

# SPECTRAL RIGIDITY ON $\mathbb{T}^n$

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ABSTRACT. We review the basic notions of lattice, torus, and length spectrum, proving a few basic results. We then define higher-dimensional spectra on tori, leading to higher dimensional rigidity results.

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## 1. INTRODUCTION

The notion of a "spectrum" of a manifold generally arises in two contexts. Commonly, the spectrum of a manifold refers to the eigenvalues of the laplacian; two manifolds are considered isospectral when the laplacian has the same set of eigenvalues on both manifolds. An overview of such rigidity is given in [2] and [5], with more detailed discussion in [3]. We can also consider the length spectrum of a manifold. This is a spectrum defined as the set of lengths of the shortest closed geodesics—the "straight lines" of the manifold. In general, isospectral manifolds are not necessarily length isospectral, nor vice versa; however, restricted to the space of tori, there is in fact an explicit bijection between the laplacian spectrum and length spectrum (see [5]). We thus have that two toruses are isospectral iff they are length isospectral. The focus of this paper, then, will be on the length spectrum of tori.

We will first establish preliminaries: definitions of a flat torus, isometries of tori, and proofs of simple propositions regarding tori that will prove useful. We then formally define the length spectrum, and define higher dimensional spectra as well; the utility of these higher-dimensional spectra will become apparent as we move into higher-dimensional tori. With these tools in hand, we then prove rigidity results for tori: specifically, we consider equivalencies of spectra that ensure two tori are isometric.

It is worth noting that, in general, neither isospectrality nor length isospectrality is sufficient to ensure isometry; Sunada (see [3]) developed a construction of nonisometric manifolds with the same laplacian spectrum, while explicit examples

of length isospectral, nonisometric tori are known as well (see [1], [6]). However, restricted to certain types of manifolds, rigidity can be established. In section 4, for example, we will prove isospectral 2-tori are isometric. Croke (in [4]) established a similar result for length isospectral compact manifolds without boundary, provided they were of genus  $\geq 2$  and of negative curvature. However, his result depended on a marked length spectrum—each length was paired with a conjugacy class of closed curves. He notes that without the marking of the length spectrum, such rigidity fails.

## 2. PRELIMINARIES

We begin by defining the notion of an  $n$ -dimensional Lattice and its associated torus.

**Definition 2.1.** Given  $n$  linearly independent vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , we define the  $n$ -dimensional lattice  $L$  generated by  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  to be  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$ , the integer span of the vectors. In other words,  $L = \left\{ \sum_{i=1}^n a_i \mathbf{v}_i : a_i \in \mathbb{Z} \right\}$ .

We note that two distinct sets of vectors may produce the same integer span. For example, in  $\mathbb{R}^2$ , both  $\{(1,0), (0,1)\}$  and  $\{(1,0), (1,1)\}$  generate  $\mathbb{Z}^2$  as their lattice. However, if we define the matrix

$$M_v = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$$

we do have the following result.

**Proposition 2.2.**  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  and  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  generate the same lattice  $L$  iff there exists a unimodular matrix transforming  $M_v$  to  $M_w$ .

*Proof.* If  $M_v$  and  $M_w$  generate the same lattice, there exist  $a_{ij}, b_{ij}$  such that  $\mathbf{v}_j = \sum_{i=1}^n a_{ij} \mathbf{w}_i$  and  $\mathbf{w}_j = \sum_{i=1}^n b_{ij} \mathbf{v}_i$  for each  $j$ . We then have

$$\begin{aligned} \mathbf{v}_j &= \sum_{k=1}^n a_{kj} \mathbf{w}_k \\ &= \sum_{k=1}^n a_{kj} \sum_{i=1}^n b_{ik} \mathbf{v}_i \\ &= \sum_{i=1}^n \left( \sum_{k=1}^n a_{kj} b_{ik} \right) \mathbf{v}_i \end{aligned}$$

Since the  $\mathbf{v}_i$ s are linearly independent, this implies that  $\sum_{k=1}^n a_{kj} b_{ik} = \delta_{ij}$ . If we define  $A=(a_{ij})$  and  $B=(b_{ij})$  to be  $n$ -by- $n$  matrices, this equation means  $AB=Id$ . Given that  $A$  and  $B$  have all integer entries, their determinants must be integers. Since  $\det(A)\det(B)=1$  we have that  $\det(A)=\det(B)=\pm 1$ . Furthermore, inspection reveals  $M_v = M_w A$ .

Now, say  $M_v = M_w A$  for some unimodular  $A$ . Then  $\mathbf{v}_j = \sum_{i=1}^n a_{ij} \mathbf{w}_i$ , and so each  $\mathbf{v}_j$  is certainly in the integer span of the  $\mathbf{w}_i$ s. Now, let  $A^{-1} = B = (b_{ij})$ . Note that

B is also unimodular. Then

$$\begin{aligned} \sum_{k=1}^n b_{jk} \mathbf{v}_k &= \sum_{k=1}^n b_{jk} \sum_{i=1}^n a_{ki} \mathbf{w}_i \\ &= \sum_{i=1}^n \left( \sum_{k=1}^n a_{ki} b_{jk} \right) \mathbf{w}_i \\ &= \sum_{i=1}^n \delta_{ij} \mathbf{w}_i \\ &= \mathbf{w}_j \end{aligned}$$

Thus, each  $\mathbf{w}_j$  is in the integer span of the  $\mathbf{v}_i$ s as well.  $\square$

We thus have an infinite family of basis for any given lattice. However, we may prefer one basis over another; in particular, we will be concerned with taking minimal bases. An  $n$ -dimensional minimal basis is formed by taking  $\mathbf{v}_1$  to be a minimal vector in  $L$ ,  $\mathbf{v}_2$  to be a minimal vector in  $L/\mathbb{Z}[\mathbf{v}_1]$ , and in general  $\mathbf{v}_i$  to be a minimal vector in  $L/\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}]$  for all  $i \leq n$ .

Given an  $n$ -dimensional lattice  $L$ , we can of course define an equivalence relation on  $\mathbb{R}^n$  as follows: for  $\mathbf{w}_i, \mathbf{w}_j \in \mathbb{R}^n$ ,  $\mathbf{w}_i \sim \mathbf{w}_j$  iff  $(\mathbf{w}_i - \mathbf{w}_j) \in L$ . This leads us to our definition of a torus.

**Definition 2.3.** Given an  $n$ -dimensional lattice  $L$ , we define the  $n$ -dimensional torus  $\mathbb{T}^n$  generated by  $L$  to be  $\mathbb{R}^n/L$ , the quotient of  $\mathbb{R}^n$  by the lattice  $L$ .

