BROWNIAN MOTION AND HAUSDORFF DIMENSION

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Abstract. In this paper, we develop Brownian motion and discuss its basic properties. We then turn our attention to the "size" of Brownian motion by defining Hausdorff dimension and its relationship to Brownian motion. This leads to the final result of the paper that for $n \geq 2$, both the range and graph of Brownian motion have Hausdorff dimension 2.

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Introduction

The physical process of Brownian motion was first observed in 1827 by botanist Robert Brown. While examining pollen grains in water under a microscope, Brown noticed that the individual grains seemed to jitter around randomly. Since then, the mathematical model of Brownian motion has become unquestionably the most important stochastic process in both pure and applied probability.

In this paper, we rigorously define and construct Brownian motion, and examine its properties as a continuous time stochastic process. We then turn our attention to the area of the set visited by Brownian motion. There are famous examples of space-filling curves parameterized in one dimension which map onto the unit square in $\mathbb{R}^2$, which leads to the question If Brownian motion is space-filling. We prove this is not the case, in that the Lebesgue measure of the range of two-dimensional Brownian motion is zero. To answer more general questions about the size of Brownian motion in higher dimensions, we develop the idea of Hausdorff dimension, a powerful tool used to study the size of irregular and fractal sets. The final result of the paper shows that in dimensions 2 and higher, both the range and graph of Brownian motion have Hausdorff dimension 2. In $\mathbb{R}^2$, this has the interpretation that Brownian motion is "almost" space filling.

1. Probability

We begin with a terse review of definitions from probability using measure theory.

Definition 1.1. A **σ-algebra** over a set Ω is a collection $\mathcal{A}$ of subsets of Ω satisfying

- $\Omega \in \mathcal{A}$
- $A \in \mathcal{A} \implies A^c \in \mathcal{A}$
- $A_1, A_2, ... \in \mathcal{A} \implies A_1 \cup A_2 \cup ... \in \mathcal{A}$

Definition 1.2. A **probability space** is a triple $(\Omega, \mathcal{A}, \mathbb{P})$ consisting of

- the sample space Ω, an arbitrary set
- the collection of all possible events, a σ-algebra $\mathcal{A}$ over Ω, &
- a probability measure $\mathbb{P}$, satisfying
  - for all $A \in \mathcal{A}$, $0 \leq \mathbb{P}(A) \leq 1$
  - $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$
  - for $A_1, A_2, ...$ disjoint sets in $\mathcal{A}$, we have

$$\mathbb{P}\left(\bigcup_{j=0}^{\infty} A_j\right) = \sum_{j=0}^{\infty} \mathbb{P}(A_j).$$

Definition 1.3. A real-valued **random variable** on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a function $X : \Omega \mapsto \mathbb{R}$ such that

$$\{\omega : X(\omega) \leq r\} \in \mathcal{A} \text{ for all } r \in \mathbb{R}$$

i.e. X is measurable with respect to $\mathcal{A}$.

Definition 1.4. If X and Y are random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we define the **expectation** of X , denoted $\mathbb{E}[X]$ by

$$\mathbb{E}[X] = \int_{\Omega} X \, d\mathbb{P}$$

Additionally we define the **covariance** of X and Y as $\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ and similarly the variance of X $\text{var}(X) = \text{cov}(X, X)$

We now define the normal distribution, one of the most important distributions in probability and statistics. As we will see, Brownian motion is intimately linked with the normal distribution.

Definition 1.5. A random variable X is said to be **normally distributed** with expectation μ and variance σ^2 if for all $x \in \mathbb{R}$

$$\mathbb{P}\{X \leq x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\left[-\frac{(u-\mu)^2}{2\sigma^2}\right] du$$

If $\mu = 0$ and $\sigma = 1$, then X is said to have a standard normal distribution.

Lemma 1.6. *Let X be a standard normally distributed random variable. Then for all $x > 0$*

$$\mathbb{P}\{|X| \geq x\} \leq \frac{1}{x}\sqrt{\frac{2}{\pi}} e^{-x^2/2}$$

Proof.

$$\mathbb{P}\{X > x\} = \frac{1}{2\pi} \int_{x}^{\infty} e^{-u^2/2} \, du \leq \frac{1}{2\pi} \int_{x}^{\infty} \frac{u}{x} e^{-u^2/2} \, du \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The desired inequality follows from the fact that the integrand is even. $\square$

Lemma 1.7. *Let X be a standard normally distributed random variable. Then for all $x > 0$*

$$\mathbb{P}\{X > x\} \geq \frac{x}{x^2+1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Proof. Define

$$f(x) = xe^{-x^2/2} - (x^2+1) \int_x^\infty e^{-u^2/2}\, du$$

Observe that $f(0) > 0$ and $f(x) \to 0$ as $x \to \infty$. Moreoever,

$$f'(x) = (1-x^2+x^2+1)e^{-x^2/2} - 2x \int_x^\infty e^{-u^2/2}\, du = -2x \left(\int_x^\infty e^{-u^2/2}\, du - \frac{e^{-x^2/2}}{x} \right)$$

which is positive for $x > 0$ by the preceeding lemma. Since f increases to zero, we have that $f(x) \leq 0$ proving the lemma. ☐

Definition 1.8. A **stochastic process** $\{F(t) : t \geq 0\}$ is a collection of (uncountably many) random variables $\omega \mapsto F(t, \omega)$ defined on a single probability space $(\Omega, \mathcal{A}, \mathbb{P})$. More intuitively, a stochastic process can be thought of as a *random function* with sample functions defined by $t \mapsto F(t, \omega)$.

Definition 1.9. For a stochastic process $\{F(t) : t \geq 0\}$ we define the **marginal distributions** as the laws of all finite dimensional random vectors

$$(F(t_1), F(t_2), ..., F(t_n)), \text{ for all } 0 \leq t_1 \leq t_2 \leq ... \leq t_n.$$

Note that it suffices to describe to describe the joint law of $F(0)$ and all the increments

$$(F(t_1) - F(t_0), ..., F(t_n) - F(t_n - 1)), \text{ for all } 0 \leq t_1 \leq t_2 \leq ... \leq t_n.$$

2. Brownian motion

In this section we rigorously define and construct Brownian motion, the stochastic process that will be the main focus of this paper. We then examine some of its invariance and continuity properties.

