BASIC ALGEBRAIC TOPOLOGY: THE FUNDAMENTAL GROUP OF A CIRCLE

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Abstract. The goal of this paper is to explore basic topics in Algebraic Topology. The paper will first begin by exploring the concept of loops on topological spaces and how one can look at these loops and form a group. From there, it moves to discuss covering spaces giving basic definitions and constructing the universal cover. After establishing this, covering spaces are used to calculate the fundamental group of a circle, one of the most foundational calculations in algebraic topology.

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1. The Fundamental Group

The fundamental group will be defined in terms of loops. However, it will be more useful to begin with slightly more generality.

Definition 1.1. Let X be a topological space with $a, b \in X$. A *path* in X from a to b is a continuous function $f\colon [0,1] \longrightarrow X$ such that $f(0) = a$ and $f(1) = b$. The points a and b are called the *endpoints*.

Definition 1.2. Given a path f in a topological space X, the *inverse path* of f is $\overline{f}(s) = f(1-s)$.

With these definitions, it seems that even on simple topological spaces, we have a massive number of paths and we desire a way in which we can relate one path to another. This is done by continuously deforming one path into another. However, we will require that the two paths share the same endpoints and that during this deformation, the endpoints remain fixed. This is an idea that is made more precise with the following definition.

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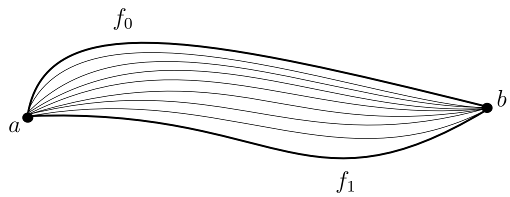
Definition 1.3. Let X be a topological space with two paths f_0 and f_1 that have endpoints $a, b \in X$. A *homotopy* from f_0 to f_1 is a family of paths $f_t \colon [0,1] \longrightarrow X$ such that for all $t \in [0,1]$, f_t satisfies the following:

(1) $f_t(0) = a$ and $f_t(1) = b$.
(2) The map $F \colon [0,1] \times [0,1] \longrightarrow X$ defined by $F(s,t) = f_t(s)$ is continuous.

When there exists a homotopy between f_0 and f_1, these two paths are said to be *homotopic* with the notation that $f_0 \simeq f_1$.

The *homotopy class* of f, denoted $[f]$, is the equivalence class of a path f under the equivalence relation of homotopy.

A visual representation of this is below:



Example 1.4. In $\mathbb{R}^n$, any two paths f_0 and f_1 that have the same endpoints are homotopic via the *linear homotopoy* defined by $f_t(s) = (1-t)f_0(s) + tf_1(s)$. This means that each $f_0(s)$ travels along the line segments to $f_1(s)$ at a constant speed.

Proposition 1.5. *Given a topological space X with two endpoints $a, b \in X$, path homotopy is an equivalence relation on the set of all paths from a to b.*

Proof. To show that $\simeq$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

Let X be a topological space and consider some $a, b \in X$. Let f, g, h be paths from a to b. It is obvious that $f \simeq f$ by the constant homotopy. It is also clear that if $f \simeq g$ via a homotopy f_t, then $g \simeq f$ via the homotopy f_{1-t}.

For transitivity assume that $f \simeq g$ via a homotopy f_t and $g \simeq h$ via a homotopy g_t. Then we can see that $f \simeq h$ via the homotopy h_t that is defined by f_{2t} on $[0, \frac{1}{2}]$ and g_{2t-1} on $[\frac{1}{2}, 1]$. It is clear that that the associated map $H(s,t)$ is continuous since, by assumption, it is continuous when restricted to the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and it agrees at $t = \frac{1}{2}$. □

Having found a way to relate two paths with the same end points in a topological space, this still leaves us with a lot of paths in a topological space because either endpoint could be anything. This brings rise to the notion that we only want to consider one point. Thus, we have the following definition.

Definition 1.6. A *loop* in a topological space X is a path f such that $f(0) = x_0 = f(1)$ for some $x_0 \in X$. The starting and ending point, x_0, is called the *basepoint*.

