THE STONE REPRESENTATION THEOREM FOR BOOLEAN ALGEBRAS

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Abstract. The Stone Representation Theorem for Boolean Algebras, first proved by M. H. Stone in 1936 ([4]), states that every Boolean algebra is isomorphic to a field of sets. This paper motivates and presents a proof.

Contents

1. Introduction

The original motivation for developing topology was a desire to generalize ideas in geometry such as shape and distance. The notion that topology could illuminate algebra was resisted by such luminaries as Birkhoff, who once, when asked about the use of topological methods in algebra, allegedly responded, "I don't consider this algebra, but this doesn't mean that algebraists can't do it" [3]. Nevertheless, in 1936 Marshall Stone published a paper [4] proving an unexpected equivalence between Boolean algebras and certain topological spaces, now called *Stone spaces* in his honor. This present paper proves the main result of [4], the Stone Representation Theorem for Boolean Algebras. Section 2 introduces Boolean algebras and tools for working with them, and Section 3 introduces Stone spaces and proves the theorem. This paper assumes that the reader is comfortable with basic topology. Experience with propositional calculus will be helpful, but not necessary.

Johnstone [3] proves the Stone Representation Theorem using tools of category theory. That approach is cleaner than ours, but also requires greater abstraction. This paper draws definitions and from Johnstone, but most of the proofs in this paper are based on proofs from Halmos [1]. Exceptions are noted as they occur.

2. Boolean Algebras

We begin by defining posets and distributive lattices; with that terminology in place, we can define Boolean algebras as a specific type of distributive lattice.

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Definition 2.1. A partially-ordered set, often abbreviated *poset*, is a set A together with a relation $\leq$ that is

(1) *reflexive*: for all $a \in A$, $a \leq a$;
(2) *transitive*: for all $a, b, c \in A$, if $a \leq b$ and $b \leq c$, then $a \leq c$; and
(3) *antisymmetric*: for all $a, b, c \in A$, if $a \leq b$ and $b \leq a$ then $a = b$.

Suppose A is a poset. The reason we say A is *partially* ordered, not *totally* ordered, is that two arbitrary elements in A are not necessarily comparable. We may very well have elements a and b in A such that $a \nleq b$ and $b \nleq a$. For example, consider the set of sets $X = \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$. We can put a partial order on X by saying $a \leq b$ if $a \subseteq b$. For example, we can say $\{0\} \leq \{0, 1\}$ since $\{0\} \subseteq \{0, 1\}$, and $\{0\} \leq \{0, 2\}$. Furthermore, $\{1\} \leq \{0, 1\}$. However, $\{1\} \nleq \{0, 2\}$. Those two sets simply cannot be compared under our partial order.

If S is a finite subset of a poset A, we define the *join* of S as the element $\bigvee S$ (unique if it exists) such that

(1) for all $s \in S$, $s \leq \bigvee S$; and
(2) for any element $x \in A$ such that $s \leq x$ for all $s \in S$, we have that $\bigvee S \leq x$.

The join is also referred to as a *least upper bound* or *supremum*. Dual to the join of S is its *meet*, written $\bigwedge S$. The meet (or *greatest lower bound*, or *infimum*) is defined identically to the join, but with inequalities reversed.

Notation 2.2. If we are considering a set with only two elements, say $\{a, b\}$, we write its join as $a \vee b$ and its meet as $a \wedge b$. If we are considering the empy set $\varnothing$, we use the symbol 0 to denote its join and 1 to denote its meet. It follows from the definition that 0 is the smallest element of a poset and 1 is the greatest.

Definition 2.3. A *lattice* is a poset A such that every finite subset of A has a meet and a join. We say a lattice is *complete* if it has meets and joins of arbitrary, not just finite, sets. We say it is *distributive* if the meet and the join obey the following distributive law.

$$\text{For all } a, b, c \in A, \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{2.4}$$

Examples 2.5. Some examples are in order.

