\lambda_1(X)$ we will use the Rayleigh quotient method.

$$\lambda_1(X) = \inf_{f \in \mathbb{R}_0(V)} \left(\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} \right) = \inf_{f \in \mathbb{R}_0(V)} \left(\frac{\langle df, df\rangle}{\langle f, f\rangle} \right).$$

Next we show why this method works. Let $f, g \in L^2(X)$ and f a minimizer of the Rayleigh Quotient. The graphs and sums are finite allowing us to ignore issues of convergence. Define $w_\epsilon(x) = f(x) + \epsilon g(x)$. Then the Rayleigh Quotient is

$$Q(w_\epsilon) = h(\epsilon) = \frac{\sum_{x \in V} (dw_\epsilon(e))^2}{\sum_{x \in V} w_\epsilon^2(x)}.$$

By assumption this function has a minimum at $\epsilon = 0$.

$$h(\epsilon) = \frac{\sum_{e \in E} \big((df(e))^2 + \epsilon^2 (dg(e))^2 + 2\epsilon df(e) dg(e)\big)}{\sum_{x \in V} f^2(x) + \epsilon^2 g^2(x) + 2\epsilon f(x) g(x)}$$

If we differentiate with respect to ϵ term by term we find

$$\begin{aligned} h'(\epsilon) = {} & \frac{\left(\sum_{e \in E} 2\epsilon (dg(e))^2 + 2df(e)dg(e)\right)\left(\sum_{x \in V} f^2(x) + \epsilon^2 g^2(x) + 2\epsilon f(x) g(x)\right)}{\left(\sum_{x \in V} (f^2(x) + \epsilon^2 g^2(x) + 2\epsilon f(x) g(x))\right)^2} \\ & - \frac{\left(\sum_{x \in V} \left(2\epsilon g^2(x) + 2f(x)g(x)\right)\right)\left(\sum_{e \in E} \left((df(e))^2 + \epsilon^2 (dg(e))^2 + 2\epsilon df(e)dg(e)\right)\right)}{\left(\sum_{x \in V} (f^2(x) + \epsilon^2 g^2(x) + 2\epsilon f(x) g(x))\right)^2} \end{aligned}$$

$$0 = h'(0) = \frac{\left(\sum_{e \in E} 2df(e)dg(e)\right)\left(\sum_{x \in V} f^2(x)\right) - \left(\sum_{x \in V} 2f(x)g(x)\right)\left(\sum_{e \in E} (df(e))^2\right)}{\left(\sum_{x \in V} f^2(x)\right)^2}$$

$$\begin{aligned} \sum_{e \in E} df(e)dg(e) = \left(\sum_{x \in V} f(x)g(x)\right)\left(\frac{\sum_{e \in E}(df(e))^2}{\sum_{x \in V} f^2(x)}\right) &= Q(f)\left(\sum_{x \in V} f(x)g(x)\right) \\ &= \sum_{x \in V} Q(f) f(x) g(x) \end{aligned}$$

$$\sum_{x \in V} \left(\deg(x) f(x) g(x) - \sum_{y \in V} \delta_{xy} f(x) g(y) - Q(f) f(x) g(x) \right) = 0$$

This must hold for all graphs. In particular, it should hold for any subset of the vertices of a graph because the sums will also be finite on these subsets. Therefore we have the equality

$$\deg(x) f(x) g(x) - \sum_{y \in V} \delta_{xy} f(x) g(y) - Q(f) f(x) g(x) = 0.$$

This is true for all $g \in \mathbb{R}(V)$ and if we pick $g(x) = f(x)$ then

$$0 = \deg(x)f^2(x) - \sum_{y \in V} \delta_{xy} f(x)f(y) - Q(f)f^2(x)$$
$$Q(f)f = \deg(x)f(x) - \sum_{y \in V} \delta_{xy} f(y) = \Delta f(x).$$

□

Now we begin the process of connecting $\lambda_1(X)$ and $h(X)$. We desire to show bounding $\lambda_1(X)$ away from zero is equivalent to bounding $h(X)$ away from zero. Then if we construct graphs such that $\lambda_1(X) > 0$ we know they are expander graphs. The next two propositions can be found as 4.2.5 and 4.2.6 of [7]. In addition the author has relied on Theorem 2.3 of [2] for guidance.

Proposition 3.7 (Cheeger's Inequality for Graphs)**.** *Let X be a finite graph with $\deg(x) \leq m$ for every vertex x. Then $\lambda_1(X) \geq \frac{h^2(X)}{2m}$.*

Proof. Consider $g \in \mathbb{R}_0(V)$ an eigenfunction of the Laplacian with eigenvalue $\lambda_1(X)$ and $\|g\| = 1$. It will be easier to bound the Cheeger constant from below if we consider a subset of the vertices that is less than or equal to half of the total number of vertices. The Cheeger constant is a ratio of the edges between a subset of the vertices and the complement of the subset divided by the number of vertices in the smaller subset and its complement. If we consider a subset with at most half the total number of vertices, then the Cheeger ratio for this subset will always be greater than the Cheeger constant. One method to do this is to consider the set of vertices on which f is strictly positive. Define the set of positive values of g to be $V^+ = \{v \in V \mid g(v) > 0\}$. Now we take a new function $f(v) = \begin{cases} g(v) & \text{if } v \in V^+ \\ 0 & \text{else.} \end{cases}$
We can assume without loss of generality $|V^+| \leq \frac{1}{2}|V|$. If $|V^+| > \frac{1}{2}|V|$ then we can consider the set $V^- = \{v \in | \, g(v) < 0\}$ of cardinality less than or equal to $\frac{1}{2}|V|$.