We now desire to define an operation on paths such that, given two paths, the product path essentially traverses each path twice as quickly so that the entire path is traversed in the unit interval. This notion is made more formal now:

Definition 1.7. Given two paths $f, g \colon [0,1] \longrightarrow X$ such that $f(1) = g(0)$, the *product path*, $f \cdot g$ is defined by the formula

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Remark 1.8. This operation respects homotopy classes. If $f_0 \simeq f_1$ via f_t and $g_0 \simeq g_1$ via g_t such that $f_0(1) = g_0(0)$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ via $f_t \cdot g_t$.

With this operation, we question if we can form an algebraic structure. With the following theorem, we see that we can indeed form a group out of the basic notions that we have just laid forth.

Theorem 1.9. *Given a topological space, X, the set of homotopy classes $[f]$ of loops $f\colon [0,1] \longrightarrow X$ at the basepoint x_0 forms a group under the product $[f] \cdot [g] = [f \cdot g]$.*

Proof. Since, the product defined respects homotopy classes by the above remark, $[f] \cdot [g] = [f \cdot g]$ is well-defined and closure has been satisfied.

Consider three loops $f, g, h\colon [0,1] \longrightarrow X$ with the basepoint $x_0 \in X$. We define a reparametrization of a path f to be a composition $f\phi$ where $\phi\colon [0,1] \longrightarrow [0,1]$ is any continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$. We notice that a reparametrization preserves homotopy classes because $f\phi \simeq f$ via the linear homotopy. From this, we can see that $f \cdot (g \cdot h)$ is a reparametrization of $(f \cdot g) \cdot h$ via the function

$$\phi(s) = \begin{cases} \frac{1}{2}, & 0 \le s \le \frac{1}{2} \\ s - \frac{1}{4}, & \frac{1}{2} \le s \le \frac{3}{4} \\ 2s - 1, & \frac{3}{4} \le s \le 1 \end{cases}$$

Therefore, we can conclude that $[f]\cdot([g]\cdot[h]) = [f\cdot(g\cdot h)] = [(f\cdot g)\cdot h] = ([f]\cdot[g])\cdot[h]$.

The identity element of our group is the homotopy class of the constant loop $e_{x_0}(s) = x_0$ for all $s \in [0,1]$. So if f is a loop with basepoint x_0, then we can see that $f \cdot e_{x_0}$ is a reparametrization of f by the map

$$\phi(s) = \begin{cases} 2s, & 0 \le s \le \frac{1}{2} \\ 1, & \frac{1}{2} \le 2 \le 1 \end{cases}$$

Similarly, $e_{x_0} \cdot f$ is a reparametrization of f and thus we see that the homotopy class of the constant map is the two-sided identity.

To see that $f \cdot \overline{f}$ is homotopic to e_{x_0} we will use a homotopy $h_t = f_t \cdot g_t$ where f_t is the path that equals f on the $[0, 1-t]$ and stationary on $[1-t, 1]$, while g_t is the inverse path of f_t. Since $f_0 = f$ and $f_1 = e_{x_0}$, we can see that $f \cdot \overline{f} \simeq e_{x_0}$. Similarly, we see that $\overline{f} \cdot f \simeq e_{x_0}$. Therefore, for any loop f with basepoint x_0, $[\overline{f}]$ is the two-sided inverse of $[f]$. □

This group is the *fundamental group* and we denote it $\pi_1(X, x_0)$. We noticed that throughout the entire proof, we chose a specific basepoint, which raises the question of whether this group depends on basepoint. As we will show now, if X is path-connected, then the fundamental group of X is independent of basepoint up to an isomorphism.

Definition 1.10. A topological space X is *path-connected* if for every $x, y \in X$, there exists a continuous path f such that $f(0) = x$ and $f(1) = y$.

Theorem 1.11. *Let X be a path-connected topological space and consider some $x_0, x_1 \in X$. Then there exists an isomorphism between $\pi_1(X, x_1)$ and $\pi_1(X, x_0)$.*

Proof. Since X is path-connected, let h be a path from x_0 to x_1 with its inverse path $\overline{h}$ from x_1 back to x_0. We then can associate each loop f based at x_1 to the loop $h \cdot f \cdot \overline{h}$ based at x_0.