Definition 2.1. A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a (one-dimensional) **Brownian motion** with start $x \in \mathbb{R}$ if the following holds:

- $B(0) = x$
- the process has **independent increments**, i.e. for all times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$ the increments $B(t_n) - B(t_n - 1), B(t_n - 1) - B(t_n - 2), ..., B(t_2) - B(t_1)$ are independent random variables.
- for any $t \geq 0, h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation 0 and variance h.
- almost surely, the function $t \mapsto B(t)$ is continuous.

A Brownian motion is called standard if $x = 0$.

It is not at all obvious from this definition whether the conditions imposed lead to a contradiction. As we will see, this is not the case and Brownian motion is well-defined.

Theorem 2.2. *Standard Brownian motion exists.*

Proof. It suffices to show that $B[0,1]$ exists. Define

$$\mathcal{D}_n = \{ \tfrac{k}{2^n} : 0 \leq k \leq 2^n \} \text{ and } \mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$$

and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independent, standard normally distributed random variables can be defined. One can easily verify that $\mathcal{D}$ is dense in $[0,1]$. For each $n \in \mathbb{N}$ we define $B(d), d \in \mathcal{D}_n$ such that

- for $r < s < t$ in $\mathcal{D}_n$, $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$, and is independent of $B(s) - B(r)$.
- the vectors $(B(d) : d \in \mathcal{D}_n)$ and $(Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n)$ are independent.

For $\mathcal{D}_0 = \{0, 1\}$ this is trivial. Proceeding inductively we can define a stochastic process with the desired properties. Now, define

$$F_0(t) = \begin{cases} 0 & \text{if } t = 0 \\ Z_1 & \text{if } t = 0 \\ \text{ linear} & \text{in between} \end{cases}$$

and for each $n \geq 0$,

$$F_n(t) = \begin{cases} 0 & \text{if } t \in \mathcal{D}_{n-1} \\ 2^{-(n+1)/2} Z_t & \text{if } t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1} \\ \text{linear} & \text{in between} \end{cases}$$

These functions are continuous on $[0,1]$ and for all n and $d \in \mathcal{D}_n$,

$$B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d)$$

By the definition of Z_d and by 1.6, for $c > 0$ and large n,

$$\mathbb{P}\{|Z_d| \geq c\sqrt{n}\} \leq \exp\left(\frac{-c^2 n}{2}\right) \tag{2.3}$$

so that the series

$$\sum_{j=0}^{\infty} \mathbb{P}\{\text{there exists } d \in D_j \text{ with } |Z_d| \geq c\sqrt{n}\} \leq \sum_{j=0}^{\infty} (2^n + 1) \exp\left(\frac{-c^2 n}{2}\right),$$

converges whenever $c > \sqrt{2 \log 2}$. For any such c, by the Borel-Cantelli lemma there exists a random (but almost surely finite) N such that for all $n \geq N$ and $d \in D_n$ we have $|Z_d| < c\sqrt{n}$. Hence for all $n \geq N$,

$$||F_n||_{\infty} < c\sqrt{n} 2^{-n/2} \tag{2.4}$$

Where $||.||$ denotes the supremum norm. This upper bound implies that almost surely the series

$$B(t) = \sum_{j=0}^{\infty} F_j(t) \tag{2.5}$$

converges uniformly. Hence $B(t)$ is continuous on $[0,1]$ as a uniformly convergent sequence of continuous functions is continuous. It follows from the properties of B on a dense set $\mathcal{D}$ and continuity that B has the correct marginal distributions. $\square$

Definition 2.6. A stochastic process $\{B(t) : t \geq 0\}$ in $\mathbb{R}^n$ is called an **n-dimensional Brownian motion** started at $(x_1, ..., x_n)$ if

$$B(t) = (B_1(t), B_2(t), ..., B_n(t))$$

where $B_1, ..., B_n$ are independent linear Brownian motions started at $x_1, ..., x_n$ respectively. An n-dimensional Brownian motion started at the origin is called standard.

For the remainder of this section, we will deal solely with one-dimensional Brownian motion. We now examine some useful invariance properties of Brownian motion.

Theorem 2.7. *(Scaling invariance) Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion and let $a > 0$. Then the process $\{X(t) : t \geq 0\}$ defined by $X(t) = 1/aB(a^2t)$ is also a standard Broownian motion.*

Proof. The properties of continuity and independence of increments remain unchanged. Additionally, $X(t) - X(s) = 1/a(B(a^2t) - B(a^2s))$ is normally distributed with expectation zero and variance $1/a^2(a^2t - a^2s) = t - s$. □

Theorem 2.8. *(Time Inversion) Suppose $\{B(t) : t \geq 0\}$ is a standard Brownian motion. Then the process $\{X(t) : t \geq 0\}$ defined by*

$$X(t) = \begin{cases} 0 & \text{if } t = 0 \\ tB(1/t) & \text{if } t > 0 \end{cases}$$

is also a standard Brownian motion.

Proof. The finite dimensional marginals $(B(t_1), ..., B(t_n))$ of Brownian motion are Gaussian random vectors, and therefore characterized by $\mathbb{E}[B(t_i)] = 0$ and $\text{Cov}(B(t_i), B(t_j)) = t_i$ for $0 \leq t_i \leq t_j$. Clearly $\{X(t) : t \geq 0\}$ is also a Gaussian process and the Gaussian random vectors $(X(t_1), ..., X(t_n))$ have expectation 0. The covariances for $t > 0, h \geq 0$ are given by

$$\text{Cov}(X(t+h), X(t)) = t(t+h)\text{Cov}(B(1/(t+h)), B(1/t)) = t(t+h)\frac{1}{t+h} = t.$$

Hence the law for all finite dimensional marginals is the same as for standard Brownian motion. The paths $t \mapsto X(t)$ are clearly continuous for all $t > 0$. Since $X(t)$ is continous on $(0, \infty)$, taking the limit as $t \to 0$ we have that $X(t)$ is continous on $[0, \infty)$. Hence $\{X(t) : t \geq 0\}$ is a standard Brownian motion. □

Corollary 2.9. *(Law of large numbers for Brownian motion). Almost surely,*

$$\lim_{t \to \infty} \frac{B(t)}{t} = 0$$

Proof. Let $\{X(t) : t \geq 0\}$ be as defined above. Then we see that

$$\lim_{t \to \infty} B(t)/t = \lim_{t \to \infty} X(1/t) = X(0) = 0$$

□

We now turn our attention to the continuity properties of Brownian motion. We focus on Hölder continuity which will later provide us with an us upper bound on the Hausdorff dimension.

