LIE ALGEBRAS AND LIE BRACKETS OF LIE GROUPS–MATRIX GROUPS

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ABSTRACT. The goal of this paper is to study Lie groups, specifically matrix groups. We will begin by introducing two examples: $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$. Then in each section we will prove basic results about our two examples and then generalize these results to general matrix groups.

CONTENTS

1. INTRODUCTION

Lie groups were initially introduced as a tool to solve or simplify ordinary and partial differential equations. The most important example of a Lie group (and it turns out, one which encapsulate almost the entirety of the theory) is that of a matrix group, i.e., $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$. First, we discover the relationship between the two matrix groups. The process in doing so will guide us through the development from $GL_n(\mathbb{R})$ to $SL_n(\mathbb{R})$ and show us several similarities between the two. Then we go deeper into other specific properties of those two matrix groups. For instance, their tangent spaces and the derivatives of the tangent spaces. Eventually, we generalize the results to general matrix groups.

2. THE LIE ALGEBRA OF $SL_n(\mathbb{R})$, $GL_n(\mathbb{R})$ AND IN GENERAL

Definition 2.1. For a given n, the *general linear group* over the real numbers is the group of all $n \times n$ invertible matrices. We denote it by $GL_n(\mathbb{R})$.

Exercise 2.2. Show that $GL_n(\mathbb{R})$ forms a group under matrix multiplication. Show also that $GL_n(\mathbb{R})$ is exactly the set of square matrices with nonzero determinant.

Definition 2.3. The *special linear group*, $SL_n(\mathbb{R})$, is the group of all square matrices that have determinant 1.

Exercise 2.4. $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

Now we know that $SL_n(\mathbb{R})$ is closed algebraically inside $GL_n(\mathbb{R})$.

Recall that $M_n(\mathbb{R})$ can be viewed as a metric space that inherits the Euclidean norm from $\mathbb{R}^{n^2}$. As a reminder, the Euclidean distance between two square matrices is defined to be

$$d\left(\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}, \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,1} & \cdots & b_{n,n} \end{pmatrix}\right) \tag{2.1}$$

$$\begin{aligned} &= d((a_{1,1}, \cdots, a_{1,n}, \cdots, a_{n,1}, \cdots, a_{n,n}), (b_{1,1}, \cdots, b_{1,n}, \cdots, b_{n,1}, \cdots, b_{n,n})) \\ &= \sqrt{(a_{11} - b_{11})^2 + \cdots + (a_{1n} - b_{1n})^2 + \cdots + (a_{n1} - b_{n1})^2 + \cdots + (a_{nn} - b_{nn})^2}. \end{aligned}$$

That is to say, a subgroup $G \subset GL_n(\mathbb{R})$ can be considered as a subset of Euclidean space. Therefore, we can ask ourselves whether G is topologically open, closed, path-connected, or compact, etc.

Definition 2.5. A *matrix group* is a subgroup G of $GL_n(\mathbb{R})$, which is topologically closed in $GL_n(\mathbb{R})$, meaning that if a sequence of matrices in G has a limit in $GL_n(\mathbb{R})$, then that limit must lie in G. In other words, G contains all of its non-singular limit points.

Proposition 2.6. *$SL_n(\mathbb{R})$ is a matrix group.*

Proof. We first need to prove that the function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous. Since for all A in $M_n(\mathbb{R})$,

$$\det(A) = \sum_{j=1}^{n+1} (-1)^{j+1} \cdot A_{1j} \cdot \det(A[1, j]), \tag{2.2}$$

where $A[i, j]$ is the matrix obtained by crossing out row i and column j from A, $\det(A)$ is an n-degree polynomial in the entries of A. And we hereby just use the fact that polynomials are continuous. Then it is clear that $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a continuous function with real outputs. Moreover, because the single-element set $\{0\}$ is closed, $SL_n(\mathbb{R}) = \det^{-1}(\{1\})$ is closed subsequently in $M_n(\mathbb{R})$, which also means that $SL_n(\mathbb{R})$ is closed in $GL_n(\mathbb{R})$. As a result, we can conclude that $SL_n(\mathbb{R})$ is a matrix group. $\square$

A major reason why we can discuss the tangent space of a matrix group is that a matrix group $G \subset GL_n(\mathbb{R})$ is a subset of the Euclidean space $M_n(\mathbb{R})$ and a Lie group, such that it is a smooth manifold.