Define the map $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ by $[f] \longmapsto [h \cdot f \cdot \overline{h}]$. Therefore, if f_t is a homotopy of loops based at x_1, then $h \cdot f_t \cdot \overline{h}$ is a homotopy of loops based at x_0. Thus, β_h is well-defined. We know that β_h is a homomorphism because $\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = \beta_h[f] \cdot \beta_h[g]$. We finally know that β_h is an isomorphism because with the inverse $\beta_{\overline{h}}$, we have $\beta_h \beta_{\overline{h}}[f] = \beta_h[\overline{h} \cdot f \cdot h] = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f]$. Similarly, $\beta_{\overline{h}} \beta_h[f] = [f]$. □

Because of the proceeding theorem, if a space is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition 1.12. A topological space X is *simply-connected* if it is path-connected and has a trivial fundamental group.

Definition 1.13. A topological space X is *semi-locally simply-connected* if for every $x \in X$, there exists a neighborhood $U \ni x$ such that any loop f with basepoint x is homotopic to the trivial loop.

Definition 1.14. Let X and Y be topological spaced and consider a continuous map $f \colon X \longrightarrow Y$ that takes a basepoint $x_0 \in X$ to a basepoint $y_0 \in Y$. We write $f \colon (X, x_0) \longrightarrow (Y, y_0)$. Define the *induced homomorphism* $f_* \colon \pi_1(X, x_o) \longrightarrow \pi_1(Y, y_0)$ by $[\alpha_{x_0}] \longmapsto [f \circ \alpha_{x_0}]$ for loops α_{x_0} based at x_0.

We notice that this homomorphism satisfies the following properties:

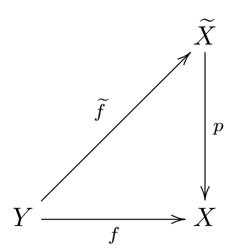
(1) $f_*[e_{x_0}] = [e_{y_0}]$
(2) $(f \circ g)_* = f_* \circ g_*$
(3) $(f^{-1})* = f_*^{-1}$

2. Covering Spaces

Definition 2.1. Given a topological space X, a *covering space* of X is a space $\widetilde{X}$ together with a map $p \colon \widetilde{X} \longrightarrow X$ such that there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$ each of which maps homeomorphically onto U_α by p.

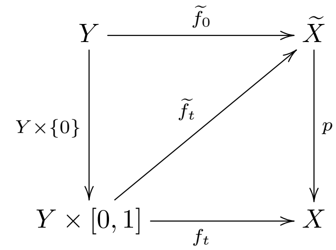
Definition 2.2. Let X and Y be two topological spaces with a covering space $p \colon \widetilde{X} \longrightarrow X$. Consider a map $f \colon Y \longrightarrow X$. Then, the map $\widetilde{f} \colon Y \longrightarrow \widetilde{X}$ is said to be a *lift* of f if $p \circ \widetilde{f} = f$.

A visual representation of this follows:



We now have our important lemma.

Lemma 2.3 (The Homotopy Lifting Property)**.** *Given a covering space $p \colon \widetilde{X} \longrightarrow X$, a homotopy $f_t \colon Y \longrightarrow X$, and a map $f_0 \colon Y \longrightarrow X$ that lifts to $\widetilde{f}_0$, there exists a unique homotopy $\widetilde{f}_t \colon Y \longrightarrow \widetilde{X}$ that lifts f_t.*



Proof. We first note that by properties of homotopies, this is equivalent to showing that given a homotopy $F\colon Y \times [0,1] \longrightarrow X$ and a lift $\widetilde{f}_0$ of the map $f_0 = F|_{Y\times\{0\}}$, there exists a unique lift $\widetilde{F}\colon Y \times [0,1] \longrightarrow \widetilde{X}$ of F such that $\widetilde{F}|_{Y\times\{0\}} = \widetilde{f}_0$.

Since p is a covering map, choose a open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$ each of which maps homeomorphically onto U_α by p.

Fix some $y \in Y$. For every $t \in [0,1]$ consider some neighborhood V_t of y and open interval I_t of t such that $F(V_t \times I_t) \subset U_\alpha$ for some α. We can cover $[0,1]$ with finitely many I_t and we can let V be the intersection of the corresponding V_t. We can now choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $F(V \times [t_i, t_{i+1}]) \subset U_{\alpha_i}$ for some α_i.