In order to get the best bound possible on the Cheeger constant we must choose these subsets properly so that we get a relation between $\lambda_1(X)$ and the Cheeger constant. We get such a relation if we choose our subsets to be $L_i = \{x \in V \mid f(x) \geq \beta_i\}$ where the β_i denote the sequence of finitely many distinct values of f union 0. We define the sequence $\{\beta_i\}_{i=1}^r$ by $\beta_0 = 0$ and $\beta_i < \beta_{i+1}$ for $i > 0$. The L_i provide us with a lower bound on the Cheeger constant because our earlier assumption $|V^+| \leq \frac{1}{2}|V|$ implies $|L_i| \leq \frac{1}{2}|V|$. For each L_i, $h(X) \leq \frac{|E(L_i, L_i^c)|}{|L_i|}$. In order to relate $h(X)$ to $\lambda_1(X)$ we need to relate $h(X)$ to the values of the eigenfunction. We will need to define the proper constant in order to do so. Before defining this constant we provide some motivation behind the choice of constant. While f is not an eigenfunction, we can say something about the Laplacian of f in relation to λ_1. In order to do this we will take the inner product of f and Δf and relate this to the eigenfunction g for which we know $\Delta g = \lambda_1 g$.

$$\begin{aligned}
\langle df, df\rangle &= \sum_{x\in V} f(x) \sum_{y\in V} \delta_{xy}(f(x) - f(y)) \\
&= \sum_{x\in V^+} g(x) \sum_{y\in V} \delta_{xy}(g(x) - g(y)) - \sum_{x\in V^+} g(x) \sum_{y\in V^-} \delta_{xy}(g(x) - g(y)) \\
&\leq \sum_{x\in V^+} g(x) \sum_{y\in V} \delta_{xy}(g(x) - g(y)) \\
&= \sum_{x\in V^+} g(x)\Delta g(x) = \lambda_1(X) \sum_{v\in V^+} g^2(x) = \lambda_1(X)\langle f, f\rangle
\end{aligned}$$

In order to get the inequality in the third line we used the fact $g(y) < 0$ for $y \in V^-$ and therefore the right hand side is always negative.

Now we can see we must bound the Cheeger constant above by something involving the ratio $\frac{\langle df,df\rangle}{\langle f,f\rangle}$. The constant $A = \sum_{e\in E} |f^2(e^+) - f^2(e^-)|$ is a good choice because if we expand it out to understand what it means we will see that A^2 is bounded above by some constant times the product $\langle df, df\rangle \cdot \langle f, f\rangle$.

$$\begin{aligned}
A &= \sum_{e\in E} |f^2(e^+) - f^2(e^-)| = \sum_{e\in E} |f(e^+) + f(e^-)||f(e^+) - f(e^-)| \\
&\leq \left(\sum_{e\in E} (f(e^+) + f(e^-))^2\right)^{1/2} \left(\sum_{e\in E} (f(e^+) - f(e^-))^2\right)^{1/2} \\
&= \left(\sum_{e\in E} (f^2(e^+) + f^2(e^-) + 2f(e^+)f(e^-))\right)^{1/2} \langle df, df\rangle^{1/2} \\
&\leq \left(2\sum_{e\in E} (f^2(e^+) + f^2(e^-))\right)^{1/2} \langle df, df\rangle^{1/2} \\
&\leq \sqrt{2m}\langle f, f\rangle^{1/2} \sqrt{\lambda_1(X)}\langle f, f\rangle^{1/2} = \sqrt{2m\lambda_1(X)}\langle f, f\rangle \\
\lambda_1(X) &\geq \frac{A^2}{2m\langle f, f\rangle^2}
\end{aligned}$$

We used the Cauchy-Schwarz Inequality to get the second and fourth lines. To get from line 4 to line 5 we noticed in the sum $\sum_{e\in E} f^2(e^+) - f^2(e^-)$ each vertex gets counted as many times as its degree and the degree of each vertex is at most m and the fact proved earlier, $\langle df, df\rangle \leq \lambda_1(X)\langle f, f\rangle$.

Next we need to relate A to the Cheeger constant to complete the proof. Now we return to the family of sets $\{L_i\}$ which we carefully defined before and show their importance.

We will desire to compute $f^2(x) - f^2(y)$ for x and y adjacent vertices. It may seem like a complicated process to go through and find which β_i corresponds to $f(x)$ and likewise for $f(y)$. However, we can use the identity

$$\beta_i^2 - \beta_{i-j}^2 = (\beta_i^2 - \beta_{i-1}^2) + (\beta_{i-1}^2 - \beta_{i-2}^2) + \cdots + (\beta_{i-j-1}^2 - \beta_{i-j}^2)$$

and we see how advantageous this is to simplify our notation when computing a sum.

$$A = \sum_{i=1}^{r} \sum_{e \in E(L_i, L_i^c)} \beta_i^2 - \beta_{i-1}^2 = \sum_{i=1}^{r} |E(L_i, L_i^c)|(\beta_i^2 - \beta_{i-1}^2)$$

As we commented before about the Cheeger ratio we know $|E(L_1, L_i^c)| \geq h(X)|L_i|$ for $i > 0$.