Definition 2.7. Let G be a matrix group. The *Lie algebra* of G is the tangent space to G at I, which is denoted by

$$T_I G := \{\gamma'(0) | \gamma : (-\epsilon, \epsilon) \to G \text{ is differentiable with } \gamma(0) = I\}. \tag{2.3}$$

We denote the Lie algebra as $\mathfrak{g}(G)$ or, when there is no confusion, $\mathfrak{g}$.

Example 2.8. As **Figure** 1 suggests, if $S \subset \mathbb{R}^3$ is the graph of a differentiable function of two variables, then $T_y S$ is a 2-dimensional subspace of R^3 at y, the red dot on the graph. In general, $T_y S$ does not necessarily have to be a subspace of R^n. However, when S, a matrix group, is a subgroup of $GL_n(\mathbb{R})$, we will see that $T_y S$ is a subspace of $M_n(\mathbb{R})$.

Proposition 2.9. *The Lie algebra $\mathfrak{g}$ of a matrix group $G \subset GL_n(\mathbb{R})$ is not only a vector subspace but also a real subspace of $M_n(\mathbb{R})$.*

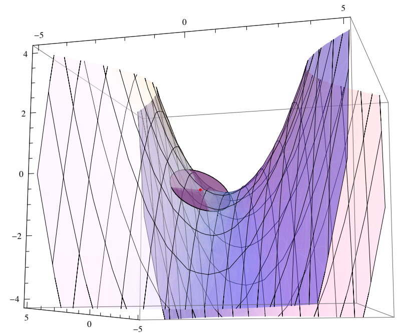

FIGURE 1. $S \subset R^3$ is the graph of a differentiable function of two variables. $T_y S$ is a 2-dimensional subspace of R^3 at y.

Proof. Since $\mathfrak{g}$ is the tangent space of matrix group G, which has elements of $n \times n$ matrices, then, trivially, the elements of $\mathfrak{g}$ are $n \times n$ matrices. Now we must show that $\mathfrak{g}$ is closed under scalar multiplication and addition. Let λ be a real number, and let A be an element of $\mathfrak{g}$. Then, by definition, $A = \gamma'(0)$ for some differentiable path γ in G, and $\gamma(0) = I$. Consider the path $\sigma : (-\frac{\epsilon}{\lambda}, \frac{\epsilon}{\lambda}) \to G$ given by $\sigma(t) = \gamma(\lambda t)$. This is in G and passes through I. It has initial velocity vector $\sigma'(0) = \lambda \cdot \gamma'(0) = \lambda A$, which proves that λA belongs to $\mathfrak{g}$.

Next, let A, B be in $\mathfrak{g}$. This means that $A = \alpha'(0)$ and $B = \beta'(0)$ for some differentiable paths α and β in G with $\alpha(0) = \beta(0) = I$. We construct the product path $\delta(t) := \alpha(t) \cdot \beta(t)$, which lies in G because G is closed under multiplication. This new path satisfies $\delta(0) = I$ and

$$\begin{aligned} \delta'(0) &= \delta'(0) \cdot \beta(0) + \gamma(0) \cdot \beta'(0) \\ &= A \cdot I + I \cdot B \\ &= A + B \end{aligned}$$

by the product rule. This shows that $A + B$ is also in $\mathfrak{g}$. $\square$

Now we will take a look at our examples and figure out what the Lie algebras of $SL_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ are.

Theorem 2.10. *The Lie algebra $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$ of $SL_n(\mathbb{R})$ is the set of all matrices in $M_n(\mathbb{R})$ with trace 0.*

The proof relies on the important fact that the trace is the derivative of the determinant; more precisely,

Lemma 2.11. *If $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ is differentiable and $\gamma(0) = I$, then*

$$\left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)) = trace(\gamma'(0)). \tag{2.4}$$