We would of course like to have some notion of equivalence when discussing tori. We first note that if  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are tori generated by  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  and  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ , respectively, the linear map formed by sending each  $\mathbf{v}_i$  to  $\mathbf{w}_i$  is a homeomorphism from  $\mathbb{T}_1$  to  $\mathbb{T}_2$  (with respect to the quotient topology). Further, this map is in fact a group homomorphism from  $(\mathbb{T}_1, +)$  to  $(\mathbb{T}_2, +)$ , considered as quotient groups of  $\mathbb{R}^n$ . Thus, all  $n$ -dimensional tori are both homomorphic and homeomorphic; this equivalence is far too weak for our purposes.

However, we may construct a metric on  $\mathbb{T}$  as follows: for any equivalence classes  $[x], [y]$  in  $\mathbb{T}$ ,  $d_{\mathbb{T}}([x], [y]) = \inf_{x \in [x], y \in [y]} \|x - y\|$ . Later it will be proven that  $L$  is necessarily closed and discrete; the inf may then be replaced by a min. Simple inspection should reveal the topology induced by  $d_{\mathbb{T}}$  is equivalent to the quotient topology on  $\mathbb{T}$ . We thus have the following equivalence relation on tori:

**Definition 2.4.** Given two tori  $\mathbb{T}_1$  and  $\mathbb{T}_2$ , generated respectively by  $L_1$  and  $L_2$ , we say  $\mathbb{T}_1 \cong \mathbb{T}_2$  if  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are isometric with respect to the metrics  $d_{\mathbb{T}_1}$  and  $d_{\mathbb{T}_2}$ .

**Theorem 2.5.**  $\mathbb{T}_1 \cong \mathbb{T}_2$  iff there exists an  $A \in O(n)$  such that  $A(L_1) = L_2$ .

*Proof.* If  $A$  is an orthogonal map that sends  $L_1$  to  $L_2$ , then  $A$  is well defined on  $\mathbb{T}_1, \mathbb{T}_2$ , as it sends every element in an equivalence class  $[x]$  in  $\mathbb{T}_1$  to an equivalence class  $A[x]$  in  $\mathbb{T}_2$ . This map from  $[x]$  into  $A[x]$  is in fact bijective. Furthermore, since orthogonal maps preserve dot products, they preserve the metric on  $\mathbb{R}^n$ . Then if  $a \in [a], b \in [b]$ ,  $d(a, b) = d(Aa, Ab)$ , and the metric induced by the tori are thus preserved as well.

Conversely, if  $A$  is an isometry from  $\mathbb{T}_1$  to  $\mathbb{T}_2$ , we may extend it to an orthogonal map from  $L_1$  to  $L_2$ . Consider  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  a basis for  $L_1$ . Then every equivalence class of  $\mathbb{T}_1$  contains a unique vector of the form  $\sum_{i=1}^n \alpha_i \mathbf{v}_i$ , where  $0 \leq \alpha_i < 1$ . Pick an  $N$  sufficiently large to ensure that  $\mathbf{v}_i/N$  is the unique vector of smallest norm in

$[\mathbf{v}_i/N]$  and  $(\mathbf{v}_i + \mathbf{v}_j)/N$  is the unique vector of smallest norm in  $[(\mathbf{v}_i + \mathbf{v}_j)/N]$  for all  $0 < i < j \leq n$ . Then we may define  $\mathbf{w}_i$  as  $N$  times the unique vector of smallest norm in  $A[\mathbf{v}_i]$ . Since  $A$  is an isometry,  $\|\mathbf{v}_i\| = \|\mathbf{w}_i\|$ . Further,  $\|\mathbf{v}_i + \mathbf{v}_j\| = \|\mathbf{w}_i + \mathbf{w}_j\|$ , implying  $\mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{w}_i \cdot \mathbf{w}_j$  and the  $\mathbf{w}_i$ s are linearly independent. By construction of the  $\mathbf{w}_i$ s, we know  $\mathbb{Z}[\mathbf{w}_1, \dots, \mathbf{w}_n] \subseteq L_2$ . Now, if  $a \notin \mathbb{Z}[\mathbf{w}_1, \dots, \mathbf{w}_n]$ ,  $a = \sum_{i=1}^n \alpha_i \mathbf{w}_i$ , where the  $\alpha_i$  are not all integers,  $[a]$  contains  $\sum_{i=1}^n \alpha_i \mathbf{w}_i$ , where  $0 \leq \alpha_i < 1$ . However, this implies  $[a]$  is the image of some nonzero equivalence class in  $\mathbb{T}_1$ ,  $[\sum_{i=1}^n \alpha_i \mathbf{v}_i]$ , and thus  $\|[a]\| > 0$  and  $a$  is not a lattice point.  $\square$

**Theorem 2.6.**  $\mathbb{T}_1 \cong \mathbb{T}_2$  if there exist  $\mathbb{Z}$ -bases  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  and  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  generating  $\mathbb{T}_1$  and  $\mathbb{T}_2$ , respectively, such that  $\mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{w}_i \cdot \mathbf{w}_j$  for all  $i, j \leq n$ .

*Proof.* The map formed by sending  $\mathbf{v}_i$  to  $\mathbf{w}_i$  and extending linearly is an orthogonal map.

Conversely, if  $T$  is an orthogonal map that sends  $L_1$  to  $L_2$  and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is a basis for  $L_1$ ,  $T\mathbf{v}_1, T\mathbf{v}_2, \dots, T\mathbf{v}_n$  is a basis for  $L_2$  that satisfies the required properties.  $\square$

### 3. SPECTRUMS ON TORI

We wish to define the length spectrum of a torus; that is, the set of possible lengths of closed curves. Additionally, we want this set to count multiplicities. We do this as follows.

**Definition 3.1.** For a torus  $\mathbb{T}$  generated by a lattice  $L$ , define  $m_\ell = |\{\mathbf{v} \in L : \|\mathbf{v}\| = \ell\}|$

**Definition 3.2.** For a torus  $\mathbb{T}$  generated by a lattice  $L$ , define  $\mathcal{L}(\mathbb{T}) = \{(\|\mathbf{v}\|, m_{\|\mathbf{v}\|}) : \mathbf{v} \in L\}$ . We call  $\mathcal{L}(\mathbb{T})$  the geodesic length spectrum of  $\mathbb{T}$ . We will also occasionally denote the length spectrum by  $\mathcal{L}(L)$ .

**Proposition 3.3.** If  $\mathbb{T}_1 \cong \mathbb{T}_2$ , then  $\mathcal{L}(\mathbb{T}_1) = \mathcal{L}(\mathbb{T}_2)$ .

