KNOTS AND BRAIDS

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Abstract. In this paper we examine two physically-inspired objects, knots and braids. The two are intimately related because when we connect the ends of a braid, we end up with a knot or link. We show that braids can be defined algebraically, geometrically, and topologically, and we determine when two braids will yield the same knot. Finally, we prove that every knot or link is the closure of some braid.

Contents

1. Introduction

A knot is a circle embedded in $\mathbb{R}^3$. In the late 1800s Lord Kelvin suggested that atoms might represent knots in the ether, with different elements corresponding to different types of knots. Once this idea was shown to be false, knot theory remained as a beautiful mathematical theory in its own right. Since then, several practical applications of knot theory have come to light, including DNA knotting and other topics in biology, chemistry and physics. For a further reference on some history, see chapter one of Adams [1].

In this paper we focus on the connections between knot and braid theory. We begin by defining knots and knot projections, and we give explicit examples of knot invariants. We then define the braid group in three different ways and prove the equivalence of these definitions. In particular, we show that a braid can be defined in a purely algebraic manner, which makes it more convenient to write out and manipulate than a knot. Finally, by introducing theorems of Markov and Alexander, we show how related knots and braids really are.

2. Preliminary Definitions

Let X be a topological space.

Definition 2.1. A *path* in X is a continuous function $f : [0, 1] \to X$.

Date: AUGUST 26, 2011.

Definition 2.2. Two paths f_0 and f_1 in X are *homotopic* if there exists a continuous family of paths $f_t(x)$ in X for $t \in [0,1]$. If $f_t(1) = x_0$ for some $x_0 \in X$ and for all $t \in [0,1]$ we say there is a *based homotopy* taking one path to the other.

Definition 2.3. A *loop* f in X with base point x_0 is a path in X with $f(0) = f(1) = x_0$. (Equivalently, a loop is a map from S^1 into X).

Definition 2.4. An *isotopy* of X is a continuous family of diffeomorphisms ϕ_t of X for $t \in [0,1]$.

In particular, given a path f in X and an isotopy ϕ_t of X, we get a family of homotopic paths $f_t = \phi_t \circ f$.

Definition 2.5. The *fundamental group* of X at a point $x_0 \in X$ is the set of based homotopy classes of loops in X with base point x_0, with group operation defined by concatenation. In other words, if f and g are loops, we have

$$(f \circ g)(t) = \begin{cases} f(2t), & t < 1/2 \\ g(2t-1), & t \geq 1/2 \end{cases} .$$

The trivial loop is the constant path $f(t) \equiv x_0$, and the inverse of a loop is then simply the same loop in the reverse direction: $f^{-1}(t) = f(1-t)$. It is straightforward to check that these operations are well-defined on equivalence classes. If X is path connected, the fundamental group is independent of base point. We denote the fundamental group of X with basepoint x_0 by $\pi_1(X, x_0)$ or simply $\pi_1(X)$ when independence from basepoint is clear.

To give an example, the fundamental group of the circle, $\pi_1(S^1)$, is isomorphic to $\mathbb{Z}$ and is generated by one complete loop around the circle. Given any loop, counting the number of times it winds around gives the integer multiple of the generator that represents this loop.

3. Knots

Definition 3.1. A *knot* is a smooth embedding of S^1 into $\mathbb{R}^3$. Two knots K_1 and K_2 are considered the same if there is an isotopy of $\mathbb{R}^3$ taking K_1 to K_2.

Intuitively, we can think of a knot as a piece of infinitesimally thin string that is knotted and then has the ends glued together. Two knots are the same if one can be deformed into the other without ungluing the ends.

Definition 3.2. A *projection* of a knot K is the image of K under a projection $P : \mathbb{R}^3 \to \mathbb{R}^2$ onto some affine plane, such that:
(1)The preimage of each point in $P(K)$ contains at most two points.
(2) There are only finitely many points in $P(K)$ with two preimages.
We call such points “crossings” and keep track of “over” and “under” strands. See Figure 1 for an example of a projection.

Definition 3.3. A *link* is a smooth embedding of a disjoint union of circles into $\mathbb{R}^3$.

In other words, a link is a collection of disjoint knots, possibly linked to one another.