We now induct on i to form the lift $\widetilde{F}\colon V \times [0,t_i] \longrightarrow \widetilde{X}$. The base case, $i = 0$ is merely $\widetilde{f}_0$. Now assume that $\widetilde{F}\colon V \times [0,t_i] \longrightarrow \widetilde{X}$ has been constructed. Since $F(V \times [0,t_i]) \subset U_{\alpha_i}$, we know that there is a set $\widetilde{U}_{\alpha_i}$ of $\widetilde{X}$ containing $\widetilde{F}(y,t_i)$ such that $\widetilde{U}_{\alpha_i}$ is homeomorphic to U_{α_i} under p. By replacing $V \times \{t_i\}$ with $V \times \{t_i\} \cap (\widetilde{F}|_{V\times\{t_i\}})^{-1}(\widetilde{U}_{\alpha_i})$ we can assure that $\widetilde{F}(V \times \{t_i\}) \subset \widetilde{U}_{\alpha_i}$. We then can define $\widetilde{F}|_{V\times[t_i,t_{i+1}]}$ to be $p^{-1}F$ where p^{-1} is the homeomorphism $p^{-1}\colon U_{\alpha_i} \longrightarrow \widetilde{U}_{\alpha_i}$. Thus, we have defined the map $\widetilde{F}\colon V \times [0,1] \longrightarrow \widetilde{X}$ that lifts $F|_{V\times[0,1]}$ for some neighborhood V of y.

Now, let $\widetilde{F}, \widetilde{F'}$ be two lifts of $F\colon \{y_0\}\times[0,1] \longrightarrow X$ such that $\widetilde{F}(y_0,0) = \widetilde{F'}(y_0,0)$. As above, choose a finite partition of $[0,1]$ such that $F(\{y_0\} \times [t_i,t_{i+1}]) \subset U_{\alpha_i}$ for some α_i. We claim that $\widetilde{F} = \widetilde{F'}$ on $[0,t_i]$ for all i, and we proceed by induction. The base case is true by assumption. Now assume that this is true for i. Since $\widetilde{F}(\{y_0\} \times [0,t_i])$ and $\widetilde{F'}(\{y_0\} \times [0,t_i])$ are connected sets and $\widetilde{F}(y_0,t_i) = \widetilde{F'}(y_0,t_i)$, they both must lie in the same open set $\widetilde{U}_{\alpha_i}$ in $\widetilde{X}$ that is homeomorphic to U_{α_i} under p. Since $p|_{\widetilde{U}_{\alpha_i}}$ is injective, we have that $p\widetilde{F} = F = p\widetilde{F'}$ which implies that $\widetilde{F} = \widetilde{F'}$ on $\{y_0\} \times [t_i,t_{i+1}]$.

We now put this information together and prove the theorem. We notice that the $\widetilde{F}$'s constructed above on varying $V \times [0,1]$ are unique when restricted to $\{y\}\times[0,1]$ and must agree when two $V \times [0,1]$ overlap. Thus, we obtain a well-defined $\widetilde{F}$ for all of $Y \times [0,1]$. We know that this is continuous since it is continuous on each $V \times [0,1]$ and it is unique because it is unique on $\{y\} \times [0,1]$ for all $y \in Y$. □

Corollary 2.4 (The Path Lifting Property). *Let $p\colon \widetilde{X} \longrightarrow X$ be a covering space and consider a path f such that $f(0) = x_0$. Given a point $\widetilde{x}_0$ in the fiber of x_0, there exists a unique lift $\widetilde{f}$ of f such that $\widetilde{f}(0) = \widetilde{x}_0$. In particular, every lift of a constant path is constant.*

Proposition 2.5. *Given a covering space $p\colon \widetilde{X} \longrightarrow X$, the map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is injective.*

Proof. Suppose that $[\widetilde{f}] \in \ker p_*$. Then there exists a homotopy f_t of $p\widetilde{f}$ to the trivial loop e_{x_0}. Then, by the homotopy lifting property, there is a unique homotopy lift $\widetilde{f}_t$ of f_t between the loops $\widetilde{f}$ and $e_{\widetilde{x}_0}$. Thus, $[\widetilde{f}] = [e_{\widetilde{x}_0}]$ and $\ker p_*$ is trivial. □

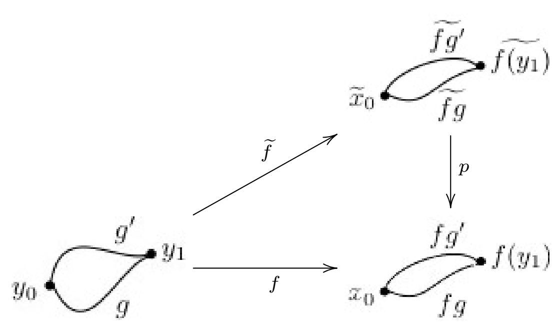
Definition 2.6. A topological space X is *locally path-connected* if for every point $x \in U \subset X$ such that U is open, there is an open set $V \subset U$ containing x that is path connected.