*Proof.* Orthogonal transformations preserve both norm and multiplicities.  $\square$

**Theorem 3.4.**  $\{\|\mathbf{v}\| : \mathbf{v} \in L\}$  is a discrete, closed set in  $\mathbb{R}$ .

*Proof.* We will prove  $\{\|\mathbf{v}\| : \mathbf{v} \in L\}$  has no accumulation points; closure and discreteness readily follow. We first show that there exists a positive lower bound for the norms of nonzero vectors in  $L$ . Consider  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ , a basis for  $L$ . Consider the orthogonal component of  $\mathbf{v}_i$ ,  $\mathbf{v}_i^\perp = \mathbf{v}_i - \sum_{j \neq i} \mathbf{v}_j * (\mathbf{v}_i \cdot \mathbf{v}_j) / \|\mathbf{v}_j\|^2$ . Since the  $\mathbf{v}_i$ s are linearly independent, this vector is necessarily nonzero. Furthermore, every linear combination involving  $\mathbf{v}_i$  must have norm at least  $\|\mathbf{v}_i^\perp\|$ . Taking  $\epsilon = \min_{1 \leq i \leq n} \|\mathbf{v}_i^\perp\|$ , we see that any nonzero element of  $L$  must have norm at least  $\epsilon$ .

This implies that for any two vectors  $\mathbf{v}, \mathbf{w} \in L$ ,  $\|\mathbf{v} - \mathbf{w}\| \geq \epsilon$ , since  $\mathbf{v} - \mathbf{w} \in L$ . Thus, there is a minimum distance between all vectors in  $L$ . Thus, given any  $N$ ,  $\{\mathbf{v} \in L : \|\mathbf{v}\| \leq N\}$  is a finite set. Then for any  $N$ , there can be no accumulation point in  $\{\|\mathbf{v}\| : \mathbf{v} \in L\}$  less than  $N$ , and so no accumulation point exists.  $\square$

**Corollary 3.5.**  $m_\ell$  is finite for all  $\ell$ .

For higher dimensional tori, we may define additional spectra as well. Consider some  $\mathbb{T}^n$  generated from a lattice  $L$ . If  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is a linear independence subset of  $L$ ,  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  can be considered to generate a  $k$ -dimensional sub-torus of  $\mathbb{T}^n$ .

**Definition 3.6.** For an  $n$ -dimensional torus  $\mathbb{T}^n$  generated from a lattice  $L$ , we define  $\mathcal{L}^k(\mathbb{T}^n) = \{(\mathbb{T}, m_{\mathbb{T}}) : \mathbb{T} \text{ is a } k\text{-dimensional sub-torus of } \mathbb{T}^n\}$ . We call  $\mathcal{L}^k(\mathbb{T}^n)$  the  $k$ -torus spectrum of  $\mathbb{T}^n$ .

In this case,  $m_{\mathbb{T}}$  is the number of subtori isometric to  $\mathbb{T}$ . We note that, since there are only a finite number of vectors of length  $\ell_i$  in any  $\mathbb{T}^n$ , there are only a finite number of  $k$ -tori with a preferred basis of vectors with norms  $\ell_1, \ell_2, \dots, \ell_k$ , and thus  $m_{\mathbb{T}}$  is finite.

#### 4. LENGTH SPECTRUM RIGIDITY

How much information is encoded in the length spectrum alone?

**Theorem 4.1.** If  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are 2-dimensional tori, and  $\mathcal{L}(\mathbb{T}_1) = \mathcal{L}(\mathbb{T}_2)$ , then  $\mathbb{T}_1 \cong \mathbb{T}_2$ .

*Proof.* By Thm 2.4, there exists a minimal nonzero vector in  $L_1$ , and a minimal nonzero vector in  $L_2$ . Call these vectors  $\mathbf{v}_1$  and  $\mathbf{w}_1$  respectively. Note that  $\|\mathbf{v}_1\| = \|\mathbf{w}_1\|$  necessarily, since their norms are minimal in equivalent sets.

Define  $L_1' = L_1 - \mathbb{Z}[\mathbf{v}_1]$ ,  $L_2' = L_2 - \mathbb{Z}[\mathbf{w}_1]$ .  $L_1'$  and  $L_2'$  are no longer lattices, of course, but we may still discuss the length spectrum of these sets,  $\mathcal{L}(L_1')$  and  $\mathcal{L}(L_2')$ . In constructing  $L_1'$  from  $L_1$ , we removed one vector of length zero, two vectors of length  $\|\mathbf{v}_1\|$ , and in general two vectors of length  $k\|\mathbf{v}_1\|$ . Similarly, in constructing  $L_2'$  from  $L_2$ , we removed one vector of length zero, two vectors of length  $\|\mathbf{w}_1\|$ , and in general two vectors of length  $k\|\mathbf{w}_1\|$ . Since  $\|\mathbf{v}_1\| = \|\mathbf{w}_1\|$ , the lengths of vectors removed from  $L_1$  and  $L_2$  are identical in both magnitude and multiplicity. Thus,  $\mathcal{L}(L_1') = \mathcal{L}(L_2')$ .

Since  $\{\|\mathbf{v}\| : \mathbf{v} \in L_1'\} \subseteq \{\|\mathbf{v}\| : \mathbf{v} \in L_1\}$ , we know  $\{\|\mathbf{v}\| : \mathbf{v} \in L_1'\}$  is a discrete set as well; the same holds for  $L_2'$ . We can thus again pick minimal nonzero vectors in  $L_1'$  and  $L_2'$ . Call these vectors  $\mathbf{v}_2$  and  $\mathbf{w}_2$ , respectively. Naturally,  $\|\mathbf{v}_2\| = \|\mathbf{w}_2\|$ . Note that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent by construction, as are  $\mathbf{w}_1$  and  $\mathbf{w}_2$ .

Define  $L_1'' = L_1' - \mathbb{Z}[\mathbf{v}_2]$ ,  $L_2'' = L_2' - \mathbb{Z}[\mathbf{w}_2]$ . By the same argument as above,  $\mathcal{L}(L_1'') = \mathcal{L}(L_2'')$ , and the set of lengths in  $L_1''$  or in  $L_2''$  are discrete. Thus we can once more pick minimal vectors,  $\mathbf{v}_3$  and  $\mathbf{w}_3$ .

We claim that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a  $\mathbb{Z}$ -basis for  $L_1$ ; that is,  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2] = L_1$ . Since they are linearly independent, we know for any  $\mathbf{v} \in L_1$ ,  $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$  for some  $\alpha_1, \alpha_2 \in \mathbb{R}$ . Since  $L_1$  is closed under vector addition,  $(\alpha_1 + m)\mathbf{v}_1 + (\alpha_2 + n)\mathbf{v}_2 \in L_1$  for all  $m, n \in \mathbb{Z}$ . Pick  $m, n$  such that  $|\alpha_1 + m| \leq 1/2$  and  $|\alpha_2 + n| \leq 1/2$ . Then  $\mathbf{v}' = (\alpha_1 + m)\mathbf{v}_1 + (\alpha_2 + n)\mathbf{v}_2 \in L_1$ . Since  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent, strict inequality holds in the triangle inequality; we thus have

$$\|\mathbf{v}'\| \leq |\alpha_1 + m| \|\mathbf{v}_1\| + |\alpha_2 + n| \|\mathbf{v}_2\| \leq \|\mathbf{v}_1\|/2 + \|\mathbf{v}_2\|/2 \leq \|\mathbf{v}_2\|$$

This contradicts the minimality of  $\mathbf{v}_2$  unless  $\mathbf{v}' = \mathbf{0}$ . Thus  $\alpha_1, \alpha_2 \in \mathbb{Z}$ , and  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a  $\mathbb{Z}$ -basis for  $L_1$ . Similarly,  $\{\mathbf{w}_1, \mathbf{w}_2\}$  is a  $\mathbb{Z}$ -basis for  $L_2$ .