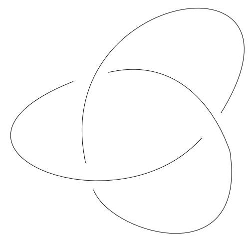

Figure 1. A projection of the trefoil knot.

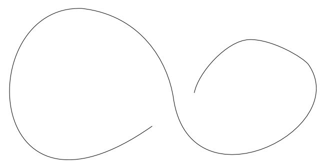

Figure 2. A projection of the unknot.

Definition 3.4. The knot which is represented by a projection with no crossings is the *unknot*. However, there are also projections of the unknot with crossings. One projection is shown in Figure 2.

A fundamental problem in knot theory is classification, that is, determining when two knots are equivalent. One approach to this problem is to find knot invariants, which differentiate between non-equivalent knots. We'll start our study of invariants with Reidemeister moves.

A Reidemeister move is a local modification of a knot projection that changes one or two crossings. There are three possible moves, depicted below. One can show that after applying a Reidemeister move to a projection, the result is a projection of an equivalent knot.

Definition 3.5. The *Reidemeister moves* are illustrated in figures 3, 4 and 5.

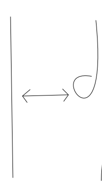

Figure 3. Type 1 Reidemeister move. A twist or untwist.

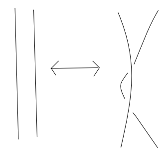

Figure 4. Type 2 Reidemeister move. One strand passing over or under another strand.

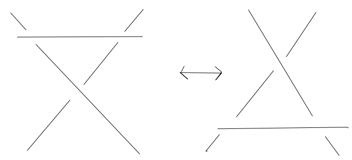

Figure 5. Type 3 Reidemeister move. A strand passing over or under a crossing.

The following theorem shows why Reidemeister moves are so important.

Theorem 3.6. *Two knots are equivalent if and only if there is a series of Reidemeister moves taking one to another.*

A proof of this can be found in chapter one of Lickorish [5]. Thus, any property of a knot which is preserved by Reidemeister moves will be a knot invariant. One such property is tricolorability.

Definition 3.7. A projection is *tricolorable* if it can be colored in such a way that at every crossing either three colors or one color meet at the crossing. Color can only change at an undercrossing, and three colors must be used. A knot is *tricolorable* if there exists a tricolorable projection of that knot.

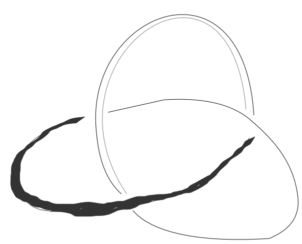

Figure 6. The trefoil knot is tricolorable.

It turns out that the tricolorability of a knot is preserved under all three Reidemeister moves, so it is a knot invariant. In particular, if any projection of a knot is tricolorable, then all projections must be. This also lets us see that there exist nontrivial knots. The unknot is not tricolorable, so any knot which is tricolorable cannot be trivial. The trefoil knot is tricolorable, so must be nontrivial. However, there are far more than two equivalence classes of knots, so there are many distinct knots with the same tricolorability status. For example, the figure eight knot is not tricolorable, but also not trivial. (Figure 7). So in order to distinguish more knots, we need stronger invariants. One such is the knot group.

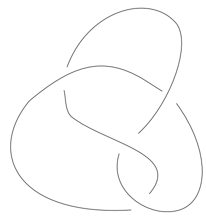

FIGURE 7. A figure eight knot. No matter how one colors this knot, there is no way to get it to satisfy tricolorability requirements.

Definition 3.8. The *knot group* of a knot K is the fundamental group of the knot complement. In other words, it is the group $\pi_1(\mathbb{R}^3 \setminus K)$.

For example, the knot group of the trivial knot is isomorphic to the integers. The fundamental groups of two isotopic spaces are isomorphic, so the fundamental groups of equivalent knots are isomorphic. Thus the knot group is a knot invariant.

However, two distinct knots can have isomorphic fundamental groups. Consider, for example, the mirror image of the trefoil knot. The trefoil knot is not equivalent to its mirror image, but their fundamental groups are isomorphic.