Proof. We need to use the property of the determinants of any square matrix A that we had mentioned above, **Equation** 2.2. Then we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)) &= \left.\frac{d}{dt}\right|_{t=0} \sum_{j=1}^{n+1} (-1)^{j+1} \cdot \gamma(t)_{1j} \cdot \det(\gamma(t)[1,j]) \\
&= \sum_{j=1}^{n} (-1)^{j+1} (\gamma'(0)_{1j} \cdot \det(\gamma(0)[1,j]) + \gamma(0)_{1j} \cdot \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)[1,j])) \\
&= \gamma'(0)_{11} + \left.\frac{d}{dt}\right|_{t=o} \det(\gamma(t)[1,1]) \quad (\textit{since } \gamma(0) = I).
\end{aligned}$$

Re-applying the above argument to compute $\left.\frac{d}{dt}\right|_{t=o} \det(\gamma(t)[1,1])$ and repeating n times gives us,

$$\left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)) = \gamma'(0)_{11} + \gamma'(0)_{22} + \cdot + \gamma'(0)_{nn}.$$

∎

*Proof. of **Theorem** 2.10.* If $\gamma : (-\epsilon, \epsilon) \to SL_n(\mathbb{R})$ is differentiable with $\gamma(0) = I$, the lemma above suggests that trace$(\gamma(0)) = 0$. This proves that every matrix in $\mathfrak{sl_n}(\mathbb{R})$ has trace zero.

Conversely, suppose A is in $M_n(\mathbb{R})$ and it has trace zero. We construct a path

$$\sigma(t) := I + tA. \tag{2.5}$$

We see that $\sigma(0) = I$ and $\sigma(0)' = A$. But this path is not entirely in $\mathfrak{sl_n}(\mathbb{R})$.

So now we define another path α(t) as the result of multiplying each entry in the first row of σ(t) by $1/\det(\sigma(t))$. It is written as

$$\alpha(t) = \begin{pmatrix} \frac{ta_{11}+1}{det(\sigma(t))} & \frac{ta_{12}}{det(\sigma(t))} & \cdots & \frac{ta_{1n}}{det(\sigma(t))} \\ ta_{21} & ta_{22}+1 & \cdots & ta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ta_{n1} & ta_{n2} & \cdots & ta_{nn}+1 \end{pmatrix}, \tag{2.6}$$

from which we can compute the value of $\alpha(t)$ and $\alpha'(t)$ when t = 0. Since we already know from **Equation** 2.5 that if $\sigma(0) = I$, then $\det(\sigma(0)) = 1$. Therefore, $\alpha(0) = I$. Notice that α(t) is a differentiable path in $SL_n(\mathbb{R})$. And it is also straightforward to see that $\alpha'(0) =$ A. This proves that every trace-zero matrix is in $\mathfrak{sl_n}(\mathbb{R})$. Hence, we conclude that the Lie algebra $\mathfrak{sl_n}(\mathbb{R})$ of $SL_n(\mathbb{R})$ is the set of all matrices in $M_n(\mathbb{R})$ with trace 0. ∎

It is an easy exercise, using the above Theorem, to prove that $\mathfrak{sl_n}(\mathbb{R})$ is a vector space without referring to **Proposition** 2.9.

Let us pause ourselves a little bit here and rewind ourselves back to the statement of T_IG being a real subspace of $M_n(\mathbb{R})$ when G is a matrix group in $GL_n(\mathbb{R})$. We shall now see that $\mathfrak{gl_n}(\mathbb{R})$ is in fact all of $M_n(\mathbb{R})$.

Proposition 2.12. *The lie algebra $\mathfrak{gl_n}(\mathbb{R})$ of $GL_n(\mathbb{R})$ is equal to $M_n(\mathbb{R})$.*

Proof. Given **Proposition** 2.9 it suffices to show that $M_n(\mathbb{R}) \subset \mathfrak{gl}_\mathfrak{n}(\mathbb{R})$.

In order to prove that $M_n(\mathbb{R})$ is a subset of $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$ works, we want to show that for every A in $M_n(\mathbb{R})$, A also belongs to $M_n(\mathbb{R})$, which means that there exists a path $\gamma : (-\epsilon, \epsilon) \to GL_n(\mathbb{R})$ such that $\gamma(0) = I$ and $\gamma'(0) = A$. Now we explicitly construct a path γ satisfying these properties. Consider $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ given by $\gamma(t) := I + tA$. Indeed, $\gamma(0) = I$ and $\gamma'(0) = A$. It remains to show is that the image of γ lies entirely in $GL_n(\mathbb{R})$.