Now, minimality demands that  $\|\mathbf{v}_1\|^2 \geq 2|\mathbf{v}_1 \cdot \mathbf{v}_2|$ . If not, we would have

$$\begin{aligned} \|\mathbf{v}_1 \pm \mathbf{v}_2\|^2 &= \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 \pm 2(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &< \|\mathbf{v}_2\|^2 \end{aligned}$$

which of course contradicts the minimality of  $\mathbf{v}_2$ . The same is true for the  $\mathbf{w}$ s:  $\|\mathbf{w}_1\| \geq 2|\mathbf{v}_1 \cdot \mathbf{v}_2|$ .

Now, consider  $\mathbf{v}_3$ . We know that  $\mathbf{v}_3 = m\mathbf{v}_1 + n\mathbf{v}_2$  for some  $m, n \in \mathbb{Z} - \{0\}$ . We claim  $|m| = |n| = 1$ . Without loss of generality, assume  $m > 0$  (if necessary, replace  $\mathbf{v}_3$  with its negative). Further, let us restrict  $n > 0$  as well. The  $n < 0$  case will follow readily.

If  $n > 0$ , then  $(\mathbf{v}_1 \cdot \mathbf{v}_2) \leq 0$ , since  $(\mathbf{v}_1 \cdot \mathbf{v}_2) > 0$  would imply  $\|m\mathbf{v}_1 - n\mathbf{v}_2\| < \|m\mathbf{v}_1 + n\mathbf{v}_2\|$ , and we specified  $\mathbf{v}_3$  as minimal. If  $n \leq m$ , we then have

$$\begin{aligned} \|(m-1)\mathbf{v}_1 + n\mathbf{v}_2\|^2 &= \|m\mathbf{v}_1 + n\mathbf{v}_2\|^2 + (1-2m)\|\mathbf{v}_1\|^2 - 2n(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &= \|\mathbf{v}_3\|^2 + (1-2m)\|\mathbf{v}_1\|^2 + 2n|\mathbf{v}_1 \cdot \mathbf{v}_2| \\ &\leq \|\mathbf{v}_3\|^2 + (1-2m)\|\mathbf{v}_1\|^2 + n\|\mathbf{v}_1\|^2 \\ &< \|\mathbf{v}_3\|^2 \end{aligned}$$

which implies  $((m-1)\mathbf{v}_1 + n\mathbf{v}_2) \notin L_1''$ , or  $m=1$ . Then, we have

$$\begin{aligned} \|\mathbf{v}_1 + (n-1)\mathbf{v}_2\|^2 &= \|\mathbf{v}_1 + n\mathbf{v}_2\|^2 + (1-2n)\|\mathbf{v}_2\|^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &= \|\mathbf{v}_3\|^2 + (1-2n)\|\mathbf{v}_2\|^2 + 2|\mathbf{v}_1 \cdot \mathbf{v}_2| \\ &\leq \|\mathbf{v}_3\|^2 + (1-2n)\|\mathbf{v}_2\|^2 + \|\mathbf{v}_2\|^2 \\ &< \|\mathbf{v}_3\|^2 \end{aligned}$$

which implies that  $n=1$  as well. The  $m \leq n$  case follows similar lines, and the  $n < 0$  case simply reverses the signs of both  $n$  and  $(\mathbf{v}_1 \cdot \mathbf{v}_2)$ , leaving the overall proof unchanged. By perhaps replacing  $\mathbf{v}_2$  by its negative, we thus have  $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ . We can do the same for the  $\mathbf{w}$ s, giving  $\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_2$ .

We then have that  $\|\mathbf{v}_1\| = \|\mathbf{w}_1\|$ ,  $\|\mathbf{v}_2\| = \|\mathbf{w}_2\|$ , and  $\|\mathbf{v}_1 + \mathbf{v}_2\| = \|\mathbf{w}_1 + \mathbf{w}_2\|$ , from which it immediately follows that  $(\mathbf{v}_1 \cdot \mathbf{v}_2) = (\mathbf{w}_1 \cdot \mathbf{w}_2)$ . By Thm 2.6, the tori are thus equivalent.  $\square$

We note that this proof relied heavily on the 2-dimensional nature of the tori, and is thus difficult to generalize to higher dimensions. In fact, an explicit example of nonisometric 16-tori with equivalent length spectra due to J Milnor has been known since the sixties; more recently, Conway and Sloane constructed nonisometric, length isospectral 4-tori (see [1] and [6] for details). Furthermore, a construction by Sunada allows one to find an infinite number of pairs of distinct tori with the same length spectrum (see [3]).

## 5. HIGHER DIMENSIONAL RIGIDITY

We want to prove a slightly weaker rigidity result for 3-dimensional tori; first, we need a lemma.

**Lemma 5.1.** *Consider two 3-tori,  $\mathbb{T}_1$  and  $\mathbb{T}_2$ , such that either  $\mathcal{L}(\mathbb{T}_1) = \mathcal{L}(\mathbb{T}_2)$  or  $\mathcal{L}^2(\mathbb{T}_1) = \mathcal{L}^2(\mathbb{T}_2)$ . If  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are generated by minimal bases  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  and  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$  respectively,  $\|\mathbf{v}_i\| = \|\mathbf{w}_i\|$ , and  $\|\mathbf{v}_i \times \mathbf{v}_j\| = \|\mathbf{w}_i \times \mathbf{w}_j\|$ , then  $\mathbb{T}_1 \cong \mathbb{T}_2$ .*

*Proof.* We know  $\|\mathbf{v}_i \times \mathbf{v}_j\|^2 = \|\mathbf{v}_i\|^2\|\mathbf{v}_j\|^2 - (\mathbf{v}_i \cdot \mathbf{v}_j)^2$ ; since the norms are equivalent, we then have that  $|\mathbf{v}_i \cdot \mathbf{v}_j| = |\mathbf{w}_i \cdot \mathbf{w}_j|$ . By replacing  $\mathbf{w}_1$  with its negative if necessary, we can ensure that  $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2$ . If the remaining two dot products are equal, by Thm 2.6 we're done. If the dot products in  $L_1$  are both equal to the negatives of the dot products in  $L_2$ , then we can replace  $\mathbf{w}_3$  with  $-\mathbf{w}_3$ , and the dot products are then equal.