(1) The symbols $\wedge$ and $\vee$ suggest an analogy with the set operations $\cap$ and $\cup$. This suspicion is fruitful: take an arbitrary nonempty set X. Its *power set* $\mathcal{P}(X)$ is the collection of all subsets of X. Take $0 = \varnothing$ and $1 = X$. Define meets ($\wedge$) to be set intersections ($\cap$), joins ($\vee$) to be set unions ($\cup$), and complementation ($\neg$) to be set complementation (^{c}). For sets $A, B \in \mathcal{P}(X)$, say $A \leq B$ if $A \subseteq B$. Then $\mathcal{P}(X)$ is a complete distributive lattice.
(2) Similarly, given a topological space X, the collection $\Omega(X)$ of all its open sets ordered by inclusion is a complete distributive lattice; the join of arbitrary sets is their union, and the meet is the interior of their intersection.

Definition 2.6. Suppose that A is a distributive lattice, and suppose that, for $a \in A$, there exists an element $\neg a \in A$ such that

(1) $a \wedge \neg a = 0$, and
(2) $a \vee \neg a = 1$.

Then we call $\neg a$ a *complement* of a. Any distributive lattice in which every element has a complement is called a *Boolean algebra*.

Proposition 2.7. *Complements are unique in a Boolean algebra.*

Proof. Suppose for some some element a we had two complements x and y. Then

$$\begin{aligned} x &= x \wedge (x \vee a) \\ &= x \wedge 1 \\ &= x \wedge (y \vee a) \\ &= (x \wedge y) \vee (x \wedge a) \text{ (since Boolean algebras are distributive)} \\ &= (x \wedge y) \vee 0 = x \wedge y. \end{aligned}$$

An almost identical argument proves that $y = x \vee y$, so $x = y$. □

Since complements are unique, we may consider $\neg$ as a function $\neg : A \to A$ that takes each element of a Boolean algebra A to its complement.

Proposition 2.8. *The following statements hold in Boolean algebras.*

(1) Meets and joins are *idempotent*: $a \wedge a = a$, and $a \vee a = a$.
(2) Meets and joins are *commutative*: $a \wedge b = b \wedge a$, and $a \vee b = b \vee a$.
(3) Meets and joins are *associative*: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, and $a \vee (b \vee c) = (a \vee b) \vee c$.
(4) Meets and joins are *distributive*: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, and $a \vee (b \wedge c) = (a \wedge b) \vee (a \wedge c)$.
(5) $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.
(6) $a \wedge 0 = 0$ and $a \vee 1 = 1$.
(7) $a \wedge 1 = a$ and $a \vee 0 = a$.
(8) Every element is its own double complement: $\neg\neg a = a$.
(9) $\neg 0 = 1$ and $\neg 1 = 0$.

Proof. For each of these statments, we need prove only half; the other half follows dually by exchanging $\vee$ with $\wedge$, 0 with 1, and $\leq$ with $\geq$. (1) and (2) follow immediately from the definitions of meet and join. For (3), note that $(a \vee b) \vee c$ and $a \vee (b \vee c)$ are both upper bounds for $\{a, b, c\}$, so they must be greater than or equal to the *least* upper bound, $\bigvee\{a, b, c\}$. On the other hand, they must be less than or equal to it, since, for example, $(a \vee b) \vee c \leq (a \vee b \vee c) \vee c = (a \vee b \vee c)$. (4) is true because a Boolean algebra is distributive by definition. (5) is the definition of a complement. (6) follows directly from the definitions of 0 and 1. (7) is immediate from the definitions of meet, join, 0, and 1. (8) is true by uniqueness of complements: $\neg a \wedge \neg\neg a = 0$ and $\neg a \wedge a = 0$, so $a = \neg\neg a$. (9) follows from (6) and uniqueness of complements. □

Examples 2.9. Here are a few examples of Boolean algebras.

(1) The set $\{0, 1\}$ (with $0 \leq 1$) is a Boolean algebra. We call this set **2**.
(2) Any complete lattice is a Boolean algebra.