Definition 2.10. A function f:$[0,\infty) \to \mathbb{R}$ is said to be **locally α-Hölder continuous** at $x \geq 0$ if there exists a $\delta > 0$ and $c > 0$ such that

$$|f(y) - f(x)| \leq c|y - x|^\alpha, \text{ for all } y \geq 0 \text{ with } |y - x| \leq \delta.$$

If $\alpha = 1$, then f is said to be **locally-Lipschitz continuous** at x.

Note that α Hölder continuity gets stronger as α gets larger.

Proposition 2.11. *There exists a constant $C > 0$ such that, almost surely, for sufficiently small $h > 0$ and all $0 \leq t \leq 1 - h$,*

$$|B(t + h) - B(t)| \leq C\sqrt{h \log(1/h)}$$

Proof. We use several bounds from the proof of 2.2. Recall from (2.5) that

$$B(t) = \sum_{j=0}^{\infty} F_j(t)$$

where each F_j is piecewise linear. The derivative of each F_j exists alsmost everywhere and by (2.4), for any $c > \sqrt{2\log(2)}$ there exists a (random) $N \in \mathbb{N}$ such that for all $j > N$

$$||F_j'||_\infty \leq \frac{2|||F_j||_\infty}{2^{-j}} \leq 2c\sqrt{j}2^{j/2}$$

Now applying the mean value theorem, we have that for all $l > N$

$$|B(t + h) - B(t)| \leq \sum_{j=0}^{\infty} |F_j(t + h) - F_j(t)| \leq \sum_{j=0}^{l} h||F_j'||_\infty + \sum_{j=l+1}^{\infty} 2||F_j||_\infty.$$

Applying (2.4) again, we get that this is bounded by

$$h\sum_{n=0}^{N} ||F_n'|| + 2ch\sum_{n=N}^{l} \sqrt{n}2^{n/2} + 2c\sum_{n=l+1}^{\infty} \sqrt{n}2^{-n/2}.$$

Now suppose that h is (random and) small enough such that the first summand is smaller than $\sqrt{h\log(1/h)}$ and that l defined by $2^{-l} < h \leq 2^{-l+1}$ exceeds N. For this choice of l the second and third summands are bounded by constant multiples of $\sqrt{h\log(1/h)}$ as both sums are dominated by their largest element. Hence we have a deterministic bound on $|B(t + h) - B(t)|$ as desired. □

Corollary 2.12. *If $\alpha < 1/2$, then, almost surely, Brownian motion is everywhere locally α-Hölder continuous.*

Proof. Let C be defined as in 2.11. Applying the theorem to the Brownian motions $\{B(t) - B(k) : t \in [k, k + 1]\}$ where $k \in \mathbb{N}$, we see that almost surely for every k there exists an h_k such that for all $t \in [k, k+1]$ and h with $0 < h < (k+1-t) \wedge h_k$

$$|B(t + h) - B(t)| \leq C\sqrt{h\log(1/h)} \leq Ch^\alpha.$$

Translating gives the full result. □

Proposition 2.13. *For every constant $c < \sqrt{2}$, almost surely for every $\epsilon > 0$ there exist $0 < h < \epsilon$ and $t \in [0, 1 - h]$ with*

$$|B(t + h) - B(t)| \geq c\sqrt{h\log(1/h)}.$$

Proof. Let $c < \sqrt{2}$ and for integers $k, n \geq 0$, define the events

$$A_{k,n} = \left\{ B((k+1)e^{-n}) - B(ke^{-n}) > c\sqrt{n}e^{-n/2} \right\}.$$

Then by 1.7 for any $k \geq 0$ we have

$$\mathbb{P}(A_{k,n}) = \mathbb{P}\{B(e^{-n}) > c\sqrt{n}e^{-n/2}\} = \mathbb{P}\{B(1) > c\sqrt{n}\} \geq \frac{c\sqrt{n}}{c^2 n + 1} e^{-c^2 n/2}.$$

By our assumption on c, we have $e^n \mathbb{P}(A_{k,n}) \to \infty$ as $n \to \infty$. Therefore using the fact that $1 - x \leq e^{-x}$ for all x,

$$\mathbb{P}\left(\bigcap_{k=0}^{\lfloor e^n - 1 \rfloor} A_{k,n}^c \right) = (1 - \mathbb{P}(A_{0,n}))^{e^n} \leq \exp(e^n \mathbb{P}(A_{0,n}) \to 0.$$

Now letting $h = e^{-n}$ we have that for any $\epsilon > 0$, the probability that for all $h \in (0, \epsilon)$ and $t \in [0, 1-h]$, we have $|B(t+h) - B(t)| \leq c\sqrt{h \log(1/h)}$ is zero. □

Corollary 2.14. *If $\alpha > 1/2$, then almost surely, at every point Brownian motion fails to be locally α-Hölder continuous.*

Proof. Let $\epsilon < 1/e$. Then by 2.13 there exist $0 < h < \epsilon$ $c > 0$ and $t \in [0, 1-h]$ with

$$|B(t+h) - B(t)| \geq \sqrt{h \log(1/h)} \geq \sqrt{h}\sqrt{\log(1/h)}.$$

Letting $\epsilon \to 0$ we see that there exists an h such that the above inequality holds. Since $\sqrt{\log(1/h)}$ diverges, we have that Brownian motion fails to be α-Holder continuous at t. □

These results show that the bound from 2.12 of $\alpha < 1/2$ is the optimal for Hölder exponent of Brownian motion. As we will see later, this is closely related to the fact that in dimensions 2 and higher, Brownian motion has Hausdorff dimension 2.

3. The Markov property

In this section we examine an important property of Brownian motion, the strong Markov property, which intuitively states that at every well-defined (possibly random) time, Brownian motion starts anew. We will then use it to show several other properties such as the reflection principle.