If one of the dot products are equal, while one is equal to the negative, we may take the negative of  $\mathbf{w}_3$  if appropriate, to ensure that  $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_3$ , while  $\mathbf{v}_2 \cdot \mathbf{v}_3 = -\mathbf{w}_2 \cdot \mathbf{w}_3$ .

Minimality of the bases demands that  $2|\mathbf{w}_1 \cdot \mathbf{w}_2| \leq \|\mathbf{w}_1\|^2$ ,  $2|\mathbf{w}_1 \cdot \mathbf{w}_3| \leq \|\mathbf{w}_1\|^2$ ,  $2|\mathbf{w}_2 \cdot \mathbf{w}_3| \leq \|\mathbf{w}_2\|^2$ , and  $-2(\mathbf{w}_1 \cdot \mathbf{w}_2) - 2(\mathbf{w}_1 \cdot \mathbf{w}_3) - 2(\mathbf{w}_2 \cdot \mathbf{w}_3) \leq \|\mathbf{w}_1\|^2 + \|\mathbf{w}_2\|^2$ . These conditions impose a monotonicity on our lattice:  $\|a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3\| > \|(a-1)\mathbf{w}_1 + (b-1)\mathbf{w}_2 + (c-1)\mathbf{w}_3\|$ , for  $a, b, c > 0$ . This can be demonstrated by showing that  $(2a-1)\|\mathbf{w}_1\|^2 + (2b-1)\|\mathbf{w}_2\|^2 + (2c-1)\|\mathbf{w}_3\|^2 + 2(a+b-1)(\mathbf{w}_1 \cdot \mathbf{w}_2) + 2(a+c-1)(\mathbf{w}_1 \cdot \mathbf{w}_3) + 2(b+c-1)(\mathbf{w}_2 \cdot \mathbf{w}_3)$  is greater than zero. We note this expression is

$$\begin{aligned} &\geq (2a-1-a-b+1-a-c+1)\|\mathbf{w}_1\|^2 + \\ &\quad (2b-1-b-c+1)\|\mathbf{w}_2\|^2 + (2c-1)\|\mathbf{w}_3\|^2 \\ &= (-b-c+1)\|\mathbf{w}_1\|^2 + (b-c)\|\mathbf{w}_2\|^2 + (2c-1)\|\mathbf{w}_3\|^2 \\ &\geq 0 \end{aligned}$$

with equality only holding if  $-2(\mathbf{w}_1 \cdot \mathbf{w}_2) = \|\mathbf{w}_1\|^2$ ,  $-2(\mathbf{w}_1 \cdot \mathbf{w}_3) = \|\mathbf{w}_1\|^2$ ,  $-2(\mathbf{w}_2 \cdot \mathbf{w}_3) = \|\mathbf{w}_1\|^2$  and  $\|\mathbf{w}_1\| = \|\mathbf{w}_2\|$ , which taken together contradict the first minimality condition.

Similar monotonicity results hold for  $\|a\mathbf{w}_1 + b\mathbf{w}_2 + \mathbf{w}_3\|$  versus  $\|(a-1)\mathbf{w}_1 + (b-1)\mathbf{w}_2 + \mathbf{w}_3\|$ . We then have that the minimal vector not in the three 2-tori formed by the basis vectors is necessarily of the form  $\mathbf{w}_1 \pm \mathbf{w}_2 \pm \mathbf{w}_3$ . The same holds true in  $L_1$ . We can redefine  $\mathbf{v}_i$  and  $\mathbf{w}_i$  by taking appropriate negatives to ensure that the minimal  $L_1$  vector is  $\mathbf{w}_1 + \mathbf{v}_2 + \mathbf{v}_3$  while the dot products remain equal/inverses as before. By either  $\mathcal{L}$  or  $\mathcal{L}^2$  equivalence, these two vectors must have the same norm. That is, there are  $a, b$  such that  $|a| = |b| = 1$  and  $\|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\| = \|a\mathbf{w}_1 + b\mathbf{w}_2 + \mathbf{w}_3\|$ . This is equivalent to saying

$$2(\mathbf{v}_1 \cdot \mathbf{v}_2) + 2(\mathbf{v}_1 \cdot \mathbf{v}_3) + 2(\mathbf{v}_2 \cdot \mathbf{v}_3) = 2ab(\mathbf{w}_1 \cdot \mathbf{w}_2) + 2a(\mathbf{w}_1 \cdot \mathbf{w}_3) + 2b(\mathbf{w}_2 \cdot \mathbf{w}_3)$$

or

$$2(\mathbf{w}_1 \cdot \mathbf{w}_2) + 2(\mathbf{w}_1 \cdot \mathbf{w}_3) - 2(\mathbf{w}_2 \cdot \mathbf{w}_3) = 2ab(\mathbf{w}_1 \cdot \mathbf{w}_2) + 2a(\mathbf{w}_1 \cdot \mathbf{w}_3) + 2b(\mathbf{w}_2 \cdot \mathbf{w}_3)$$

which has no solutions unless one of the  $\mathbf{w}_i \cdot \mathbf{w}_j$  is equal to zero.  $\square$

**Theorem 5.2.** *If  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are 3-dimensional tori such that  $\mathcal{L}^2(\mathbb{T}_1) = \mathcal{L}^2(\mathbb{T}_2)$ , then  $\mathbb{T}_1 \cong \mathbb{T}_2$*

*Proof.* In  $L_1$ , pick  $\mathbf{v}_1$  and  $\mathbf{v}_2$  as above; that is,  $\mathbf{v}_1$  is a minimal vector,  $\mathbf{v}_2$  is a minimal vector not in  $\mathbb{Z}[\mathbf{v}_1]$ .  $\mathbf{v}_1$  and  $\mathbf{v}_2$  generate a torus, with a counterpart in  $L_2$ . In this  $L_2$  torus, by Thm 2.6, we may find basis vectors  $\mathbf{w}_1, \mathbf{w}_2$  such that  $\|\mathbf{v}_1\| = \|\mathbf{w}_1\|$ ,  $\|\mathbf{v}_2\| = \|\mathbf{w}_2\|$ , and  $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2$ . Choose  $\mathbf{v}_3$  to be a minimal vector not in  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$ , and  $\mathbf{w}_3$  to be a minimal vector not in  $\mathbb{Z}[\mathbf{w}_1, \mathbf{w}_2]$ . Note that  $\|\mathbf{v}_3\| = \|\mathbf{w}_3\|$  due to  $\mathcal{L}^2$  equivalence.