We introduce a few tools for working with Boolean algebras. The first is the notion of an *ideal*.

Definition 2.10. Let A be a Boolean algebra. An *ideal* is a set $I \subset A$ such that

(1) $0 \in I$,
(2) if $a \in I$ and $b \in I$, then $a \vee b \in I$, and
(3) if $a \in I$ and $b \in A$, then $a \wedge b \in I$.

We say an ideal $I \subset A$ is *proper* if $I \neq A$, and *maximal* if it is proper and contains no proper ideals except for itself.

Example 2.11. Given a topological space X, $\Omega(X)$ (the lattice of open sets ordered by inclusion) is an ideal in $\mathcal{P}(X)$.

Theorem 2.12. *In a Boolean algebra A, an ideal $I \subset A$ is maximal iff for every $a \in A$, either $a \in I$ or $\neg a \in I$, but not both.*

Proof. First, let a_0 be an element of A and let I be an ideal of A containing neither a_0 nor $\neg a_0$. We will show that I is not maximal. To do so, consider the set J of all elements of the form $a \vee b$, where $a \leq a_0$ and $b \in I$. Then it is easy to check that J is an ideal that contains a_0. Furthermore, every element in I is contained in J, that is, $I \subset J$. If we can show $J \neq A$, we will have proven that I is not maximal. But J cannot be A because it cannot contain $\neg a_0$. If it did, we could find a and b ($a \leq a_0$, $b \in I$) such that $\neg a_0 = a \vee b$. But then $\neg a_0 = \neg a_0 \wedge \neg a_0 = (a \vee b) \wedge \neg a_0 = b$, contradicting the assumption that $\neg a_0 \notin I$. Hence, for all $a \in A$, any maximal ideal must contain either a or $\neg a$. Furthermore, no maximal ideal contains both: if an ideal K contains both a and $\neg a$, then it contains $a \vee \neg a = 1$, so for any element $b \in A$, K contains $b \wedge 1 = b$, so $K = A$. Therefore K is not a proper ideal, so it is not a maximal ideal. Thus, every maximal ideal contains precisely one of a or $\neg a$.

Conversely, suppose $I \subset A$ is an ideal such that, for all $a \in A$, either $a \in I$ or $(\neg a) \in I$. We will show that I is maximal by showing that the only ideal properly containing it is A. Let J be an ideal properly containing I. Then there is an element a such that $b \in J$ but $b \notin I$, so by the assumption $\neg b \in I$, whence $\neg b \in J$. But we've already shown that if both b and $\neg b$ are in an ideal, the ideal must be the entire algebra. □

Lemma 2.13 (Maximal Ideal Theorem)**.** *Every proper ideal in a Boolean algebra is contained in some maximal ideal.*

Proof. This proof follows Halmos and Givant [2], page 72.

Let B be a Boolean algebra with some proper ideal I. Further assume B is countable. (The proof for uncountable Boolean algebras is similar, but requires the axiom of choice.) We may then enumerate the elements in B: p_0, p_1, p_2, et cetera. We then inductively define a sequence of ideals. Define $J_0 = I$. Now, suppose J_n has been defined. If $\neg p_n \in J_n$, then define $J_{n+1} = J_n$. Otherwise, define

$$J_{n+1} = \{p \vee q : p \leq p_n \text{ and } q \in J_n\}.$$

We may verify that this set is in fact a proper ideal. It follows each J_n is proper and included in the proper ideal J_{n+1}. Therefore, the union M of all these ideals is proper (since no proper ideal contains 1, the union of proper ideals is again proper), and $I \subset M$.

It follows from Theorem 2.12 that M is maximal. For if $\neg p_n \in J_n$, then $\neg p_n \in M$. On the other hand, if $\neg p_n \notin J_n$, then $p_n \in J_{n+1} \subset M$. Hence, for each p_n, either p_n or $\neg p_n$ is in M. □

It follows in particular that, for every $a \in A$, there is some maximal ideal containing a. To see this, consider the set

$$\downarrow(a) = \{b \in A : b \leq a\}. \tag{2.14}$$

This set, called the *principal ideal generated by* a, is a proper ideal that contains a and, by the above lemma, is in turn contained in some maximal ideal.