Proposition 2.7 (The Lifting Criterion)**.** *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ be a covering space and $f\colon (Y, y_0) \longrightarrow (X, x_0)$ be a map such that Y is path-connected*[1]*. Then a lift $\widetilde{f}\colon (Y, y_0) \longrightarrow (\widetilde{X}, \widetilde{x}_0)$ of f exits if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. The forward direction is obvious because $f_* = p_*\widetilde{f}_*$.

In the reverse direction, Consider some $y_1 \in Y$. Since Y is path connected, there exists a path g from y_0 to y_1 which implies that there is a path fg in X from x_0 to $f(y_1)$. Since p is a covering map, choose a point in the fiber of $f(y_1)$ and call it $\widetilde{f(y_1)}$. Then by the Path Lifting Property, there is a path $\widetilde{fg}$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{f(y_1)}$. We then define $\widetilde{f}(y_1) = \widetilde{fg}(1)$.

We now show this is well-defined. Consider another path g' from y_0 to y_1 such that $g \neq g'$. Let $h_0 = (fg')(\overline{fg})$, and thus $[h_0] \in f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. This means that there is a homotopy h_t from the loop h_0 to a loop h_1 that lifts to a loop $\widetilde{h}_1$. Then, by the Path Homotopy Lifting Property, h_t lifts to a homotopy $\widetilde{h}_t$. Since $\widetilde{h}_1$ is a loop at $\widetilde{x}_0$, $\widetilde{h}_0$ is a loop at $\widetilde{x}_0$, and by the uniqueness of lifts, we know that $\widetilde{h}_0$ is a loop that first traverses $\widetilde{fg}$ and then traverses $\widetilde{fg'}$ in the reverse direction. Therefore, $\widetilde{fg}(1) = \widetilde{fg'}(1)$ and $\widetilde{f}$ is well-defined. This process is shown in the following diagram:



As for the continuity of $\widetilde{f}$, if we take an open subset of X, then we know that there is a smaller open set that is mapped homeomorphically to open sets in X whose pre-image under f is open in Y since f is continuous. □

Remark 2.8. We also notice that if Y is connected, $\widetilde{f}$ is a unique lift that takes y_0 to $\widetilde{x}_0$.

[1]Hatcher [1] also includes the uncessary assumption that Y is locally path-connected

3. Construction of the Universal Cover

We now desire to construct a simply connected covering space.

Definition 3.1. The *Universal Cover* of a path-connected, locally path-connected, semi-locally simply-connected topological space X is a simply-connected covering space $\widetilde{X}$ that is a covering space of every other path-connected covering space of X.

We now consider a path-connected, locally path-connected, and semi-locally simply-connected topological space X. Choose some $x_0 \in X$. We now will define a set that we will show is a simply-connected covering space of X. Let

$$\widetilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

Define

$$p\colon \tilde{X} \longrightarrow X \quad \text{by} \quad [\gamma] \longmapsto \gamma(1)$$

Since X is path-connected, p is surjective.

Consider the collection, $\mathcal{U}$, of path-connected open sets of X such that for all $U \in \mathcal{U}$, the map $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial.

Given a $U \in \mathcal{U}$ and a path γ from x_0 to a point in U, let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ such that } \eta(0) = \gamma(1)\}$$

Notice that $p\colon U_{[\gamma]} \longrightarrow U$ is surjective since U is path-connected and injective because $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial.

We also know that $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. This is because if $[\gamma'] \in U_{[\gamma]}$, then $\gamma' = \gamma \cdot \eta$. Then elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \eta']$, and thus, these elements are in $U_{[\gamma]}$. Also, elements of $U_{[\gamma]}$ take the form $[\gamma \cdot \eta'] = [\gamma \cdot \eta \cdot \overline{\eta} \cdot \eta'] = [\gamma' \cdot \overline{\eta} \cdot \eta']$ and thus are in $U_{[\gamma']}$.