$$\begin{aligned} A &\geq h(X) \sum_{i=1}^{r} |L_i|(\beta_i^2 - \beta_{i-1}^2) = h(X) \left(|L_r|\beta_r^2 + \sum_{i=1}^{r-1} \beta_i^2(|L_i| - |L_{i+1}) \right) \\ & h(X)\langle g, g\rangle \geq h(X)\langle f, f\rangle \end{aligned}$$

In the first line we used summation by parts and in the second line we used the observation a vertex, x, is in $L_i \setminus L_{i+1}$ exactly when $f(x) = \beta_i$ for some i. Therefore we have shown $\lambda_1(X) \geq \frac{h^2(X)}{2m}$. □

Proposition 3.8 (Reverse Cheeger's Inequality.)**.** *Under the same conditions,* $h(X) \geq \lambda_1(X)/2$.

Proof. Given a finite graph, $X = (V, E)$ with $\deg(v) \leq m$ for all vertices, we place an upper bound on $\lambda_1(X)$ based on the Cheeger constant, $h(X)$. In order to show this we construct a function whose Rayleigh Quotient only picks up values of f on vertices that have an edge connecting them to the other set. This gives us a way to count the number of edges between the two sets which is exactly what the Cheeger constant measures.

Assume $|V| = n$, and the graph is connected. Divide V into two disjoint subsets A, B such that the union of A and B is all of V. Define $|A| = a$ and $|B| = b$. Assume without loss of generality $a \leq \frac{n}{2}$. Define $g(v) = \begin{cases} b & \text{if } v \in A, \\ -a & \text{if } v \in B. \end{cases}$ Notice $g \in \mathbb{R}_0(x)$. Then if we take the Rayleigh quotient we can bound $\lambda_1(X)$ from above

$$\begin{aligned} \lambda_1(x) = \inf_{f \in \mathbb{R}_0(V)} \frac{\|df\|^2}{\|f\|^2} &\leq \frac{\sum_{e \in E}(g(e^+) - g(e^-))^2}{\sum_{v \in A} g^2(v) + \sum_{v \in B} g^2(v)} \\ &= \frac{|E(A, B)|n^2}{b^2 a + a^2 b} = \frac{|E(A, B)|n^2}{nab}. \end{aligned}$$

For every subset of V we can estimate

$$\frac{|E(A, B)|n^2}{nab} = \frac{n/2}{\max(a, b)} \frac{2|E(a, b)|}{\min(a, b)} \leq \frac{2|E(A, B)|}{\min(a, b)}.$$

because the inequality $\max(a, b) \geq \frac{n}{2}$ holds. In particular the set V is finite, therefore this inequality also holds for the infimum over all subsets A of the vertices. Therefore the inequality can give us a lower bound on the Cheeger constant

$$\frac{\lambda_1(X)}{2} \leq h(X).$$

□

Thus we have proven an expander graph is a graph whose Cheeger constant is bounded away from zero or equivalently a graph whose smallest nonzero eigenvalue, $\lambda_1(X)$, is bounded away from zero.

4. Cayley Graphs and Explicit Construction of Expander Families

We present the construction of expander graphs but reserve the proof for the next section of the paper. Let G be a finitely generated group, and $S = \{s_1, \dots s_n\}$ a finite generating set. We can associate to the group the Cayley graph, $X(G, S)$ whose vertices represent elements of the group. Two vertices, $g, h \in G$ are joined by an edge if there exists some $s_j \in S$ such that $g = s_j h$. In order to explicitly construct expander families we rely on the ability to examine Cayley graphs from both graph theoretic and algebraic perspectives.

In the later sections of the paper we will prove Cayley graphs of groups satisfying relative property (T) are expander graphs. In particular we show the group $SL_n(\mathbb{Z})$ has property (T) for $n \geq 3$. We fix n and take a finite generating set S_n. It is proven in proposition 5 of [9] that two elements suffice to generate $SL_n(\mathbb{Z})$ for $n \geq 3$. In fact the set $S_n = \{A_n, B_n\}$ where

$$A_n = \begin{pmatrix} 1 & 1 & \vdots & 0_{2\times(n-2)} \\ 0 & 1 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & 0_{(n-2)\times 2} & \vdots & I_{n-2} \end{pmatrix}, B_n = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 1 \\ (-1)^{n-1} & & & & 0 & \end{pmatrix}$$

generates $SL_n(\mathbb{Z})$.

We construct an expander family from the Cayley graphs $X(SL_n(\mathbb{Z}/p\mathbb{Z}), S_n)$ where the vertices are finite index normal subgroups from the reduction modulo a prime p maps and p runs over all prime numbers.

5. Property (T) and Relative Property (T)

The first explicit construction of an expander family was discovered by Margulis in [8] who relied on Kazhdan's property (T) [5].

Definition 5.1. A locally compact group G has property (T) or is a Kazhdan group if there exists $\epsilon > 0$ and a compact subset K of G such that for every continuous nontrivial irreducible unitary representation $(\mathcal{H}, \rho)$ of G and every vector $v \in \mathcal{H}$ of norm one then $\|\rho(k)v - v\| > \epsilon$ for some $k \in K$.

There is an equivalent definition of property (T) that is easier to use in practice. Before we can state the definition we must introduce the notion of weak containment.