Another useful notion is the idea of a *homomorphism*. In general, a homomorphism is a structure-preserving mapping. For Boolean algebras specifically, a homomorphism preserves meets, joins, and complementation. Formally:

Definition 2.15. Let A and B be Boolean algebras. A *(Boolean) homomorphism* is a mapping $f : A \to B$ such that, for all $p, q \in A$:

(1) $f(p \wedge q) = f(p) \wedge f(q)$,
(2) $f(p \vee q) = f(p) \vee f(q)$, and
(3) $f(\neg a) = \neg f(a)$,

where the operations on the left side of each equation are operations in A, and the operations on the right side of each equation are operations in B.

A surjective (onto) homomorphism is called an *epimorphism*. An injective (one-to-one) homomorphism is called a *monomorphism*. A homomorphism that is both surjective and injective is called an *isomorphism*. If there exists an isomorphism between A and B, we say that they are *isomorphic*.

If f is a homomorphism from A to B, the *kernel* of f is the subset of A that f maps to the zero element of B. In symbols, $\ker(f) = \{x \in A : f(x) = 0\}$. Kernels are key to an unexpected connection between ideals and homomorphisms:

Lemma 2.16 (Homomorphism Theorem)**.** *Every proper ideal is the kernel of some epimorphism between Boolean algebras.*

Proof. This proof follows Johnstone [3], Lemma I 2.1.

Let A be a Boolean algebra, let I be any proper ideal of A, and define a relation $\equiv_I$ by $a \equiv_I b$ iff there exist i and j in I such that $a \vee i = b \vee j$. Then $\equiv_I$ is an equivalence relation. Given $a \in A$, let $[a]$ be the set $\{b : b \equiv_I a\}$. Then the collection of all such sets is a Boolean algebra B, and every element $a \in A$ corresponds to precisely one element of B, namely $[a]$. Let $f : A \to B$ be the function that maps a to $[a]$; then f is an epimorphism, and its kernel is $[0]$, which is simply I. □

3. Stone Representation Theorem for Boolean Algebras

Our goal is to find a connection between the algebraic construct of Boolean algebras and the topological construct of Stone spaces. Let us therefore turn away from algebra and briefly discuss some topology. We recall some definitions.

- A topological space X is *Hausdorff* if, for any distinct x and y in X, there exist disjoint open sets $P, Q \subset X$ such that $x \in P$ and $y \in Q$. Succinctly, distinct points have disjoint neighborhoods.
- A topological space X is *compact* if every open cover of X has a finite subcover.
- A Hausdorff space X is *totally disconnected* if every open set is the union of the clopen sets it contains.

Definition 3.1. A *Stone space* is a topological space that is Hausdorff, compact, and totally disconnected.

We present a few examples of Stone spaces.

Proposition 3.2. *Recall that* $\mathbf{2}$ *is the set* $\{0, 1\}$*. Let* A *be an arbitrary nonempty set. Consider the set* $\mathbf{2}^A = \{f : f$ *is a function from* A *to* $\mathbf{2}\}$*. Then* $\mathbf{2}^A$ *is a Stone space.*

Proof. Endow $\mathbf{2}$ with the discrete topology. The set $\mathbf{2}^A$ is homomorphic to the Cartesian product of $\mathbf{2}$ with itself, with one copy for each element of A; give it the product topology. Tychonoff's Theorem guarantees that this set is compact and Hausdorff. To show that it is a Stone space, we need to show that it is totally disconnected. It is sufficient to show that every open set in a subbase for the topology on $\mathbf{2}^A$ is clopen.