Theorem 3.1. *(Markov property) Suppose that $\{B(t) : t \geq 0\}$ is an n-dimensional Brownian motion started at $x \in \mathbb{R}^n$ and let $s > 0$. Then the process $\{B(t+s) - B(s) : t \geq 0\}$ is again a Brownian motion started at the origin and is independent of the process $\{B(t) : 0 \leq t \leq s\}$.*

Proof. It is trivial to check that $\{B(t+s) - B(s) : t \geq 0\}$ satisfies the definition of an n-dimensional Brownian motion started at the origin. The independence statement follows directly from the independence of increments of a Brownain motion. □

Before proceeding any further, it is necessary to introduce some measure-theoretic terminology that relates directly to stochastic processes.

Definition 3.2. A **filtration** defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of σ-algebras $\{\mathcal{F}(t) : t \geq 0)\}$ such that $\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F}$ whenever $s < t$. A probability space together with a filtration is sometimes called a **filtered probability space**. A stochastic process $\{X(t) : t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called **adapted** if $X(t)$ is $\mathcal{F}(t)$-measurable for any $t \geq 0$.

We now define two filtrations of Brownian motion, $\mathcal{F}^0$ and $\mathcal{F}^+$ which will be defined in terms of $\mathcal{F}^0$. The latter will be of greater use to us in proofs, but as we shall see, they do not differ too much.

Definition 3.3. Suppose $\{B(t) : t \geq 0\}$ is a Brownian motion defined on some probability space, then we can define a filtration $(\mathcal{F}^0(t) : t \geq 0)$ by letting $\mathcal{F}^0(t)$ be the σ-algebra generated by the random variables $\{B(s) : 0 \leq t \leq s\}$.

Intuitively, this σ-algebra contains all the information available from observing the process up to time t, and is hence adapted to the filtration. By the markov property, the process $\{B(t+s) - B(s) : t \geq 0\}$ is independent of $\mathcal{F}^0(s)$. We now define a slightly larger σ-algebra.

Definition 3.4. Define the σ-algebra $\mathcal{F}^+(s)$ by

$$\mathcal{F}^+(s) = \bigcap_{t<s} \mathcal{F}^0(t).$$

Clearly, the family $\{\mathcal{F}^+(t) : t \geq 0\}$ is again a filtration and $\mathcal{F}^+(s) \supset \mathcal{F}^0(s)$. Intuitively, $\mathcal{F}^+(s)$ is a bit larger than $\mathcal{F}^0(s)$ by allowing an infinitesimal glance into the future.

Proposition 3.5. *For every $s \geq 0$ the process $\{B(t+s) - B(s) : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$.*

Proof. By continuity, $B(t+s) - B(s) = \lim_{n\to\infty} B(s_n + t) - B(s_n)$ for $\{s_n\}$ strictly increasing and $s_n \to s$. Now by the markov property of Brownian motion, for any $t_1, ..., t_m \geq 0$, the vector $(B(t_1+s) - B(s), ..., B(t_m+s) - B(s)) = \lim_{n\to\infty} B(t_1 + s_n) - B(s_n), ..., B(t_m) - B(s_n)$ is independent of $\mathcal{F}^+(s)$, and hence so is the process $\{B(t+s) - B(s) : t \geq 0\}$. □

Alternatively, we can say that conditional on $\mathcal{F}^+(s)$, the process $\{B(t+s) : t \geq 0\}$ is a Brownain motion started at $B(s)$. Given that $\mathcal{F}^+(s)$ shares the property of $\mathcal{F}^0(s)$ of being independent of all events to the right of s, one might ask by how much $\mathcal{F}^+$ can differ from $\mathcal{F}^0$. To do this we look at the **germ σ-algebra** $\mathcal{F}^+(0)$ which intuitively comprises all events defined on an infinitesimally small interval to the right of the origin. As we will see in the following result, the two σ-algebras do not differ significantly.

Theorem 3.6. *(Blumenthal's 0-1 law). Let $x \in \mathbb{R}^n$ and $A \in \mathcal{F}^+(0)$. Then $\mathbb{P}_x(A) \in \{0, 1\}$.*

Proof. By the previous theorem for $s = 0$ we see that any event A defined in terms of $\{B(t) : t \geq 0\}$ is independent of $\mathcal{F}^+(0)$. In particular this applies to $A \in \mathcal{F}^+(0)$, which therefore is independent of itself, which can only happen in A has probability zero or one. □

We now define the crucial property that distinguishes $\mathcal{F}^+(t) : t \geq 0$ from $\mathcal{F}^0(t) : t \geq 0$.

Definition 3.7. A filtration $\{\mathcal{F}(t) : t \geq 0\}$ defined on a probability space is said to be **right-continuous** if

$$\bigcap_{\epsilon \geq 0} \mathcal{F}(t + \epsilon) = \mathcal{F}(t).$$

It is easy to see that $\{\mathcal{F}^+(t) : t \geq 0\}$ is right-continuous. This property will make our lives much easier in the coming proofs. We now define a class of random times for which right-continuity will be important.

Definition 3.8. A random variable T with values in $[0, \infty]$, defined on a probability space with filtration $(\mathcal{F}(t) : t \geq 0)$ is called a **stopping time** if $\{T < t\} \in \mathcal{F}(t)$ for every $t \geq 0$. It is called a **strict stopping time** if $\{T \geq t\} \in \mathcal{F}(t)$ for every $t \geq 0$.

It is clear that every strict stopping time is also a stopping time. This follows from

$$T < t = \bigcup_{n=1}^{\infty} \{T \leq t - 1/n\} \in \mathcal{F}(t).$$

Proposition 3.9. *Every stopping time T with respect to the filtration $\{\mathcal{F}^+(t) : t \geq 0\}$ or indeed with respect to any right-continuous filtration, is automatically a strict stopping time.*

Proof. Suppose that T is a stopping time. Then

$$\{T \leq t\} = \bigcap_{k=1}^{\infty} \{T < t + 1/k\} \in \bigcap_{k=1}^{\infty} \mathcal{F}^+(t + 1/k) = \mathcal{F}^+(t).$$

which uses only the right-continuity of $\{\mathcal{F}^+(t) : t \geq 0\}$. □

We now give some examples of stopping times (and hence strict stopping times) with respect to $\mathcal{F}^+(t)$.