Definition 5.2. Let G be a locally compact group and σ and ρ two continuous unitary representations of G on a Hilbert space, $\sigma, \rho : G \to \mathcal{U}(\mathcal{H})$. For every vector v in Hilbert space of norm 1 we associate a coefficient of ρ by by the function $G : g \mapsto \langle v, \rho(g)v \rangle$ where the bilinear form is the scalar product in the Hilbert space. This measures the difference in angles and in essence how much the vector is moved by the representation. We say ρ is weakly contained in σ, denoted by $\rho \propto \sigma$ if every coefficient of ρ is a uniform limit on compact sets of G of coefficients of σ.

We will use ρ_0 to denote the trivial representation throughout the paper. If $\rho = \rho_0$ then it is clear $\rho_0 \propto \sigma$ if and only if for every $\epsilon > 0$ and compact subset K of G then there exists $v \in \mathcal{H}_\sigma$ such that $\|v\| = 1$ and $\|\sigma(g)v - v\| < \epsilon$ for all

$g \in K$. When a representation weakly contains the trivial representation we say it has almost invariant vectors. A group has property (T) if and only if every unitary representation with almost invariant vectors contains a nonzero invariant vector.

Property (T) is in fact a stronger condition than we need for the remainder of the paper. We only need the condition in 5.1 to hold for some subset of the representations of G. We call this weaker condition relative property (T). First we define what we mean by the set of representations of G.

Definition 5.3. Let G be a locally compact group. A unitary representation $(\mathcal{H}, \rho)$ of G is called irreducible if the only G-invariant closed subspaces of $\mathcal{H}$ are the trivial ones, $\{0\}$ and $\mathcal{H}$. The set of equivalence classes of irreducible representation of G is called the unitary dual of G and denoted by $\hat{G}$. It is not immediately clear why this is a set. For the proof see Remark C.4.13 in [1].

Definition 5.4. If R is a subset of $\hat{G}$ we say G has property $(T : R)$ or relative property (T) with respect to R if there exists an $\epsilon > 0$ and compact subset K of G such that for every nontrivial irreducible unitary representation $(\mathcal{H}, \rho) \in R$ and every vector $v \in \mathcal{H}$ of norm one, $\|\rho(k)v - v\| > \epsilon$ for some $k \in K$.

Definition 5.5. Given a finitely generated group Γ and a family of finite index normal subgroups $\mathcal{L} = \{N_i\}$, let $R = \{\phi \in \hat{\Gamma} \mid \ker \phi \supseteq N_i \text{ for some } i\}$. We say Γ has property (τ) if Γ has property $(T : R)$ with respect to the family of all finite index normal subgroups.

With this condition in hand we are finally ready to prove that we can construct expanders from the Cayley graphs of quotient subgroups of groups satisfying property (T).

Theorem 5.6. *Let Γ be a finitely generated group and S a finite set of generators including the inverse of each element. Let $\mathcal{L} = \{N_i\}$ be a family of finite index normal subgroups of Γ. Then the following conditions are equivalent:*

i. *The group Γ has property (τ) with respect to $\mathcal{L}$.*
ii. *There exists $\epsilon_2 > 0$ such that all the Cayley graphs $X_i = X(\Gamma/N_i, S)$ are $([\Gamma : N_i], |S|, \epsilon_2)$ -expanders.*
iii. *There exists $\epsilon_3 > 0$ such that $h(X_i) \geq \epsilon_3$.*
iv. *There exists $\epsilon_4 > 0$ such that $\lambda_1(X_i) \geq \epsilon_4$.*

Proof. The equivalence $(ii) \iff (iii)$ is proved in 2.6. The proof of the equivalence $(iii) \iff (iv)$ is shown in 3.7 and 3.8.

Now we prove the implication $(i) \implies (iii)$. The strategy is very similar to the proof of the reverse Cheeger's Inequality. We take our vector in the Hilbert space to be a function on the vertices of the graph whose Rayleigh Quotient only picks up values on vertices in one of the two subsets connected by an edge to the other subset and vice versa. In essence this function is a minimizer for $\lambda_1(X)$. Finally, we use property (τ) to show that the expander inequality must be satisfied for this graph.

Let $N \in L$ and $\mathcal{H} = L^2(\Gamma/N)$ the vector space of complex valued functions on the finite set $V = \Gamma/N$ with the norm $\|f\|^2 = \sum_{x \in V} |f(x)|^2$. Now we can decompose $\mathcal{H}$ as the direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C}$ where $\mathcal{H}_0 = \{f \in \mathcal{H} \mid \sum_{x \in V} f(x) = 0\}$ and $\mathbb{C}$ denotes the constant functions on the vertices. In other words we can represent any function $f \in \mathcal{H}$ as a pair (u, v) where u is a function with average value 0 and

v is a constant function with average value equal to the average value of f. The vector space $\mathcal{H}$ is a Γ-module and $\gamma \in \Gamma$ acts on $\mathcal{H}$ by $(\gamma f)(x) = f(x\gamma)$. The action of Γ on Γ/N is transitive. Therefore, only the constant functions $\mathbb{C}$ are Γ-invariant functions on V and $\mathcal{H}_0$ does not contain the trivial representation. The group Γ has property (τ) which enables us to conclude $\mathcal{H}_0$ does not have invariant functions. Given the group Γ and finite generating set S, we can find an $\epsilon > 0$ such that for every N_i and $f \in \mathcal{H}_0$, $\|\gamma f - f\| > \epsilon \|f\|$ for some $\gamma \in S$.