We claim that there can be at most four vectors of norm  $\|\mathbf{v}_3\|$  not in  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$  (and similarly for  $L_2$ ). Minimality of  $\mathbf{v}_3$  demands that  $\|\mathbf{v}_3\|^2 \leq \|\mathbf{v}_3 \pm \mathbf{v}_1\|^2$ , which means that

$$2|\mathbf{v}_1 \cdot \mathbf{v}_3| \leq \|\mathbf{v}_1\|^2$$

Similarly, since  $\|\mathbf{v}_3\|^2 \leq \|\mathbf{v}_3 \pm \mathbf{v}_2\|^2$

$$2|\mathbf{v}_2 \cdot \mathbf{v}_3| \leq \|\mathbf{v}_2\|^2$$

and since  $\|\mathbf{v}_3\|^2 \leq \|\mathbf{v}_3 + \mathbf{v}_2 + \mathbf{v}_1\|^2$

$$-2(\mathbf{v}_1 \cdot \mathbf{v}_2) - 2(\mathbf{v}_1 \cdot \mathbf{v}_3) - 2(\mathbf{v}_2 \cdot \mathbf{v}_3) \leq \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2$$

Since  $\|n\mathbf{v}_1 + \mathbf{v}_3\|^2 = n^2\|\mathbf{v}_1\|^2 + 2n(\mathbf{v}_1 \cdot \mathbf{v}_3) + \|\mathbf{v}_3\|^2$ , we have that  $\|n\mathbf{v}_1 + \mathbf{v}_3\| = \|\mathbf{v}_3\|$  iff  $n^2\|\mathbf{v}_1\| = -2n(\mathbf{v}_1 \cdot \mathbf{v}_3)$ , which in turn happens iff  $n=\pm 1$  and equality holds in  $2|\mathbf{v}_1 \cdot \mathbf{v}_3| \leq \|\mathbf{v}_1\|^2$ . Note that if  $n=+1$  works,  $n=-1$  will not. Similarly, when we consider  $n\mathbf{v}_2 + \mathbf{v}_3$ , we find a similar condition must hold. Finally, if  $\|n\mathbf{v}_1 + m\mathbf{v}_2 + \mathbf{v}_3\| = \|\mathbf{v}_3\|$ , then we have that

$$n^2\|\mathbf{v}_1\|^2 + m^2\|\mathbf{v}_2\|^2 + 2mn(\mathbf{v}_1 \cdot \mathbf{v}_2) + 2n(\mathbf{v}_1 \cdot \mathbf{v}_3) + 2m(\mathbf{v}_2 \cdot \mathbf{v}_3) = 0$$

Without loss of generality, assume  $n, m > 0$ , replacing  $\mathbf{v}_1, \mathbf{v}_2$  with their negatives if necessary. Assume  $m$  or  $n$  is greater than one. We want to show that this implies  $\|(n-1)\mathbf{v}_1 + (m-1)\mathbf{v}_2 + \mathbf{v}_3\| < \|\mathbf{v}_3\|$ , or

$$(n-1)^2\|\mathbf{v}_1\|^2 + (m-1)^2\|\mathbf{v}_2\|^2 + 2(m-1)(n-1)(\mathbf{v}_1 \cdot \mathbf{v}_2) + 2(n-1)(\mathbf{v}_1 \cdot \mathbf{v}_3) + 2(m-1)(\mathbf{v}_2 \cdot \mathbf{v}_3) < 0$$

which is in turn equivalent to showing

$$(2n-1)\|\mathbf{v}_1\|^2 + (2m-1)\|\mathbf{v}_2\|^2 + 2(n+m-1)(\mathbf{v}_1 \cdot \mathbf{v}_2) + 2(\mathbf{v}_1 \cdot \mathbf{v}_3) + 2(\mathbf{v}_2 \cdot \mathbf{v}_3) > 0$$

but we know from the above relations that this is

$$\begin{aligned} &\geq (2n-2)\|\mathbf{v}_1\|^2 + (2m-2)\|\mathbf{v}_2\|^2 + 2(n+m-2)(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &\geq (2n-2)\|\mathbf{v}_1\|^2 + (2m-2)\|\mathbf{v}_2\|^2 - (n+m-2)\|\mathbf{v}_1\|^2 \\ &> 0 \end{aligned}$$

Minimality established while proving the lemma ensures that the coefficient in front of  $\mathbf{v}_3$  must be 1.

We then have that the only possibility for a vector of the form  $\|n\mathbf{v}_1 + m\mathbf{v}_2 + \mathbf{v}_3\|$  is  $\|\mathbf{v}_3 \pm \mathbf{v}_2 \pm \mathbf{v}_1\|$ . Further, inspection will reveal that two vectors of that form are not possible. For example,  $\|\mathbf{v}_3 + \mathbf{v}_2 + \mathbf{v}_1\| = \|\mathbf{v}_3\|$  demands that  $\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 = -2[\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3]$ , while  $\|\mathbf{v}_3 + \mathbf{v}_2 - \mathbf{v}_1\| = \|\mathbf{v}_3\|$  demands that  $\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 = -2[-\mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3]$ . Together, these require that  $\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3 = 0$ , which in turn would mean that  $-2\mathbf{v}_2 \cdot \mathbf{v}_3 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2$ , despite the restriction that  $-2(\mathbf{v}_2 \cdot \mathbf{v}_3) \leq \|\mathbf{v}_2\|^2$  that minimality demands.

Thus, there can be at most four vectors in  $L_2/\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$  with norm  $\|\mathbf{v}_3\|$ .

We now work in cases.

Case 1: There is exactly one vector in  $L_1$  not in  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$  with norm  $\|\mathbf{v}_3\|$  (up to taking a negative). Then the same is true in  $L_2$ .

Case 1a:  $\|\mathbf{v}_1\| < \|\mathbf{v}_2\|$ . In this case, there is exactly one 2-torus generated by vectors with norm  $\|\mathbf{v}_1\|$  and  $\|\mathbf{v}_3\|$  outside of  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$ ; the same holds true in  $L_2$ . By  $L^2$  equivalence, these are thus equivalent tori. Now, the only vectors outside of multiples of  $\mathbf{w}_1$  with norm  $\|\mathbf{w}_2\|$  are  $\mathbf{w}_2$  and possibly  $\mathbf{w}_1 + \mathbf{w}_2$ . In this case, redefine  $\mathbf{w}_2$  to be the vector that generates the 2-torus equivalent to the  $\mathbf{v}_2 - \mathbf{v}_3$  torus. Note this redefined  $\mathbf{w}_2$  still forms the same 2-torus with  $\mathbf{w}_1$ . We thus have vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  and  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  such that  $\|\mathbf{v}_i\| = \|\mathbf{w}_i\|$  and  $\|\mathbf{v}_i \times \mathbf{v}_j\| = \|\mathbf{w}_i \times \mathbf{w}_j\|$ . By Lemma 5.2, the tori are thus equivalent.