We now can say that

$$\{U_{[\gamma]} \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

forms a basis for the topology of $\widetilde{X}$. It is clear that the base elements cover $\widetilde{X}$. Now consider two elements $U_{[\gamma]}$ and $V_{[\gamma']}$, and an element $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$. Then by the above, we know that $U_{[\gamma'']} = U_{[\gamma]}$ and $V_{[\gamma'']} = V_{[\gamma']}$. Thus, if $W \subset U \cap V$ and $\gamma''(1) \in W$, we have $[\gamma''] \in W_{[\gamma'']} \subset U_{[\gamma]} \cap V_{[\gamma']}$.

We now desire to show that $p\colon U_{[\gamma]} \longrightarrow U$ is a homeomorphism. We have already stated that it is a bijection. Also, for any $[\gamma'] \in U_{[\gamma]}$ such that $\gamma'(1) \in V \subset U$, we have $V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}$. Therefore, $p(V_{[\gamma']}) = V$ and $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. This implies that p is continuous and thus is a homeomorphism.

We also know that p is a covering map since for any fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for differing $[\gamma]$ partition the pre-image of U. This is because if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma]} = U_{[\gamma']} = U_{[\gamma'']}$.

Having just shown that p is a covering map, we now show that $\widetilde{X}$ is simply-connected. A natural basepoint for $\widetilde{X}$ is $[e_{x_0}]$. So consider a loop f in $\widetilde{X}$ with the basepoint $[e_{x_0}]$. Thus, pf is a loop in X with basepoint x_0. Let γ_t be the path in X restricting γ to the interval $[0, t]$. Then $[\gamma_t]$ for varying t form 0 to 1 forms a path in $\widetilde{X}$ lifting γ and starting at $[x_0]$ and ending at $[\gamma]$. Since $[\gamma]$ was arbitrary, we can see that $\widetilde{X}$ is path-connected. Now to show that $\pi_1(\widetilde{X}) = 0$. We can show that $p_*(\pi_1(\widetilde{X}))$ is trivial since p_* is injective. Elements of the image of p_* are represented by loops γ in X based at x_0 that lift to loops in $\widetilde{X}$ based at $[x_0]$. By $t \longmapsto [\gamma_t]$, we can see that $[\gamma_1] = [x_0]$ and since $\gamma_1 = \gamma$, $[\gamma] = [x_0]$ and thus γ is nullhomotopic. Therefore p_* is trivial.

4. Classification of Covering Spaces

We next aim to classify all covering spaces of a fixed topological space X. The main thrust of our classification will arise through the Galois correspondence between path-connected covering spaces of X and subgroups of $\pi_1(X)$. This correspondence assigns each path-connected covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ to the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X)$. We first ask whether the trivial subgroup is realized. Since this function is always injective, this amounts to wondering if our topological space has a simply-connected covering space. As we have just shown, under the mild connectivity assumptions, we know that every topological space has a simply-connected covering space, and thus the trivial subgroup is realized.

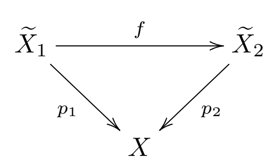
Proposition 4.1. *Let X be a path-connected, locally path-connected, and semi-locally simply-connected topological space. Then for every subgroup $H \subset \pi_1(X)$, there exists a covering space $p\colon X_H \longrightarrow X$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$ for a suitably chosen basepoint $\widetilde{x}_0 \in X_H$.*

Proof. Consider the simply-connected covering space, $\widetilde{X}$. For $[\gamma], [\gamma'] \in \widetilde{X}$, define the equivalence relation $[\gamma] \sim [\gamma']$ to mean $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \overline{\gamma'}] \in H$. It is easy to check that this is an equivalence relation since H is a subgroup. Let $X_H = \widetilde{X}/\sim$. So, if any two points in $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in the quotient group X_H, then the entire neighborhood is identified. This allows us to conclude that the natural map $X_H \longrightarrow X$ induced by $[\gamma] \longmapsto \gamma(1)$ is a covering space.