Now we will construct a function, f that is a minimizer for $\lambda_1(X)$. Break up the graph into disjoint subsets of the vertices A and B where $|A| = a$, $|B| = b$, and $|V| = n$. Then define $f(v) = \begin{cases} b & \text{if } v \in A \\ -a & \text{if } v \in B \end{cases}$ and let $\gamma \in \Gamma$. We see by a computation

$$\|\gamma f - f\|^2 = (b+a)^2 |E_\gamma(A,B)|$$

where $E_\gamma(A,B) = \{v \in V \mid v \in A \text{ and } v\gamma \in B\} \cup \{v \in V \mid v \in B \text{ and } v\gamma \in A\}$. By property (τ) we know there is some $\epsilon > 0$ such that $\|\gamma f - f\|^2 > \epsilon^2 \|f\|^2$. Notice

$$\{v \in V \mid v \in A \text{ and } v\gamma \in B\}| \leq |\partial A|$$
$$\{v \in V \mid v \in B \text{ and } v\gamma \in A\}| \leq |\partial A|$$
$$\frac{1}{2}|E_\gamma(A,B)| \leq |\partial A|.$$

Then by property (τ) we know

$$\epsilon_1^2 \|f\|^2 \leq \|\gamma f - f\|^2 = (b+a)^2 |E_\gamma(A,B)| \leq 2n^2 |\partial A|$$
$$|\partial A| \geq \frac{\epsilon_1^2 \|f\|^2}{2n^2} = \frac{\epsilon_1^2 (ab^2 + ba^2)}{2n^2} = \epsilon_1^2 \frac{ab}{2n} = \epsilon_1^2 \frac{n-a}{n} a = \epsilon_1^2 \left(1 - \frac{|A|}{n}\right) |A|$$

Therefore the Cayley graph is an expander graph for the group satisfying Property (τ).

To conclude the proof of the theorem we must prove the implication $(iv) \implies (i)$. If $f \in L_0^2(X_i)$ we can alternatively think of f as an element of $\mathbb{C}[\Gamma/N_i] = \sum_{v \in V} \alpha_v v$ where $\alpha_v \in \mathbb{C}$. This holds because of the isomorphism $L^2(X_i) \simeq \mathbb{C}[\Gamma/N_i]$. We assume $\|f\| = 1$ and $\|\Delta f\| \geq \epsilon_4$. The action of the Laplacian on an element of the group algebra of Γ/N_i is right multiplication by $k \cdot e - \sum_{s \in S} s$ where e is the identity element of Γ, $k = |S|$, and each s acts on $f(x)$ to evaluate f at a vertex adjacent to x. Then if we apply the triangle inequality we find

$$\epsilon_4 \leq \|\Delta f\| = \|f \cdot \sum_{s \in S} (e - s)\| = \|\sum_{s \in S} f(e-s)\| \leq \sum_{s \in S} \|f - f \cdot s\|.$$

There is at least one $s \in S$ such that $\frac{\epsilon_4}{k} \leq \|f - f \cdot s\|$. By definition $f \cdot s = R(s) f$ where $R(s)$ denotes the right regular representation of Γ on $L_0^2(\Gamma/N_i)$. The finite quotient Γ/N_i appears in the right regular representation because ρ factors through it. Therefore we can choose $\epsilon_1 = \epsilon_4 / k$ in order to satisfy $\epsilon_1 \leq \|f - \rho f\|$. □

Nothing has been said in this proof about the selection of the generating set. In the next lemma we show the selection of the generating set does not matter.

Lemma 5.7. *If Γ is a finitely generated discrete group with property (T) then for every finite set S of generators of Γ every nontrivial irreducible representation $(\mathcal{H}, \rho)$ of Γ there exists an $s \in S$ such that for every vector $v \in \mathcal{H}$ with $\|v\| = 1$, $\|\rho(s)v - v\| > \epsilon$.*

Proof. By definition of property (T) there exists $\epsilon > 0$ and compact subset K of Γ such that for every nontrivial irreducible unitary representation $(\mathcal{H}, \rho)$ of Γ and vector $v \in \mathcal{H}$, there exists $k \in K$ such that $\|\rho(k)v - v\| > \epsilon\|v\|$. We can write $w \in K$ as a word in the elements of S of at most finite length because a compact subset of a discrete group contains finitely many elements. Call l the maximal length of an element of K in words from S. If $(\mathcal{H}, \rho)$ is a representation and $v \in \mathcal{H}$ a vector such that $\|\rho(s)v - v\| < \epsilon\|v\|$ for every $s \in S$, then $\|\rho(k)v - v\| < l\epsilon\|v\|$ for every $k \in K$. This means if S has almost invariant vectors then K must also have almost invariant vectors. Equivalently, if K does not have almost invariant vectors then S also does not have almost invariant vectors. □

6. $SL_3(\mathbb{R})$ has Property (T)

Now we must show the group chosen in our construction of expanders has property (T). In order to do so we first show $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ has property $(T : R)$ where $R = \{\rho \in \mathbb{R}^2 \rtimes \hat{S}L_2(\mathbb{R}) \mid \rho\mid_{\mathbb{R}^2} \text{ is nontrivial }\}$. Before we can start we need to introduce the notion of an amenable group. The proof will hinge on the fact $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ is not amenable. It is interesting to note, the Cayley graph of an amenable group is the exact opposite of an expander graph.