Case 1b:  $\|\mathbf{v}_1\| = \|\mathbf{v}_2\|$ . In this case, redefine  $\mathbf{w}_1$  to be the vector of norm  $\|\mathbf{v}_1\|$  in the original span of  $\mathbf{w}_1, \mathbf{w}_1$  that together with  $\mathbf{w}_3$  forms the torus equivalent to  $\mathbf{v}_1 - \mathbf{v}_3$ . Throw out the span of  $\mathbf{v}_1, \mathbf{v}_3$  and  $\mathbf{w}_1, \mathbf{w}_3$  in the  $\mathcal{L}^2$  spectrum, and redefine  $\mathbf{w}_2$  to be the vector of norm  $\|\mathbf{v}_2\|$  in the  $\mathbb{Z}$ -span of the original  $\mathbf{w}_1, \mathbf{w}_1$  that forms the same 2-torus with  $\mathbf{w}_3$  as  $\mathbf{v}_2 - \mathbf{v}_3$ . Note that these redefined  $\mathbf{w}_1, \mathbf{w}_2$  form the same torus as before. We are thus left with bases as above.

Case 2: There are exactly 2 vectors in  $L_1$  not in  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$  with norm  $\|\mathbf{v}_3\|$ . Then the same is true in  $L_2$ . Further, the 2-tori formed by the two vectors in each lattice are equivalent.

Now, up to sign changes, there are only four possibilities for vectors of norm  $\|\mathbf{v}_3\|$  in  $L_2$ :  $\mathbf{w}_3, \mathbf{w}_1 + \mathbf{w}_3, \mathbf{w}_2 + \mathbf{w}_3$ , and  $\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ . The same is true in  $L_1$ . We can adjust these easily enough; if the second vector is  $\mathbf{v}_3 - \mathbf{v}_1$ , we simply replace  $\mathbf{v}_1$  with its negative, leaving the  $\mathbf{v}_1 - \mathbf{v}_2$  torus unchanged. We can do the same if the second vector is  $\mathbf{v}_3 - \mathbf{v}_2$ , or  $\mathbf{w}_3 \pm \mathbf{w}_2 \pm \mathbf{w}_1$ .

Now, if the vector in  $L_1$  is  $\mathbf{v}_3 + \mathbf{v}_2$  while the vector in  $L_2$  is  $\mathbf{w}_3 + \mathbf{w}_1$ , we then have that  $-2(\mathbf{v}_2 \cdot \mathbf{v}_3) = \|\mathbf{v}_2\|^2$  and  $-2(\mathbf{w}_1 \cdot \mathbf{w}_3) = \|\mathbf{w}_1\|^2$ . Further, since the two vectors in each lattice form equivalent 2-tori, we have that  $|\|\mathbf{v}_3\|^2 + (\mathbf{v}_2 \cdot \mathbf{v}_3)| = |\|\mathbf{w}_3\|^2 + (\mathbf{w}_1 \cdot \mathbf{w}_3)|$ . We know the signs of both of these must be positive, so that  $(\mathbf{v}_2 \cdot \mathbf{v}_3) = (\mathbf{w}_1 \cdot \mathbf{w}_3)$ ; we thus have that  $\|\mathbf{v}_2\| = \|\mathbf{w}_1\|$ , and we can replace  $\mathbf{w}_1$  with  $\mathbf{w}_2$  and vice versa without affecting their 2-torus. We can do the same if the vector in  $L_1$  is  $\mathbf{v}_3 + \mathbf{v}_1$  while the vector in  $L_2$  is  $\mathbf{w}_3 + \mathbf{w}_2$ .

Finally, if the vector in  $L_1$  is  $\mathbf{v}_3 + \mathbf{v}_1$  while the vector in  $L_2$  is  $\mathbf{w}_3 + \mathbf{w}_2 + \mathbf{w}_1$ , we have that  $-2(\mathbf{v}_1 \cdot \mathbf{v}_3) = \|\mathbf{v}_1\|^2$  and  $-2(\mathbf{w}_1 \cdot \mathbf{w}_2) - 2(\mathbf{w}_1 \cdot \mathbf{w}_3) - 2(\mathbf{w}_2 \cdot \mathbf{w}_3) = \|\mathbf{w}_1\|^2 + \|\mathbf{w}_2\|^2$ . Additionally, by 2-tori equivalence, we know that  $|\mathbf{v}_1 \cdot \mathbf{v}_3 + \|\mathbf{v}_3\|^2| = |\mathbf{w}_1 \cdot \mathbf{w}_3 + \mathbf{w}_2 \cdot \mathbf{w}_3 + \|\mathbf{w}_3\|^2|$ . If they have the same sign, then  $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_3 + \mathbf{w}_2 \cdot \mathbf{w}_3 = -\|\mathbf{v}_1\|/2$ , and we would have that  $-2\mathbf{w}_1 \cdot \mathbf{w}_2 = \|\mathbf{w}_2\|^2$ , meaning that not only  $\|\mathbf{w}_1\| = \|\mathbf{w}_2\|$ , but that we can replace  $\|\mathbf{w}_2\|$  with  $\|\mathbf{w}_1 + \mathbf{w}_2\|$ , leaving both the 2-torus and the dot product unchanged. If the signs are opposite, then  $\mathbf{w}_1 \cdot \mathbf{w}_3 + \mathbf{w}_2 \cdot \mathbf{w}_3 = -2\|\mathbf{w}_2\|^2 - \mathbf{v}_1 \cdot \mathbf{v}_3 = -2\|\mathbf{w}_2\|^2 + \|\mathbf{w}_1\|/2$ , and we have that  $-2(\mathbf{w}_1 \cdot \mathbf{w}_2) = 2\|\mathbf{w}_1\|^2 - 3\|\mathbf{w}_2\|^2$ , an impossibility. Thus we may assume our vectors in  $L_1$  and  $L_2$  are of the same form.

Case 2a:  $\|\mathbf{v}_1\| < \|\mathbf{v}_2\|$ . In this case, there is only one vector in  $L_2$  with norm  $\|\mathbf{v}_1\|$ ,  $\mathbf{w}_1$ . There are possibly 2 vectors in  $\mathbb{Z}[\mathbf{w}_1, \mathbf{w}_2]$  not in  $\mathbb{Z}[\mathbf{w}_1]$  with norm  $\|\mathbf{v}_2\|$ ,  $\mathbf{w}_2$  and  $\mathbf{w}_1 + \mathbf{w}_2$ . One of these must combine with one of the  $\|\mathbf{v}_3\|$  vectors to form a 2-torus equivalent to  $\mathbf{v}_2 - \mathbf{v}_3$ ; redefine  $\mathbf{w}_2$ , if necessary, to be this vector. Redefine  $\mathbf{w}_3$  to be the vector of norm  $\|\mathbf{v}_3\|$  that forms the same 2-torus with  $\mathbf{w}_1$  as  $\mathbf{v}_1 - \mathbf{v}_3$ . Then the 2-torus corresponding to  $\mathbf{v}_2 - \mathbf{v}_3$  is either  $\mathbf{w}_2 - \mathbf{w}_3$ , or  $\mathbf{w}_2$  in combination with the second vector of norm  $\|\mathbf{v}_3\|$ . In the first case, we're done. In the second, there are three possibilities. If the second vector is  $\mathbf{w}_1 + \mathbf{w}_3$ , redefine  $\mathbf{w}_3$  to be this vector, leaving the  $\mathbf{w}_1 - \mathbf{w}_3$  torus unchanged. If the second vector is  $\mathbf{w}_2 + \mathbf{w}_3$ , then this forms the same 2-torus with  $\mathbf{w}_2$  as  $\mathbf{w}_3$ . If the second vector is  $\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ , then we have that the  $\mathbf{v}_2 - \mathbf{v}_3$  torus corresponds to  $\mathbf{w}_2 - (\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3)$ , and so the  $\mathbf{v}_2 - (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$  torus necessarily corresponds to the  $\mathbf{w}_2 - \mathbf{w}_3$  torus. Then we have both that