We now are concerned with our choice of basepoint. If we choose $\widetilde{x}_0$ to be the constant path e_{x_0} at x_0, then we see that the image of $\pi_1(X_H, \widetilde{x}_0)$ under the induced map of p is exactly H. This is because for any loop β in X based at x_0, this gets lifted to a loop in $\widetilde{X}$ that starts at $[e_{x_0}]$ and ends at $[\beta]$. Therefore, the lifted path's image is a loop if and only if $[\beta] \sim [e_{x_0}]$ which is equivalent to requiring that $[\beta] \in H$. □

Now that we have shown that every subgroup of $\pi_1(X)$ is realized by a covering space, we are interested the uniqueness of these covering spaces.

Definition 4.2. An isomorphism between two covering spaces $p_1\colon \widetilde{X}_1 \longrightarrow X$ and $p_2\colon \widetilde{X}_2 \longrightarrow X$ is a homeomorphism $f\colon \widetilde{X}_1 \longrightarrow \widetilde{X}_2$ such that the following diagram commutes:

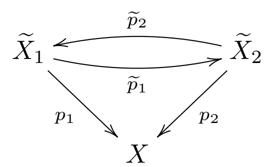


This means that f preserves the covering spaces structures taking $p_1^{-1}(x)$ to $p_2^{-1}(x)$ for each $x \in X$.

Proposition 4.3. *If X is a path-connected and locally path-connected topological space with two path-connected covering spaces $p_1\colon \widetilde{X}_1 \longrightarrow X$ and $p_2\colon \widetilde{X}_2 \longrightarrow X$,* *then there exists an isomorphism* $f\colon \widetilde{X}_1 \longrightarrow \widetilde{X}_2$ *taking a basepoint* $\widetilde{x}_1 \in p_1^{-1}(x)$ *to* $\widetilde{x}_2 \in p_2^{-1}(x)$ *if and only if* $p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.

Proof. Using the same diagram as above, the forward is shown because of the existence of the isomorphism.

In the reverse direction, we assume that $p_{1_*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2_*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. This means that we can lift p_1 to $\widetilde{p_1}\colon (\widetilde{X}_1, \widetilde{x}_1) \longrightarrow (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_2\widetilde{p}_1 = p_1$. This also means that we can lift p_2 to $\widetilde{p_2}\colon (\widetilde{X}_2, \widetilde{x}_2) \longrightarrow (\widetilde{X}_1, \widetilde{x}_1)$ such that $p_1\widetilde{p}_2 = p_2$. Since these lifts are unique by the unique lifting property, we know that $\widetilde{p}_1\widetilde{p}_2 = \mathbb{1} = \widetilde{p}_2\widetilde{p}_1$. Therefore, these lifts are inverse isomorphisms and we have the following commutative diagram:



□

This now leads us to the important classification theorem, the first part of which we have already proven.

Theorem 4.4. *Let* X *be a path-connected, locally path-connected, and semi-locally simply-connected topological space.Then there is a bijection between the set of basepoint preserving isomorphism classes of path-connected covering spaces* $p\colon (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ *and the set of subgroups of* $\pi_1(X)$*, obtained by associating the subgroup* $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subset \pi_1(X)$ *with the covering space* $p\colon (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$*. If basepoints are ignored, this gives a bijection between the set of basepoint preserving isomorphism classes of path-connected covering spaces and conjugacy classes of subgroups of* $\pi_1(X)$*.*

Proof. The first statement has been proven in the proceeding propositions. To show the second, it merely remains to show that for a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$, changing the basepoint $\widetilde{x}_0$ to another element of the fiber of x_0 corresponds exactly to changing $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to a conjugacy group of $\pi_1(X)$.

Suppose that $\widetilde{x}_1$ is a different element in the fiber of x_0, and let $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Since $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$, $\widetilde{\gamma}$ projects to a loop γ in X based at x_0. This loop γ represents some $g \in \pi_1(X)$. Let $H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$ for $i = 0, 1$. Thus, if $\widetilde{f}$ is a loop at $\widetilde{x}_0$, then $\widetilde{\overline{\gamma}}\widetilde{f}\widetilde{\gamma}$ is a loop at $\widetilde{x}_1$. This gives rise to the inclusion $g^{-1}H_0g \subset H_1$ and similarly $gH_1g^{-1} \subset H_0$. Conjugating the latter by g^{-1}, we can see that $H_1 \subset g^{-1}H_0g \subset H_1$ which allows us to conclude that $g^{-1}H_0g = H_1$. □

5. Computation of $\pi_1(S^1)$

We calculate the fundamental group of a circle S^1, and will be utilizing covering spaces.