Definition 6.1. A locally compact group G is called amenable if given $\epsilon > 0$ and a compact subset $K \subset G$, there exists a Borel set $U \subseteq G$ of positive finite left Haar measure $\lambda(U)$ such that

$$\frac{\lambda(xU\Delta U)}{\lambda(U)} < \epsilon, \qquad \forall x \in K.$$

where we have used $A \triangle B$ to denote the symmetric difference, $xU \triangle U = (xU \setminus U) \cup (U \setminus xU)$.

In this case the Cayley graph $X(G; K)$ is not an expander graph because for every $\epsilon > 0$ and compact subset K the graph has a finite subset U of the vertices such that $\|\partial U\| < \epsilon\|U\|$. We are finally ready to begin the proof.

Proposition 6.2. *For notational convenience let $G = \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$, the semi-direct with the standard action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$. Let $R = \{\rho \in \hat{G} \mid \rho\mid_{\mathbb{R}^2} \text{ is nontrivial }\}$. Then G has property $(T : R)$.*

Proof. Let $\rho \in R$ be an irreducible unitary representation of G. Notice for any irreducible unitary representation π of G, the restriction of π to $\mathbb{R}^2$ is a unitary representation of an abelian group which implies it is the directed integral of one-dimensional characters of $\mathbb{R}^2$. For finite groups this follows by an application of Schur's Lemma [3, p. 20]. Let χ be one of these characters in the integral and let M be its stabilizer in $SL_2(\mathbb{R})$. Observe M also acts on the group of characters of $\mathbb{R}^2$. Next we apply Mackey's theorem [10, Th. 7.3.1] which states we can write $\rho = \mathrm{ind}_{\mathbb{R}^2 \rtimes M}^{G}(\chi\sigma)$ where σ is an irreducible representation of M and $\chi\sigma$ the representation of $\mathbb{R}^2 \rtimes M$ defined by $(\chi\sigma)(r, m) = \chi(r)\sigma(m)$. By definition of the set R, χ is nontrivial which implies M is a proper subgroup of $SL_2(\mathbb{R})$. In addition M is conjugate to the set $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\right\}$. The group $M_1 = \mathbb{R}^2 \rtimes M$ is nilpotent because there is a short exact sequence $1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \rtimes M \hookrightarrow M \hookrightarrow 1$. The fact abelian groups are amenable allows us to induct on the step size to prove nilpotent groups are amenable. We can apply Hulanicki's Theorem [7, Th. 3.1.5] which states the trivial representation ρ_0 is weakly contained in the left regular representation L_{M_1} if and only if M_1 is amenable because $\mathbb{R}^2 \rtimes M$ is nilpotent. An application of this fact shows

$$\chi\sigma = (\chi\sigma \otimes \rho_0) \propto (\chi\sigma \otimes L_{M_1}) \simeq \dim(\chi\sigma) \cdot L_{M_1}.$$

The last relation indicates isomorphism and the $\dim(\chi\sigma)$ refers to the number of copies of the left regular representation $\chi\sigma$ is contained in. The isomorphism holds because for every representation (W, π) we get an isomorphism $\theta : L^2(G) \to L^2(G, w)$ defined by

$$\theta(f \otimes v)(g) = f(g)(\pi(g)v)$$

where

$$L^2(G, W) = \{f : G \to W \mid \int_G \|f(g)\| df < \infty\}.$$

Induction respects weak containment implying

$$\rho = \mathrm{ind}_{M_1}^G(\chi\sigma) \propto \infty \cdot \mathrm{ind}_{M_1}^G = \infty \cdot L_{SL_2(\mathbb{R})}$$

where $L_{SL_2(\mathbb{R})}$ denotes the left regular representation of $SL_2(\mathbb{R})$.

If $\rho_0 \propto \rho$ we have $\rho_0 \propto L_G$. This is a contradiction because it cannot be that the left regular representation of G weakly contains the trivial representation. This means G is amenable (result originally due to Hulanicki, can be found in [7, 3.1.5]). This is a contradiction because G is not amenable. To show this we notice the fractional linear transformation associated with each matrix in $SL_2(\mathbb{R})$ acts transitively on $\mathbb{R} \cup \{\infty\}$. Therefore we must prove a non-zero $SL_2(\mathbb{R})$ -invariant Borel measure on $\mathbb{R} \cup \{\infty\}$ does not exist. Assume an invariant measure exists called ν. The measure ν must be invariant under translations $x \to x+t$ for $t, x \in \mathbb{R}$ which are represented by the matrices $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The Lebesgue measure is the unique translation invariant measure on $\mathbb{R}$ up to scaling. This means $\nu = c_1\mu + c_2\delta_\infty$ where μ is the Lebesgue measure on $\mathbb{R}$, δ is the Dirac measure of the point at infinity, and c_1 and c_2 are positive constants. ν also must be invariant under the transformation $x \to -1/x$ represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ but neither μ nor δ_∞ are invariant under this transformation. Thus $c_1 = c_2 = 0$ and $\nu = 0$ which implies $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ is not amenable.

□

Lemma 6.3. *Let* $N = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$. *If* ρ *is a unitary representation of* E*, then every vector fixed by* $\rho(N)$ *is also fixed by* $\rho(SL_2(\mathbb{R}))$.