- Every deterministic time $t \geq 0$.
- Take $G \subset \mathbb{R}^n$ an open set and let $T = \inf\{t \geq 0 : B(t) \in G\}$.
- Take $H \subset \mathbb{R}^n$ a closed set and let $T = \inf\{t \geq 0 : B(t) \in H\}$.
- Any increasing sequence of stopping times

Definition 3.10. For every stopping time T, we define the σ-algebra

$$\mathcal{F}^+(T) = \{A \in \mathcal{A} : A \cap \{T < t\} \in \mathcal{F}^+(t) \text{ for all } t \geq 0\}.$$

Note that by the right-continuity of $\mathcal{F}^+(t)$ we may replace the event $\{T < t\}$ with $\{T \leq t\}$ without changing the definition. We are now ready to generalize the Markov property of Brownian motion to general stopping times.

Theorem 3.11. *(Strong Markov property). For every almost surely finite stopping time T, the process $\{B(T + t) - B(T) : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.*

Proof. We begin by showing the statement for stopping times T_n which discretely approximate T from above. Define T_n by

$$T_n = (m + 1)2^{-n} \text{ if } m2^{-n} \leq T < (m + 1)2^{-n}$$

In other words, we stop at the first integer multiple of 2^{-n} after T. It is easy to see that T_n is a stopping time. Let $B_k = \{B_k(t) : t \geq 0\}$ be the Brownian motion defined by $B_k(t) = B(t+k/2^n) - B(k/2^n)$, and $B_* = \{B_*(t) : t \geq 0\}$ be the process defined by $B_*(t) = B(t + T_n) - B(T_n)$. Now suppose that $E \in \mathcal{F}^+(T_n)$. Then for every event $\{B_* \in A\}$, we have

$$\mathbb{P}(\{B_* \in A\} \cap E) = \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\} \cap E \cap \{T_n = k2^{-n}\})$$

$$= \sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\} \mathbb{P}(E \cap \{T_n = k2^{-n}\})$$

using the fact that $\{B_k \in A\}$ is independent of $E \cap \{T_n = k2^{-n}\} \in \mathcal{F}^+(k2^{-n})$ by 3.5. Now by the Markov property, $\mathbb{P}\{B_k \in A\} = \mathbb{P}\{B \in A\}$ does not depend on k, hence we get

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\} \mathbb{P}(E \cap \{T_n = k2^{-n}\}) &= \mathbb{P}\{B \in A\} \sum_{k=0}^{\infty} \mathbb{P}(E \cap \{T_n = k2^{-n}\}) \\ &= \mathbb{P}\{B \in A\} \mathbb{P}(E) \end{aligned}$$

which shows that B_* is a Brownian motion independent of E, and hence independent of $\mathcal{F}^+(T_n)$ as claimed. It remains to generalize this to arbitrary stopping times T. As $T_n \to T$ we have that $\{B(s + T_n) - B(T_n) : s \geq 0\}$ is a Brownian motion independent of $\mathcal{F}^+(T_n) \supset \mathcal{F}^+(T)$. Hence the increments

$$B(s + t + T) - B(t + T) = \lim_{n \to \infty} B(s + t + T_n)$$

of the process $\{B(r+T) - B(T)\}$ are normally distributed with mean zero and variance s. The process is a Brownian motion since almost sure continuity is apparent. Moreover, all increments $B(s+t+T) - B(t+T) = \lim_{n\to\infty} B(s+t+T_n) - B(t+T_n)$ and hence the process itself are independent of $\mathcal{F}^+(T)$. □

We now state and prove the reflection principle, which states that Brownian motion reflected at some stopping time T is still a Brownian motion. The proof is a direct consequence of the strong Markov property.

Theorem 3.12. *(Reflection principle) It T is a stopping time and $\{B(t) : t \geq 0\}$ is a standard Brownian motion, then the process $\{B^*(t) : t \geq 0\}$ called* ***Brownian motion reflected at*** *T and defined by*

$$B^*(t) = \begin{cases} B(t) & \text{if } t \leq T \\ B(T) - (B(t) - B(T)) & \text{if } t > T \end{cases}$$

is also a standard Brownian motion.

Proof. If T is finite, by the strong Markov property, both

$$\{B(t + T) - B(T) : t \geq 0\} \text{ and } \{-B(t + T) - B(T) : t \geq 0\}$$

are Brownian motions independent of the beginning $\{B(t) : 0 \leq t \leq T\}$. Hence the two concatenations $\{B(t) : t \geq 0\}$ and $\{B^*(t) : t \geq 0\}$ have the same distribution. □

We now use the reflection principle to prove a useful bound on the maximum of Brownian motion.

Proposition 3.13. *Let*

$$M(t) = \max_{0 \leq s \leq t} B(s).$$

If $a > 0$ then we have $\mathbb{P}\{M(t) > a\} = 2\mathbb{P}\{B(t) > a\} = \mathbb{P}\{|B(t)| > a\}$.

Proof. Let $T = \inf\{t \geq 0 : B(t) = a\}$ and let $\{B^*(t) : t \geq 0\}$ be the Brownian motion reflected at T. Then $\{M(t) > a\}$ is the disjoint union of the events $\{B(t) > a\}$ and $\{M(t) > a, B(t) \leq a\}$ and since the latter is exactly the event $\{B^*(t) \geq a\}$ the statement follows from the reflection principle. □

This result is most useful when combined with 1.6 yielding,

$$\mathbb{P}\{M(t) > a\} \leq \frac{\sqrt{2t}}{a\sqrt{\pi}} \exp\{-\frac{a^2}{2t}\}. \tag{3.14}$$

4. The area of planar Brownian motion

In this section, we examine the area of planar Brownian motion. As the path of Brownian motion is highly irregular, one might ask if Brownian motion acts like a space-filling curve in $\mathbb{R}^2$. As we will see, this is not the case.