Then we know that for all A in $GL_n(\mathbb{R})$, $\det(A) \neq 0$. So now we just need to check whether the determinant of all A in the image of $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{R})$ is non-zero. Well, we see that $\det(\gamma(0)) = \det(I) = 1$, and since γ is continuous, for a small enough ϵ, $\gamma(-\epsilon)$ and $\gamma(\epsilon)$ are extremely close to $\gamma(0)$. In addition, since the determinant function $\det : \gamma(-\epsilon, \epsilon) \to \mathbb{R}$ is also continuous, for a small enough ϵ, $\det(\gamma(-\epsilon))$ and $\det(\gamma(\epsilon))$ are also very close to $\det(\gamma(0)) = 1$. This tells us that neither of $\det(\gamma(-\epsilon))$ and $\det(\gamma(\epsilon))$ nor any value between the two is zero, which demonstrates that A is in $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$.

Hence, we conclude that the *Lie algebra* $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$ of $GL_n(\mathbb{R})$ is equal to $M_n(\mathbb{R})$. □

After all, we can say that the tangent space of matrix groups is not only a subspace of the Euclidean space $M_n(\mathbb{R})$ but also a vector space itself. This tells us that matrix groups are "nice."

3. The Lie bracket of $SL_n(\mathbb{R})$ and in general

Since any two finite dimensional vector spaces of the same dimension are isomorphic, it is clear that if two Lie algebras of matrix groups have the same dimension then they are isomorphic as vector spaces. That we learn so little from the isomorphism class of the Lie algebra is obviously undesired. In this section, we introduce algebraic structure on the Lie algebras in the form of a Lie bracket. This helps us avoid the above situation.

Definition 3.1. Let G be a matrix group with *Lie algebra* $\mathfrak{g}$. For all g, x in G, the *conjugation map* is defined as

$$C_g x := gxg^{-1}. \tag{3.1}$$

The fact that G is a group instantly implies that $C_g x$ is in G for any g, x in G.

Definition 3.2. We define the Adjoint map as the derivative at the identity. To be more precise, if A is in $\mathfrak{g}$ then define $Ad_g A = \frac{d}{dt}\Big|_{t=0} C_g(a(t))$, where $a(t)$ is the path in G such that $a(0) = I$ and $a'(0) = A$.

$$Ad_g := d(C_g)_I = g \frac{d}{dt}\Big|_{t=0} a(t) g^{-1} = gAg^{-1}. \tag{3.2}$$

With these preliminaries out of the way we are finally ready to define the Lie bracket.

Definition 3.3. For A, B in $\mathfrak{g}$, we define the Lie bracket as

$$[A, B] = \frac{d}{dt}\Big|_{t=0} Ad_{a(t)} B, \tag{3.3}$$

where $a(t)$ is any differentiable path in G with $a(0) = I$ and $a'(0) = A$.

We want to give an explicit formula for the Lie bracket, but before we do so, we need a lemma.

Lemma 3.4. *Let $a(t)$ be a path in $GL_n(\mathbb{R})$ such that $a'(0) = A$. Then $(a(t)^{-1})' = -A$.*

Proof. Since
$$a(t) \cdot a(t)^{-1} = I,$$
then
$$a'(t)a(t)^{-1} + a(t)(a(t)^{-1})' = 0.$$
And
$$a'(t)a(t)^{-1} = -a(t)(a(t)^{-1})'.$$
When t = 0, the equality then leads to
$$A = -(a(t)^{-1})'. \tag{3.4}$$
$\square$

We will see, first in the case of $SL_n(\mathbb{R})$ and then more generally, that Ad_g preserves the Lie algebra for any g in G.

Then let us look at how the conjugation map is related to the tangent spaces of matrix groups.