Proof. Define $f(g) = \langle \rho(g)v, v\rangle$. We will show $f(g)$ is constant for all $g \in SL_2(\mathbb{R})$. If we show this then if $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity in $SL_2(\mathbb{R})$ we have $f(g) = \langle \rho(g)v, v\rangle = f(e) = \langle v, v\rangle$ and this implies $\rho(g)v = v$.

First notice $f(g) = f(n_1gn_2)$ so f is constant on the double cosets $N\backslash E/N$.

$$\begin{aligned} f(n_1gn_2) &= \langle \rho(n_1gn_2)v, n_1gn_2\rangle = \langle \rho(n_1)\rho(g)\rho(n_2)v, v\rangle \text{ since } \rho \text{ is a homomorphism} \\ &= \langle \rho(n_1)\rho(g), v\rangle = \langle \rho(g), \rho(n_1)^{-1}v\rangle = \langle \rho(g), v\rangle = f(g) \end{aligned}$$

Where we have used the identity for unitary representations, $\rho(g)^* = \rho(g)^{-1}$.

$SL_2(\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ and N stabilizes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ allowing us to identify E/N with $\mathbb{R}^2 \setminus \{0\}$. We have already decided f is constant on the double cosets $N \backslash E/N$ so we can view f as a function on $\mathbb{R}^2 \setminus \{0\}$ constant on the orbits of N. If we calculate the action of N

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}$$

we see the orbits of N in $\mathbb{R}^2 \setminus \{0\}$ are the points on the x-axis and the lines parallel to the x-axis. It is clear f is constant on the x-axis as well by continuity of f. We have shown if $p \in P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{R}) \right\}$ then $f(p) = f(e)$ in other words v is an invariant vector under P.

Next we show f is constant on the double cosets $P \backslash SL_2(\mathbb{R})/P$. This is the exact same calculation done when we showed f is constant on $N \backslash SL_2(\mathbb{R})/N$ except that here we used the fact our vector,v, is fixed by P which we have just shown.

Next we notice we can identify $SL_2(\mathbb{R})/P$ with the projective line $\mathbf{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ by the Mobius transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$. Then we have two orbits, $\mathbb{R}$ and ∞ where the ∞ orbit comes from those matrices such that $d = 0$. The function f is continuous and constant on the real line therefore f is constant on all of $SL_2(\mathbb{R})$. □

Lemma 6.4. *Let* $J = \left\{ \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\} \subseteq SL_3(\mathbb{R})$. *For any unitary representation of* G*, every vector fixed under* J *is fixed under* G*.*

Proof. The set of matrices generated by $E_1 = \left\{ A = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \text{ and } \det(A) = 1 \right\}$ and $E_2 = \left\{ A = \begin{pmatrix} 1 & 0 & b \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \text{ and } \det(A) = 1 \right\}$ generate a dense subgroup of $SL_3(\mathbb{R})$. This is clear by Lie Theoretic methods. Define $N_1 = E_i \cap J$ for $i = 1, 2$. Then we can see by the previous lemma any vector fixed by N_i is also fixed by E_i. Thus if a vector is fixed by J it must also be fixed by both E_1 and E_2 and therefore it is also fixed by $SL_3(\mathbb{R})$. □

Theorem 6.5. $SL_3(\mathbb{R})$ *has property* (T)*.*

Proof. Let ρ be a unitary representation of $SL_3(\mathbb{R})$ that weakly contains the trivial representation, ρ_0. We must find a nonzero invariant vector.

Let $H = \left\{ \begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix} \in G \right\} \simeq J \rtimes SL_2(\mathbb{R}) \simeq \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$. By assumption ρ restricted to H weakly contains the trivial representation. By proposition 3.1.11 $\rho \mid_H$ contains a vector invariant under $J \simeq \mathbb{R}^2$ and by the previous Lemma this vector is invariant under $SL_3(\mathbb{R})$. Therefore $SL_3(\mathbb{R})$ has property (T). □

Definition 6.6. A subgroup Γ of a locally compact group G is called a lattice subgroup if Γ is discrete and G/Γ has a finite G-invariant measure.

It is difficult to construct an expander family from $SL_n(\mathbb{R})$ because it does not have a finite generating set. It is reasonable to ask if a simpler group, for instance $SL_n(\mathbb{Z})$ has property (T). It turns out if we consider the lattice subgroup of a group then the lattice has property (T) if and only if the full group has property (T). In order to show this we will need to introduce the concept of a cocycle to better examine the group action.

Definition 6.7. If X is a G-space a Borel function $\alpha : X \times G \to \mathcal{U}(\mathcal{H})$ is called a cocycle if for all $g, h \in G$ the identity $\alpha(x, gh) = \alpha(x, g)\alpha(xg, h)$ holds for almost every $x \in X$. If this identity holds for all $x \in X$ then the cocycle is said to be strict.

Definition 6.8. A function $f : X \to \mathcal{H}$ is called α-invariant for a cocycle $\alpha : X \times G \to \mathcal{U}(\mathcal{H})$ if for each $g \in G$, $\alpha(x, g)f(xg) = f(x)$ for almost every $x \in X$. The function f is said to be strictly α-invariant if the equation is true for all $x \in X$.

Proposition 6.9. *Let Γ be a lattice subgroup of a locally compact group G. If G is a Kazhdan group then Γ is also a Kazhdan group.*

Proof. The proof, originally due to Kazhdan, can be found as Proposition 7.4.3 in [10].