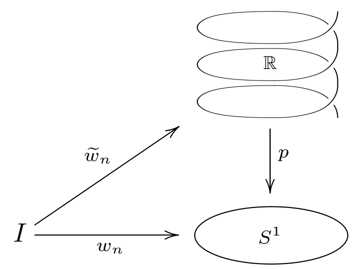
Theorem 5.1. $\pi_1(S^1) \approx \mathbb{Z}$.

Proof. We first identify the real numbers as a helix in $\mathbb{R}^3$ parameterized by $s \longmapsto (\cos 2\pi s, \sin 2\pi s, s)$. We also identify S^1 as a circle of unit radius inside of $\mathbb{R}^2$. Let

$p\colon \mathbb{R} \longrightarrow S^1$ be a map such that $p(s) = (\cos 2\pi s, \sin 2\pi s)$. This function can be thought of as a projection map from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by $(x, y, z) \longmapsto (x, y)$. It is obvious that p is a covering map.

This thus means that $\mathbb{R}$ is a covering space, and since it is simply-connected, it is the universal cover of S^1.

Consider the map, $\Phi\colon \mathbb{Z} \longrightarrow \pi_1(S^1)$ such that $\Phi(n) = [w_n]$ where $w_n\colon I \longrightarrow S^1$ is a loop in S^1 such that $w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$. By the Path Lifting Property, there is a path in $\mathbb{R}$, $\widetilde{w}_n$, that lifts w_n such that $\widetilde{w}_n(s) = ns$.



We can think of $\widetilde{w}_n$ as a path in $\mathbb{R}$ that begins at our starting point, 0, and wraps $|n|$ times around the helix.

With this helix model in mind, we can think of Φ in a different way. We now can think of Φ as sending n to the homotopy class of a loop $p\widetilde{f}$ for some path $\widetilde{f}$ in $\mathbb{R}$ such that $\widetilde{f}(0) = 0$ and $\widetilde{f}(1) = n$. We know that $\widetilde{f} \simeq \widetilde{w}_n$ via the linear homotopy. Thus, $p\widetilde{f} \simeq p\widetilde{w_n} = w_n$ which implies that $[p\widetilde{f}] = [w_n]$, which is what we desired.

We now wish to verify that Φ is a group homomorphism. Consider the transformation function $\tau_k\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $\tau_k(x) = x + k$. This means that the path $\widetilde{w}_m \cdot (\tau_k \widetilde{w}_n)$ is a path in $\mathbb{R}$ from 0 to $m + n$. Thus, Φ sends this path to the homotopy class of the image of $\widetilde{w}_m \cdot (\tau_k \widetilde{w}_n)$ under p which is just the homotopy class of $w_m \cdot w_n$. Therefore, we can see that $\Phi(n + m) = \Phi(n) \cdot \Phi(m)$ and thus Φ is a group homomorphism.

Now consider a loop $f\colon I \longrightarrow S^1$ with basepoint $(1, 0)$ that represents an element of $\pi_1(S^1)$. By the Path Lifting Property, there is a unique lift $\widetilde{f}$ of f that starts at 0 and ends at some n since $p\widetilde{f}(1) = f(1) = (1, 0)$ and $p^{-1}(1, 0) = \mathbb{N} \subset \mathbb{R}$. Therefore, by our reconceived thinking of Φ, we can see that $\Phi(n) = [p\widetilde{f}] = [f]$. Therefore, Φ is surjective.

Now suppose that $\Phi(n) = \Phi(m)$. This implies that $w_n \simeq w_m$. Let f_t be the homotopy such that $f_0 = w_n$ and $f_1 = w_m$. By the Path Homotopy Lifting Property, there exists a unique homotopy lift $\widetilde{f_t}$ of paths starting at 0. By uniqueness, we know that $\widetilde{f_0} = \widetilde{w}_n$ and $\widetilde{f_1} = \widetilde{f_1} = \widetilde{w}_m$. This implies that $\widetilde{w}_n \simeq \widetilde{w}_m$ which implies that $n = \widetilde{w}_n = \widetilde{w}_m = m$. Therefore, Φ is injective.

We now can see that Φ is an isomorphism what proves that $\pi_1(S^1) \approx \mathbb{Z}$. □

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References

[1] Hatcher, Allen. Algebraic Topology. Cambridge of Chicago Press. 2002.