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_2 + \|\mathbf{w}_2\|^2 + \mathbf{w}_2 \cdot \mathbf{w}_3$$

and

$$\mathbf{v}_1 \cdot \mathbf{v}_2 + \|\mathbf{v}_2\|^2 + \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{w}_2 \cdot \mathbf{w}_3$$

which together demand that  $\|\mathbf{v}_2\| = 0$ , an impossibility. So this is not a valid case.

Case 2b:  $\|\mathbf{v}_1\| = \|\mathbf{v}_2\|$ . In this case, we can redefine  $\mathbf{w}_1$  as the vector with norm  $\|\mathbf{v}_1\|$  that forms, with either of the two  $\|\mathbf{v}_3\|$  vectors, the same 2-torus as  $\mathbf{v}_1 - \mathbf{v}_3$ . Note this  $\mathbf{w}_1$  must be in the same plane as the original  $\mathbf{w}_1 - \mathbf{w}_2$ . Pick  $\mathbf{w}_2$  to be the other vector of norm  $\|\mathbf{v}_1\|$  that together with  $\mathbf{w}_1$  forms the same 2-torus as  $\mathbf{v}_1 - \mathbf{v}_2$ . Take appropriate negatives, so the angle between them is equal. Now, there are two possibilities: either the  $\mathbf{v}_2 - \mathbf{v}_3$  torus corresponds to a torus generated by one of  $\mathbf{w}_2, \mathbf{w}_1 + \mathbf{w}_2$ , or it corresponds to the torus generated by  $\mathbf{w}_1$  and the second

vector of norm  $\|\mathbf{v}_2\|$ . In the first case, we're done, redefining  $\mathbf{w}_2$  if necessary. The second requires more care. If the two vectors of norm  $\|\mathbf{w}_3\|$  are  $\mathbf{w}_3, \mathbf{w}_1 + \mathbf{w}_3$ , they in fact only generate one distinct torus with  $\mathbf{w}_1$ . If the second vector is  $\mathbf{w}_2 + \mathbf{w}_3$ , we have that  $(\mathbf{v}_1 \cdot \mathbf{v}_3)^2 = (\mathbf{w}_1 \cdot \mathbf{w}_3)^2$ , and that  $\mathbf{w}_2 \cdot \mathbf{w}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3$ , so we in fact have the 3-tori are equivalent. If the second vector is  $\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ , then we have that  $(\mathbf{v}_1 \cdot \mathbf{v}_3)^2 = (\mathbf{w}_1 \cdot \mathbf{w}_3)^2$  and that  $\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_2 + \mathbf{w}_1 \cdot \mathbf{w}_3 + \mathbf{w}_2 \cdot \mathbf{w}_3$ . Since all dot products here must have the same sign, we then have that  $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2, \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_3$ , and from this  $\mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{w}_2 \cdot \mathbf{w}_3$ . Thus the 3-tori are again equivalent.

Case 3: There are exactly three vectors in  $L_1$  not in  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$  with norm  $\|\mathbf{v}_3\|$ . Then the same is true in  $L_2$ . We can adjust the form of these vectors easily enough; if the vectors are  $\mathbf{v}_3, \mathbf{v}_3 \pm \mathbf{v}_1, \mathbf{v}_3 \pm \mathbf{v}_2$ , we simply replace  $\mathbf{v}_1$  and  $\mathbf{v}_2$  with their negatives where appropriate, to get  $\mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1, \mathbf{v}_3 + \mathbf{v}_2$ . If the vectors are  $\mathbf{v}_3, \mathbf{v}_3 \pm \mathbf{v}_1, \mathbf{v}_3 \pm \mathbf{v}_2 \pm \mathbf{v}_1$ , we simply redefine  $\mathbf{v}_3$  as  $\mathbf{v}_3 \pm \mathbf{v}_1$  and reduce to the previous case. None of these operations affect the 2-torus formed by  $\mathbf{v}_1, \mathbf{v}_2$ . So we can thus assume that our three vectors in  $L_1$  are  $\mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3$ , and our three vectors in  $L_2$  are  $\mathbf{w}_3, \mathbf{w}_1 + \mathbf{w}_3, \mathbf{w}_2 + \mathbf{w}_3$ .

Then we have that that  $\|\mathbf{v}_1 + \mathbf{v}_3\|^2 = \|\mathbf{v}_3\|^2$  and  $\|\mathbf{w}_1 + \mathbf{w}_3\|^2 = \|\mathbf{w}_3\|^2$ , which implies that  $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_3$ , and similarly for  $\mathbf{v}_2 \cdot \mathbf{v}_3$  and  $\mathbf{w}_2 \cdot \mathbf{w}_3$ . We already know that  $\mathbf{v}_1 \cdot \mathbf{v}_2 = \pm \mathbf{w}_1 \cdot \mathbf{w}_2$ , so by our lemma the tori are equal.

Case 4: There are exactly four vectors in  $L_1$  not in  $\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2]$  with norm  $\|\mathbf{v}_3\|$ . Then the same is true in  $L_2$ . These vectors are of course of the form  $\mathbf{v}_3, \mathbf{v}_3 \pm \mathbf{v}_1, \mathbf{v}_3 \pm \mathbf{v}_2, \mathbf{v}_3 \pm \mathbf{v}_2 \pm \mathbf{v}_1$ . By taking appropriate negatives, we can get them to be of the form  $\mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1, \mathbf{v}_3 + \mathbf{v}_2, \mathbf{v}_3 \pm \mathbf{v}_2 \pm \mathbf{v}_1$ . Doing the same in  $L_2$ , we then have that  $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{w}_1 \cdot \mathbf{w}_3$  and  $\mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{w}_2 \cdot \mathbf{w}_3$ . Since none of these operations affected the 2-torus formed by  $\mathbf{v}_1, \mathbf{v}_2$  or  $\mathbf{w}_1, \mathbf{w}_2$ , we thus have the two tori are equal.  $\square$

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