Before beginning the proof we must mention a minor caveat. We will need to use a measurable section of the natural projection map $G \to G/\Gamma$ which is only defined almost everywhere. The most we can say about the following proof is the equations and discussion are valid almost everywhere.

Let ρ be a representation of Γ that weakly contains the trivial representation. We can apply proposition 7.3.7 of [10] which tells us $\mathrm{Ind}_\Gamma^G(\rho_0) \propto \mathrm{Ind}_\Gamma^G(\rho)$ because the group Γ is a closed subgroup of G. The quotient G/Γ has a finite G-invariant measure which means the constants are in the space $L^2(G/\Gamma)$ and the trivial representation is contained in $\mathrm{Ind}_\Gamma^G(\rho)$. By assumption G is a Kazhdan group therefore $\mathrm{Ind}_\Gamma^G(\rho)$ has an invariant vector or function to be more precise in this case. Now we show this implies ρ has invariant vectors.

Take a representation $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$. There is a simple way to get a cocycle, $\alpha : G/\Gamma \times G \to \mathcal{U}(\mathcal{H})$, from ρ. We will use a fact about the composition of the natural projection map, $p : G \to G/\Gamma$ with a measurable section of it, $s : G/\Gamma \to G$ such that $s([e]) = e$. By definition for any $[g] \in G/\Gamma$ then $p \circ s([g]) = g$. It will be useful if the cocycle encodes information about how far apart the cosets $s(x)g$ and $s(xg)$ are. Both cosets project to the same element of G/Γ under the projection map p because g acts on the right and Γ mods out on the left. Therefore $\Gamma s(x)g = \Gamma s(xg)$ and we can find $\gamma \in \Gamma$ such that $es(x)g = \gamma s(xg)$ where e denotes the identity element in $\Gamma s(x)g$. Let $\beta : G/\Gamma \times G \to \Gamma$ be the function that satisfies

$$s(x)g = \beta(x, g)s(xg)$$
$$\beta(x, g) = s(x)gs^{-1}(xg).$$

Notice β must be a cocycle. It is a function of two variables and we can check that it satisfies the cocycle identity. Let $g, h \in G$

$$\begin{aligned} \beta(x, gh) &= s(x)ghs^{-1}(xgh) \\ &= s(x)gs^{-1}(xg)s(xg)hs^{-1}(xgh) = \beta(x, g)\beta(xg, h). \end{aligned}$$

Each strict cocycle corresponds to a homomorphism from $h : \Gamma \to \Gamma$ if we set $h(g) = \beta([e], g)$. For our original homomorphism ρ, we get a corresponding cocylce α by setting $\alpha(x, g) = \rho\left(s(x)gs^{-1}(xg)\right)$. One can check to make sure $\alpha([e], g) = \rho(g)$.

Let v_0 be the invariant vector in $\mathrm{Ind}_\Gamma^G(\rho)$ and define a function $\phi : G/\Gamma \to \mathcal{H}_\pi$ by $\phi(x) = \alpha^{-1}(e, x)v_0$. The correct notion of α invariant we want is $\alpha(x, g)\phi(xg) = \phi(x)$ for all $g \in G$. From the cocycle identity we know $\alpha^{-1}(e, x) = \alpha(x, g)\alpha^{-1}(e, xg)$. Therefore

$$\alpha([x], g)\phi(xg) = \alpha([x], g)\alpha^{-1}([e], xg)v_0 = \alpha^{-1}([e], x)v_0 = \phi(x).$$

The α-invariance equation along with the property $\phi([e]) \neq 0$ allow us to solve for ϕ. The group G acts transitively on G/Γ implying $\phi \neq 0$ on a set whose complement is measure zero.

We can solve for α in terms of ρ

$$\alpha([x], g) = \alpha([e]x, g) = \alpha^{-1}([e], x)\alpha([e], xg) = \rho^{-1}(x)\rho(xg).$$

The function ϕ is α-invariant which helps us to compute

$$\begin{aligned} \alpha([x], g)\phi([x]g) &= \phi([x]) \\ \rho^{-1}(x)\rho(xg)\phi([x]g) &= \phi([x]) \\ \rho(xg)\phi([x]g) &= \rho(x)\phi([x]). \end{aligned}$$

This can only be true if

$$\rho(y)\phi([y]) = \rho(x)\phi([y])$$

for almost every $y \in G$ because G acts transitively. We are looking for a unit vector. In order to ensure we find a vector of norm one we instead consider the function $\psi(x) = \phi(x)/\|\phi(x)\|$ which maps G/Γ into the unit ball in $\mathcal{H}$. Notice ψ is also α-invariant and therefore $\rho(y)\psi([y]) = \rho(x)\psi([y])$ for almost every $y \in G$. Define the vector $a = \rho(x)\psi([x])$ which by construction is of norm 1. The following calculation shows for $h \in \Gamma$, the vector a is invariant and $\rho(h)a = a$. Let $h \in \Gamma$. For almost every $y \in G$ we see $\rho(hy)\psi([hy]) = a$. By the property of a homomorphism we know $\rho(h)\rho(y)\psi([y]) = a$ because we mod out on the left. We can combine this with the fact $\rho(y)\psi([y]) = a$ for almost every y to conclude $\rho(h)a = a$. □

Corollary 6.10. *$SL_3(\mathbb{Z})$ forms a lattice in $SL_3(\mathbb{R})$ therefore $SL_3(\mathbb{Z})$ has property (T).*

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