Proposition 3.5. *For all A living in $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$, $Ad_g A$ is also in $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$.*

Proof. From **Theorem** 2.10, we had observed the fact that for all A in $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$, $trace(A) = 0$. By the remark above it suffices to show that for any invertible g we have $trace(C_g A) = 0$. We will actually show that for any A in $M_n(\mathbb{R})$ and B in $GL_n(\mathbb{R})$ we have trace $(BAB^{-1}))$ = trace(B). This easily follows from the following computation.

$$\begin{aligned} trace(A \cdot B) &= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ji} A_{ji} \\ &= trace(B \cdot A). \end{aligned} \tag{3.5}$$

And

$$\begin{aligned} trace(BAB^{-1}) &= trace(B(AB^{-1})) \\ &= trace(B(B^{-1}A)) \\ &= trace((BB^{-1})A) \\ &= trace(A). \end{aligned} \tag{3.6}$$

Apply this to the proof, we have

$$\begin{aligned} trace(C_g A) &= trace(gAg^{-1}) \\ &= trace(A) \\ &= 0. \end{aligned} \tag{3.7}$$

Accordingly, $Ad_g A$ is in $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$. $\square$

Proposition 3.6. *For all A in $\mathfrak{g}$, $Ad_g A$ is in $\mathfrak{g}$ also.*

Proof. By the remark above it suffices to find a path $\gamma(t) \in G$ such that $\gamma(0) = I$ and $\gamma'(0) = gAg^{-1}$. We construct a path $\gamma(t) \in G$ such that $\gamma(t) = C_g A = gAg^{-1}$. Then we have

$$\gamma(0) = ga(0)g^{-1} = I$$

and

$$\begin{aligned} \gamma'(t)|_{t=0} &= g(\frac{d}{dt}\bigg|_{t=0} a(t))g^{-1} \\ &= gAg^{-1}. \end{aligned}$$

Since we know that g is invertible and A is also in $\mathfrak{g}$, then gAg^{-1} must be in $\mathfrak{g}$. And this demonstrates that $Ad_g A$ is a map from $\mathfrak{g}$ to $\mathfrak{g}$. □

With that out of the way we are ready to give an explicit formula for the Lie bracket in matrix groups.

Proposition 3.7. *For all A, B in $\mathfrak{g}$, $[A, B] = AB - BA$.*

Proof. Let a(t) and b(t) be differentiable paths in G with a(0) = b(0) = I and $a'(0) = A$, $b'(0) = B$. By the definition of lie bracket and derivative of conjugation map,

$$[A, B] = \frac{d}{dt}\bigg|_{t=0} Ad_{a(t)} B = \frac{d}{dt} a(t)Ba(t)^{-1}. \tag{3.8}$$

By the product rule and **Lemma** 3.4 we are done. □

Notice $[A, B] = 0$ if and only if A and B commute. So we say that the derivative of $Ad_{a(t)}B$ measure whether two square matrices commute, and $Ad_{a(t)}B$ itself measures the failure of a square matrix g to commute with elements of G near I. For this reason if all elements of G commute with g, then Ad_g is the identity map on $\mathfrak{g}$.

We will want to show that the Lie bracket of two matrices in the Lie algebra is also in the Lie algebra. First we prove this for $SL_n(\mathbb{R})$.

Proposition 3.8. *For all A, B in $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$, $[A, B]$ also belongs to $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$.*

Proof. We already know that for all A, B in $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$, trace(A) = trace(B) = 0, then we need to show trace($[A, B]$) = 0. Given **Theorem** 2.10 and **Proposition** 3.7 all that is left to be shown is trace($AB - BA$) = 0. But trace is clearly additive and we have shown that trace (AB) = trace(BA). So we are done. □

Now we want to prove the analogous theorem for general matrix groups. First we need a technical lemma.

Lemma 3.9. *If the path $\gamma(t)$ is in $\mathfrak{g}$, then $\gamma'(0)$ is also in $\mathfrak{g}$.*

Proof. For every n in $\mathbb{N}$, we define

$$\frac{\gamma(\frac{1}{n}) - \gamma(0)}{\frac{1}{n}} = V_n.$$

It is clear that $V_n \in \mathfrak{g}$ and that as n goes to infinity, V_n converges (in the Euclidean metric) to $\gamma'(0)$. **Proposition** 2.9 that tells us that $\mathfrak{g}$ is a subspace of $M_n(\mathbb{R})$ implies that $\mathfrak{g}$ is a subspace of a finite dimensional space. Therefore, $\mathfrak{g}$ is topologically closed and we are done. □

Proposition 3.10. *For all A, $B \in \mathfrak{g}$, $[A, B]$ is in $\mathfrak{g}$*

Proof. From **Proposition** 3.6 we learnt that $Ad_g B$ is in $\mathfrak{g}$. Then following **Definition** 3.3 we know that $[A, B]$ is the derivative of $Ad_g B$ on $\mathfrak{g}$. Then using the **Lemma** 3.9 above, we know that $[A, B]$ is also in $\mathfrak{g}$. ☐

For a total completion of the proof, we will prove the following theorem, which we had used in **Lemma** 3.9.