Definition 4.1. We define the **convolution** of two functions f and g denoted $f * g$ by

$$f * g(x) = \int f(y)g(x-y)\,dy$$

Lemma 4.2. *If $A_1, A_2 \subset \mathbb{R}^2$ are Borel sets with positive area, then*

$$\mathrm{m}\left(\{\mathrm{x} \in \mathbb{R}^2 : \mathrm{m}(\mathrm{A}_1 \cap (\mathrm{A}_2 + \mathrm{x})) > 0\}\right) > 0,$$

where m *denotes the Lebesgue measure.*

Proof. We may assume without loss of generality that A_1 and A_2 are bounded. By Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbb{1}_{A_1} * \mathbb{1}_{-A_2}(x)\,dx &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{A_1}(w)\mathbb{1}_{A_2}(w-x)\,dw\,dx \\ &= \int_{\mathbb{R}^2} \mathbb{1}_{A_1}(w)\left(\int_{\mathbb{R}^2} \mathbb{1}_{A_2}(w-x)\,dx\right)dw = \mathrm{m}(\mathrm{A}_1)\mathrm{m}(\mathrm{A}_2) > 0.\end{aligned}$$

Thus $\mathbb{1}_{A_1} * \mathbb{1}_{A_2} > 0$ on a set of positive area. But

$$\mathbb{1}_{A_1} * \mathbb{1}_{A_2} = \int \mathbb{1}_{A_1}(y)\mathbb{1}_{-A_2}(x-y)\,dy = \int \mathbb{1}_{A_1}(y)\mathbb{1}_{A_{2+x}}(y)\,dy = \mathrm{m}(\mathrm{A}_1 \cap (\mathrm{A}_2 + \mathrm{x})),$$

proving the lemma. □

We are now ready to prove that the area of planar Brownian motion is zero.

Theorem 4.3. *Let $\{B(t) : t \geq 0\}$ be 2-dimensional Brownian motion. Then almost surely, $m(B[0,1]) = 0$.*

Proof. Let $X = \mathrm{m}(B[0,1])$. We first check that $\mathbb{E}[X] < \infty$. Note that a necessary condition for $X > a$ is that Brownian motion leave the square centered in the origin of sidelength $\sqrt{a}/2$. Hence by (3.14),

$$\mathbb{P}\{X > a\} \leq 2\mathbb{P}\{\max_{0 \leq t \leq 1} |W(t)| > \sqrt{a}/2\} \leq 4e^{-a/8},$$

for $a > 1$ and were $\{W(t) : t \geq 0\}$ is a standard one-dimensional Brownian motion. Hence,

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}\{X > a\}\,da \leq 4\int_1^\infty e^{-a/8}\,da + 1 < \infty.$$

Recall that $B(3t)$ and $\sqrt{3}B(t)$ are identically distributed, and hence

$$\mathbb{E}[\mathrm{m}(B[0,3])] = 3\mathbb{E}[\mathrm{m}(B[0,1])] = 3\mathbb{E}[X].$$

Now, note that $\mathrm{m}(B[0,3]) \leq \sum_{j=0}^{2} \mathrm{m}(B[j,j+1])$ with equality if and only if for $i,j \in \{0,1,2\}$ and $i \neq j$, we have $\mathrm{m}(B[i,i+1] \cap B[j,j+1]) = 0$. On the other hand, we have that $\mathbb{E}[\mathrm{m}(B[j,j+1])] = \mathbb{E}[X]$ and

$$3\mathbb{E}[X] = E[\mathrm{m}(B[0,3])] \leq \sum_{j=0}^{2} \mathbb{E}[\mathrm{m}(B[j,j+1])] = 3\mathbb{E}[X],$$

hence almost surely, the intersection of any two of the $B[j,j+1]$ has measure zero. In particular, $\mathrm{m}(B[0,1] \cap B[2,3]) = 0$ almost surely. Now, define $\{B_1(t) : 0 \leq t \leq 1\}$ by $B_1(t) = B(t)$ and $\{B_2(t) : 0 \leq t \leq 1\}$ by $B_2(t) = B(t+2) - B(2) + B(1)$. By the Markov property both Brownian motions are independent of of the random variable $Y = B(2) - B(1)$. For $x \in \mathbb{R}^2$, let $R(x)$ denote the area of the set $B_1[0,1] \cap (x + B_2[2,3])$ and note that $R(x)$ is independent of Y Then

$$0 = \mathbb{E}[\mathrm{m}(B[0,1] \cap B[2,3])] = \mathbb{E}[R(Y)] = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-|x|^2/2} \mathbb{E}[R(x)]\, dx,$$

where we are averaging with respect to the Gaussian distribution of $B(2) - B(1)$. Thus almost surely, $R(x) = 0$ almost everywhere and hence

$$\mathrm{m}\{x \in \mathbb{R}^2 : R(x) > 0\} = 0, \text{ almost surely.}$$

From 4.2 we have that almost surely $\mathrm{m}(B[0,1]) = 0$ or $\mathrm{m}(B[2,3]) = 0$. The observation that $\mathrm{m}(B[0,1])$ and $\mathrm{m}(B[2,3])$ are independent and identically distributed completes the proof. □

This rough result about the range of Brownian motion is about as well as we can do right now. It shows that the range of Brownian motion is not "large" in the sense of two-dimensional area. For a more precise description of the size of Brownian, we will develop the concept of Hausdorff dimension.

5. Hausdorff dimension

In this section we develop the ideas of Hausdorff measure and dimension to study more precisely the size of Brownian motion. The intuition is as follows. For each $\alpha > 0$ we compute the "α-dimensional measure" which will be the α-Hausdorff measure. If we pick α too small, the measure will be infinite. If we pick α too large, the measure will be zero. There will only be one value of α for which the measure will make sense, and this will be the Hausdorff dimension.

Definition 5.1. Let A be a metric space. For each $\epsilon > 0, \alpha \geq 0$ we define

$$\mathcal{H}_\epsilon^\alpha(A) = \inf \sum_{j=1}^{\infty} \mathrm{diam}(A_j)^\alpha,$$

Where the inf is over all countable collections of sets $\{A_j\}$ that cover A with $\mathrm{diam}(A_j) \leq \epsilon$. Here diam denotes the diameter of a set. Note that if $\delta < \epsilon$ then

$$\mathcal{H}_\delta^\alpha(A) \geq \mathcal{H}_\epsilon^\alpha(A)$$

since any covering of A by sets of diameter at most δ is also a covering of A by sets of at most ϵ. We now define the **α-Hausdorff measure** of A as

$$\mathcal{H}^\alpha(A) = \lim_{\epsilon \to 0} \mathcal{H}_\epsilon^\alpha(A).$$

The existence of the limit follows from the fact that monotone increasing limits exist (though possibly are ∞).