Write x_a for the value that a function $x \in \mathbf{2}^A$ takes at a point $a \in A$. The product topology is the coarsest topology for which all projections $x_a : \mathbf{2}^A \to \mathbf{2}$ are continuous. To form a subbase for this topology, take as open all sets of the form $\{x \in \mathbf{2}^A : x_a = 1\}$ and $\{x \in \mathbf{2}^A : x_a = 0\}$. A set of the first form is the complement of some set of the second form, and vice versa, so every open set in this subbase is clopen. □

For our second example, let A be a Boolean algebra, and consider the subset $\mathcal{S}(A) \subset \mathbf{2}^A$ that consists of homomorphisms from A to $\mathbf{2}$. Is this a Stone space? We can prove that it is, but first it would be reassuring to know that the set $\mathcal{S}(A)$ is nonempty; could it ever be the case that there are no homomorphisms from A to $\mathbf{2}$? The following lemma answers our queries with a resounding "no": such homomorphisms always exist.

Lemma 3.3 (Existence Theorem)**.** *If p is a nonzero element of a Boolean algebra A, then there exists a homomorphism $f : A \to 2$ such that $f(p) = 1$.*

Proof. Let I be the principal ideal generated by $\neg p$, as defined in Equation 2.14. Then by the Maximal Ideal Theorem (Lemma 2.13), there is some maximal ideal M containing $\neg p$. By Theorem 2.12, $p \notin M$. By Lemma 2.16, there exists some homomorphism $f : A \to \mathbf{2}$ whose kernel is M. Since p is not in this kernel, $f(p) = 1$. □

We can now present the second example of a Stone space:

Proposition 3.4. *Let A be a Boolean algebra, and consider the set $\mathcal{S}(A) \subset 2^A$ of homomorphisms from A to 2. Then $\mathcal{S}(A)$ is a Stone space.*

Proof. Fix $a \in A$. For all $x \in \mathbf{2}^A$, $\{x_a\}$ is open in $\mathbf{2}$, since $\mathbf{2}$ has the discrete topology, and sets of the form $\{x \in \mathbf{2}^a : x_a = 1\}$ and $\{x \in \mathbf{2}^a : x_a = 0\}$ are open in $\mathbf{2}^A$ by definition. Hence the value of x_a depends continuously on x.

Now, if $f : X \to Y$ and $g : X \to Y$ are continuous functions into a Hausdorff space Y, then the set $\{x : f(x) = g(x)\}$ is closed. (To prove this, consider the map $(f, g) : X \to Y \times Y$ and the diagonal set $\Delta = \{(y, y) : y \in Y\}$. Then Δ^c is open iff Y is Hausdorff, so Δ is closed iff Y is Hausdorff.)

Therefore, for $p \in A$, the set $\{x : x(\neg p) = \neg x(p)\}$ is closed in $\mathbf{2}^A$. Hence the intersection of all complement-preserving functions in $\mathbf{2}^A$ form a closed subset of $2^{\mathbf{A}}$.

Similarly the sets of all functions in $\mathbf{2}^A$ that preserve meets and joins are closed subsets of $2^{\mathbf{A}}$.

Intersect these three sets to get the set of functions that preserve meets, joins, and complements, that is, the set H of homomorphisms $p : A \to \mathbf{2}$. The intersection of closed subsets is closed, so H is closed and hence compact. Because $\mathbf{2}^A$ is Hausdorff and totally disconnected, so is H, and we are done. □

We now present a few definitions.

Definition 3.5. Given a Boolean algebra A, we call $\mathcal{S}(A)$ the *Stone space associated with* A.

Definition 3.6. If X is a Stone space, then the *dual algebra* of X is the class of clopen sets in X.

Definition 3.7. A *field of sets* is a Boolean algebra of sets. More formally, take an arbitrary nonempty set X and consider its power set $\mathcal{P}(X)$. A field of sets is a subset $F \subset \mathcal{P}(X)$ that is closed under finite set unions, interesections, and complementation. We will usually abbreviate the term to "field," but beware: the reader should not expect to find any connection to the fields of field theory.