Theorem 3.11. *If $U \subset W$ is a finite dimensional subspace, then U is closed.*

Proof. We need to show that if V_n converges to V when n gets arbitrarily large, and V_n belongs to the subspace $U \subset W$, V also belongs to the subspace $U \subset W$.

First of all, let us look at the orthonormal basis, $e_1, e_2, ...e_m$ of U, where m is a finite number. Then we can write each vector in U as

$$V_i = \sum_{j=1}^{m} \lambda_j^i e_j. \tag{3.9}$$

Then we want to find out whether $\{\lambda_{j_i}^i\}$ is Cauchy.

When we write out all the vectors in U, we have

$$\begin{array}{ccccccc} V_1 & = & \lambda_1^1 e_1 & + & \cdots & + & \lambda_m^1 e_m \\ \vdots & = & \vdots & + & \vdots & + & \vdots \\ V_n & = & \lambda_1^n e_1 & + & \cdots & + & \lambda_m^n e_m \end{array}$$

If V_n is Cauchy, then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|V_{n_1} - V_{n_2}| < \epsilon$ when $n_1, n_2 \geq \mathbb{N}$. Namely, if we write V_{n_1} and V_{n_2} in the summation form we just defined; for V_n, V_n is Cauchy, we will have

$$\begin{aligned} |V_{n_1} - V_{n_2}| &= |\sum_{j=1}^{m} \lambda_j^{i_1} e_j - \sum_{j=1}^{m} \lambda_j^{i_2} e_j| \\ &= |\sum_{j=1}^{m} (\lambda_j^{i_1} - \lambda_j^{i_2}) e_j| \\ &= \sum_{j=1}^{m} |\lambda_j^{i_1} - \lambda_j^{i_2}| (\textit{since } e_j \textit{ is an orthonormal basis}) \\ &< \epsilon. \end{aligned} \tag{3.10}$$

Since the sum of all absolute values is less than ϵ, then each absolute value must be less than ϵ, which leaves us that

$$|\lambda_j^{i_1} - \lambda_j^{i_2}| < \epsilon. \tag{3.11}$$

This proves that $\lambda_{j_i}^i$ is a Cauchy sequence. So it has a limit $\lambda_i \in \mathbb{R}$. Pick an $\epsilon > 0$, then for each j, there exists a N_j such that $n \geq N_j$ and $|\lambda_j^n - \lambda_j| < \epsilon$.

Now let us consider $\tilde{V} = \sum_{j=1}^{m} \lambda_j e_j$, which is the sum of all the limit points.

$$\begin{array}{ccccccc} V_1 & = & \lambda_1^1 e_1 & + & \cdots & + & \lambda_m^1 e_m \\ \vdots & = & \vdots & + & \vdots & + & \vdots \\ V_n & = & \lambda_1^n e_1 & + & \cdots & + & \lambda_m^n e_m \\ \vdots & = & \vdots & + & \vdots & + & \vdots \\ \tilde{V} & = & \lambda_1 e_1 & + & \cdots & + & \lambda_m e_m \end{array}$$

$\tilde{V}$ is, by definition, an element in U.

Now if we can show that $\tilde{V}$ is exactly V, the limit of V_i's, U will be topologically closed. As one Cauchy sequence cannot converge to two different limit points, and a topologically closed space contains all of its limit points.

We need to remind ourselves that we had stated at the beginning of the proof that U is a finite dimensional subspace. Therefore, we can clearly see that V_i does converge to $\tilde{V}$. Then we have shown that U is closed, if it is a finite dimensional subspace. If U is not a finite dimensional space, then it is almost impossible to match V with $\tilde{V}$, for the reason that there would be an unaccountable number of coordinates for us to wait them to converge, which is impossible to happen. $\square$

Now we have successfully added algebraic structure to any Lie algebra of a matrix group.

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