While the standard properties of measures such as countable additivity are clear, it is still worth justifying the term measure. What we have defined here is technically an outer measure on an arbitrary metric space. However by restricting ourselves to the Borel sets, as is frequently done in Lebesgue theory, α-Hausdorff measure indeed defines a proper measure. Additionally, it can be shown that in $\mathbb{R}^n$ the n-Hausdorff measure and the n-dimensional Lebesgue measure are equivalent up to a constant multiple.

Proposition 5.2. *Let A be a metric space and $0 \leq \alpha < \beta < \infty$, then*

- *If $\mathcal{H}^\alpha(A) < \infty$, then $\mathcal{H}^\beta(A) = 0$.*
- *If $\mathcal{H}^\beta(A) > 0$, then $\mathcal{H}^\alpha(A) = \infty$.*

Proof. For any $\epsilon > 0$ and any cover $A_1, A_2, ...$ of A with sets of of $\text{diam}(A_j) \leq \epsilon$, we have

$$\sum_j \text{diam}(A_j)^\beta \leq \epsilon^{\beta-\alpha} \sum_j \text{diam}(A_j)^\alpha.$$

Taking infimums we get

$$\mathcal{H}_\epsilon^\beta(A) \leq \epsilon^{\beta-\alpha} \mathcal{H}_\epsilon^\alpha(A).$$

Since $\epsilon^{\beta-\alpha} \to 0$ as $\epsilon \to 0$, we get the desired result. □

This result shows us that there is at most one "interesting" value of α, which leads to the following definition.

Definition 5.3. For any metric space A, we define the **Hausdorff dimension** of A, denoted $\dim(A)$ by

$$\dim(A) = \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha(A) = \infty\}$$

While this definition may seem somewhat obscure, one can easily check that it corresponds nicely with the more classical vector space definition of dimension. For instance, it can be shown that $\mathbb{R}^n$ has Hausdorff dimension n. We now turn our attention to the relationship between Hausdorff dimension and Hölder continuity.

Proposition 5.4. *Let $f : [0,1] \mapsto \mathbb{R}^n$ be an α-Hölder continuous function. Then*

$$\dim(\mathrm{Graph_f}) \leq 1 + (1-\alpha)\min\{\mathrm{n}, 1/\alpha\}.$$

Proof. Since f is α-Hölder continuous, there exists a constant C such that, if $s, t \in [0,1]$ with $|t-s| \leq \epsilon$, then $|f(t) - f(s)| \leq C\epsilon^\alpha$. Now, cover $[0,1]$ by no more than $\lfloor 1/\epsilon \rfloor$ intervals of length ϵ. The image of each such interval is contained in a ball of diameter $C\epsilon^\alpha$. One can now

- *either* cover each ball by no more than a constant multiple of $\epsilon^{n\alpha-n}$ balls of diameter ϵ,
- *or* use the fact that subintervals of length $(\epsilon/C)^{1/\alpha}$ in the domain are mapped into balls of diameter ϵ to cover the image inside the ball by a constant multiple of $\epsilon^{1-1/\alpha}$ balls of radius ϵ.

In both cases, look at the cover of the graph consisting of the product of intervals and corresponding balls in $[0,1] \times \mathbb{R}^n$ of diameter ϵ. The first construction needs a constant multiple of $\epsilon^{n\alpha-n-1}$ product sets, the second uses $\epsilon^{-1/\alpha}$ product sets, all of which have diameter of order ϵ. These coverings give the desired upper bounds. □

Proposition 5.5. *Let $f : [0,1] \mapsto \mathbb{R}^n$ be an α-Hölder continuous function. Then for any $A \subset [0,1]$, we have*

$$\dim f(A) \leq \frac{\dim(A)}{\alpha}$$

Proof. Suppose that $\dim(A) < \beta < \infty$. Then there exists a covering $A_1, A_2, ...$ such that $A \subset \cup_j A_j$ and $\sum_j |A_j|^\beta < \epsilon$. Then $f(A_1), f(A_2), ...$ is a covering of $f(A)$, and $|f(A_j)| \leq C|A_j|^\alpha$ where C is the Holder constant. Thus,

$$\sum_j |f(A_j)|^{\beta/\alpha} \leq C^{\beta/\alpha} \sum_j |A_j|^\beta < C^{\beta/\alpha}\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Hence dim $f(A) \leq \beta/\alpha$. The desired inequality follows. □

Note that by compactness, we may easily extend this result to locally α-Hölder continuous functions. These two results lead to the following bounds on the Hausdorff dimension of Brownian motion.

Theorem 5.6. *The graph of an n-dimensional Brownian motion satisfies, almost surely,*

$$\dim(\text{Graph}) \leq \begin{cases} 3/2 & \text{if } n = 1 \\ 2 & \text{if } n \geq 2 \end{cases}$$

and for any fixed set $A \subset [0, \infty)$, almost surely

$$\dim B(A) \leq \min\{2\dim(A), n\}.$$

Proof. Follows from 5.4, 5.5, 2.12, & the countable additivity of Hausdorff measures. □

We now develop two methods for finding lower bounds on the Hausdorff dimension of a set, the mass distrubution principle and the energy method. The basic idea of the mass distrubution principle is that if the measure distributes a positive mass on a set E in such a way that the local concentration is bounded from above, the set must be large so as to accomodate the entire mass.

Definition 5.7. Suppose μ is a measure on a metric space E. If $0 < \mu(E) < \infty$ we call μ a **mass distribution** on E.

Theorem 5.8. *(Mass distribution principle). Suppose E is a metric space and $\alpha \geq 0$. If there is a mass distribution μ on E and constants $C > 0$ and $\delta > 0$ such that*

$$\mu(V) \leq C|V|^\alpha$$

for all closed sets V with diameter $|V| \leq \delta$, then

$$\mathcal{H}^\alpha(E) \geq \frac{\mu(E)}{C} > 0,$$

and hence $\dim E \geq \alpha$.