A field $F \subset \mathcal{P}(X)$ is *separating* if, given any distinct $x, y \in X$, there exist disjoint sets $S, T \in F$ such that $x \in S$ and $y \in T$.

Lemma 3.8. *If X is a Stone space and F is a separating field of clopen subsets of X, then F is the dual algebra of X; that is, it is the field of* all *clopen subsets of X.*

Proof. We first show that every open set in X can be written as a union of finitely many sets of F. Since F separates points, it also separates points and closed sets. To prove this, first suppose C is a closed set and $x \notin C$ is a point of X. Since X is Hausdorff, for each point $y \in C$ we can find a set in F that contains y but not x. These sets form a cover of C; use compactness to get a finite subcover. The union of sets in this subcover is in F since fields are closed under finite intersections. Furthermore, the union does not contain x. Hence, F separates points from closed sets.

Now, let D be a clopen subset of X. Then its complement D^{c} is closed, so from the above argument we can separate each point in D from D^{c} by sets in F. Take the union of these open sets to get a cover of D. By compactness, there is a finite subcover. Since the cover is disjoint from D^{c}, the union of sets in the subcover is D itself. Hence D is the union of finitely many sets of F. Since fields are closed under finite unions, $D \in F$. □

Finally, we can state and prove the Stone Representation Theorem for Boolean Algebras.

Theorem 3.9 (Stone Representation Theorem for Boolean Algebras)**.** *Every Boolean algebra is isomorphic to the dual algebra of its associated Stone space.*

Proof. Let A be a Boolean algebra, and let B be the dual algebra of its Stone space. We need to find an isomorphism between A and B. Our culprit shall be the function f, defined by $f(p) = \{x \in \mathcal{S}(A) : x(p) = 1\}$. To convict f of being an isomorphism, we must show that is a homomorphism, that it is one-to-one, and that it is onto. Straightfoward calculation proves that f is indeed a homomorphism. First, we verify complements.

$$\begin{aligned} f(\neg a) = \{x : x(\neg a) = 1\} \text{ (by definition of } f) \\ = \{x : \neg x(a) = 1\} \text{ (since } x \text{ is a homomorphism)} \\ = \{x : x(a) = 0\} \\ = \{x : x(a) \neq 1\} \\ = \{x : x(a) = 1\}^c \\ = [f(a)]^c. \end{aligned}$$

Next, joins.

$$\begin{aligned} f(a \vee b) = \{x : x(a \vee b) = 1\} \\ = \{x : x(a) \vee x(b) = 1\} \\ = \{x : x(a) = 1\} \cup \{x : x(b) = 1\} \\ = f(a) \cup f(b). \end{aligned}$$

Finally, to show that $f(a \wedge b) = f(a) \cap f(b)$, follow the same steps as for joins, but flip every $\vee$ to $\wedge$, $\cup$ to $\cap$, and 1 to 0.

Next we must show f is one-to-one. If $f(a) = \varnothing$, (that is, if there are no homomorphisms x such that $x(a) = 1$), then by Lemma 3.3, a must be zero. Hence, f is injective. Proof: let $a - b = a \wedge \neg b$. Then $f(a) - f(b) = f(a) \cap f(b)^c$. If $f(a) = f(b)$, then

$$\begin{aligned} \varnothing = f(a) - f(b) = f(a - b), \text{ so} \\ a - b = 0, \text{ so} \\ a = b \text{ by uniqueness of complements.} \end{aligned}$$

Finally, we show that f is onto. Clearly the range of any Boolean homomorphism, and in particular of f, is itself a Boolean algebra. Hence the clopen sets of the form $\{x : x(p) = 1\}$ constitute a field, say H. We can verify that H is a separating field: if g and h are distinct homomorphisms in $\mathcal{S}(A)$, then there must exist some element $q \in A$ such that $g(q) \neq h(q)$. Hence there is some set in H that contains g but not h, so the field is separating. By Lemma 3.8, H is the dual algebra of $\mathcal{S}(A)$, which we already know as B. Hence, f maps A onto B, and is therefore an isomorphism. □

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