Proof. Suppose $U_1, U_2, ...$ is a cover of E with $|U_i| \leq \delta$. Let V_i be the closure of U_i and note that $|U_i| = |V_i|$. We have

$$0 < \mu(E) \leq \mu\left(\bigcup_i U_i\right) \leq \mu\left(\bigcup_i V_i\right) \leq \sum_i \mu(V_i) \leq C \sum_i |U_i|^\alpha.$$

Passing to the infimum over all such covers and letting $\delta \to 0$ gives the desired inequality. □

We now develop the energy method, which like the mass distribution principle will be used to provide lower bounds on the Hausdorff dimension of Brownian motion. The idea is similar to the mass distribution principle, but replaces the condition on the mass of closed sets with the formal concept of the finiteness of energy. As we will see, these lower bounds will be strong enough so that combining them with 5.6 will give us the final result.

Definition 5.9. Suppose μ is a mass distribution on a metric space E, and $\alpha \geq 0$. The α**-potential** of a point $x \in E$ with respect to μ is defined as

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{\rho(x,y)^\alpha}$$

and the α**-energy** of μ as

$$I_\alpha(\mu) = \int \phi_\alpha(x)\, d\mu(x) = \int\int \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha}$$

where ρ denotes the distance function on E. It is possible to give various physical interpretations to these quantities. For instance when $E = \mathbb{R}^3$ and $\alpha = 1$ we have Newtonian gravitational potential and energy of a mass μ. However since we are doing math, we can and will take α to be whatever we want.

Theorem 5.10. *(Energy method). Let $\alpha \geq 0$ and μ be a mass distribution on a metric space E. Then for every $\epsilon > 0$, we have*

$$\mathcal{H}^\alpha_\epsilon \geq \frac{\mu(E)}{\int\int_{\rho(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha}}.$$

Hence if $I_\alpha(\mu) < \infty$ then $\mathcal{H}^\alpha(E) = \infty$ and in particular, $\dim(E) \geq \alpha$.

Proof. Suppose $A_1, A_2, ...$ is a pairwise disjoint covering of E by sets of diameter $< \epsilon$. Then,

$$\int\int_{\rho(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \geq \sum_j \int\int_{A_j \times A_j} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \geq \sum_j \frac{\mu(A_j)^2}{|A_j|^\alpha}.$$

We can bound $\mu(E)$ as follows

$$\mu(E) \leq \sum_j \mu(A_j) = \sum_j |A_j|^{\alpha/2} \frac{\mu(A_j)}{|A_j|^{\alpha/2}}$$

Now, applying the Cauchy-Schwarz inequality, we have

$$\mu(E) \leq \sum_j |A_j|^\alpha \sum_j \frac{\mu(A_j)^2}{|A_j|^\alpha} \leq \mathcal{H}^\alpha_\epsilon(E) \int\int_{\rho(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha}.$$

Dividing both sides by the integral gives the desired inequality. If $\mathbb{E}[I_\alpha] < \infty$, the integral converges to zero, so that $\mathcal{H}^\alpha_\epsilon$ diverges to infinity. □

We are now ready to prove the final result of the paper, which will show that for $n \geq 2$, both the range and graph of Brownian motion have Hausdorff dimension 2. Intuitively, this result is stating that in dimensions two and higher, the "size" of Brownian motion is always two-dimensional. The proof will use the earlier lower bounds from 5.6 combined with an argument using the energy method.

Theorem 5.11. *Let $\{B(t) : t \geq 0\}$ be an n-dimensional Brownian motion. If $n \geq 2$, then* $\dim \mathrm{Range} = \dim \mathrm{Graph} = 2$ *almost surely*

Proof. By 5.6 it suffices to provide the necessary upper bounds. Let μ_B be the measure defined by $\mu_B(A) = \mathrm{m}(B^{-1}(A)) \cap [0,1]$, or equivalently,

$$\int_{\mathbb{R}^n} f(x) d\mu_B(x) = \int_0^1 f(B(t))\, dt$$

For all bounded measurable functions f. We want to show that for any $0 < \alpha < 2$,

$$\mathbb{E}[I_\alpha(\mu_B)] = \mathbb{E} \int \int \frac{d\mu_B(x) d\mu_B(y)}{|x-y|^\alpha} = \mathbb{E} \int_0^1 \int_0^1 \frac{ds\, dt}{|B(t) - B(s)|^\alpha} < \infty.$$

To do this we begin by evaluating the expectation

$$\mathbb{E}|B(t) - B(s)|^{-\alpha} = \mathbb{E}[(|t-s|^{1/2}|B(1)|)^{-\alpha}] = |t-s|^{-\alpha/2} \int_{\mathbb{R}^n} \frac{c_n}{|z|^n} e^{-|z|^2/2}\, dz.$$

Where c_n is a constant that comes from the multivariate normal density and depends only on n. Rather than compute the integral, we simply observe that it is a finite constant k depending on n and α only. Substituting and applying Fubini's theorem gives us that

$$\mathbb{E}[I_\alpha(\mu_B)] = k \int_0^1 \int_0^1 \frac{ds\, dt}{|t-s|^{\alpha/2}} \leq 2k \int_0^1 \frac{du}{u^{\alpha/2}} < \infty$$

Therefore $I_\alpha(\mu_B) < \infty$ since any random variable with finite expectation is finite almost surely. Hence by the energy method we have that dim Range $> \alpha$ almost surely. The lower bound on the range follows from letting $\alpha \to 2$. Since the graph can be projected onto the range by a Lipschitz map, the dimension of the graph is at least the dimension of the range. Hence if $n \geq 2$ almost surely dim Graph ≥ 2. Combining with 5.6 gives the full result. □

We are now prepared to justify the statement made in the introduction that planar Brownian motion is "almost" space filling. From the standpoint of measure, the path of Brownian motion only makes sense as a two-dimensional object. Hence Brownian motion is as large as it can possibly be while still having zero two-dimensional area.

Acknowledgments. I would like to thank my mentor Mohammad Abbas-Rezaei for suggesting this interesting topic and providing guidance as I wrote this paper. Additionally I would like to thank the University of Chicago *VIGRE* REU program for providing me with a fun summer doing math.

References

[1] Billingsley, Patrick. *Probability and Measure.* Wiley series in probability and mathematical statistics. 1995.
[2] Lawler, Gregory F. *Random Walk and the Heat Equation.* American Mathematical Society. 2010.
[3] Mörters, Peter & Yuval Peres. *Brownian Motion.* Cambridge University Press. 2010.
[4] Stein, Elias M. & Rami Shakarchi. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces.* Princeton University Press. 2005.