4. The Braid Group

We will now define a braid and give three equivalent definitions of the braid group. While some connections between braids and knots will be immediately clear, we will discuss in more detail how braids and knots are related in section 5.

Definition 4.1. In $\mathbb{R}^3$, let A be the set of points with coordinates $y = 0, z = 1$, and $x = 1, 2, 3, ..., n$, and B the set of points with $y = 0, z = 0$, and $x = 1, 2, 3, ..., n$. An *n-strand braid* is a set of n non-intersecting smooth paths connecting the n points in A to the n points in B. We say that two braids are equivalent if there is an isotopy of $\mathbb{R}^3$ taking one to the other.

One can think of a braid as a collection of strands in space with fixed endpoints that are braided around each other. This lets us define the *braid group.*

Definition 4.2. (geometric) The *n-strand braid group* consists of equivalence classes of braids as defined above with the operation that connects the bottom of the first braid to the top of the second, and rescales so that the new braid is still unit length. The trivial braid has n parallel strands, and the inverse of a braid is its mirror image.

Note that we can get a knot or link by connecting the corresponding top and bottom strands by paths in $\mathbb{R}^3$. This is called a *closed braid.* (Figure 8.)

The braid group also has a strictly algebraic definition. This gives a shorthand way to express braids, and makes the relations defining equivalent braids very clear.

Definition 4.3. (algebraic) The *n-strand braid group* is the group given by the generators $\{\sigma_1, ..., \sigma_{n-1}\}$ with the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i - j| \geq 2 \qquad (*)$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } 1 \leq i \leq n-2 \qquad (**)$$

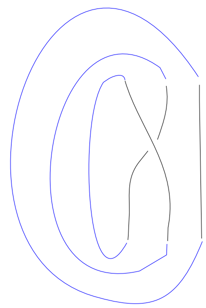

Figure 8. A closed braid. Note that this particular braid yields a link of two components.

Proposition 4.4. *The geometric and algebraic definitions of the braid group are equivalent.*

Proof. In the same manner as a knot projection, any braid in $\mathbb{R}^3$ can be represented by a 2-dimensional projection onto the $x - z$ plane, as in figure 9. Without loss of generality, we can arrange it so that only one crossing occurs in our projection for any value of z.

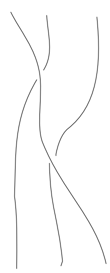

Figure 9. A projection of a braid. If the two crossings had the same z-value, a small perturbation would make them different heights. This particular projection can be described by $\sigma_1\sigma_2$.

We can then describe the braid from top to bottom as a word in the generators $\{\sigma_1, \sigma_2, ...\sigma_{n-1}\}$ where σ_i represents strand i passing over strand $i+1$, and σ_i^{-1} represents strand i passing under strand $i+1$. One can check that different braid diagrams corresponding to the same braid differ precisely by the relations specified in the algebraic definition.

For example, the inverse relations hold, because if strand j goes over strand $j+1$ and then comes back over strand $j+1$, this is isotopic to the trivial braid, so $\sigma_j\sigma_j^{-1} = e$. Also, if two crossings occur in a row at strands which are not adjacent, the order doesn't matter, as we can shift one up or down along the braid. This shows $(*)$. If the strands are adjacent, we have $(**)$. These two possibilities are illustrated in Figure 10.

$\square$

For our third definition of the braid group, we will view braids as the fundamental group of a particular space called the *n-configuration space*

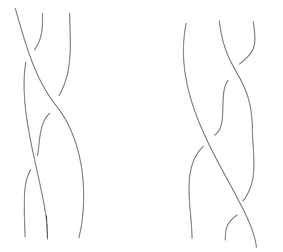

FIGURE 10. On the left we have $\sigma_i \sigma_{i+1} \sigma_i$ and on the right $\sigma_{i+1} \sigma_i \sigma_{i+1}$. There is clearly an isotopy between these two braids that only involves this segment of these three strands.

Definition 4.5. The *unordered n-configuration space* of X is the space of all unordered sets of n distinct points of X, with the natural topology.

Definition 4.6. (topological) The *n-strand braid group* is the fundamental group of the unordered n-configuration space for $\mathbb{R}^2$.

Proposition 4.7. *The geometric and topological definitions of the braid group are equivalent.*

Proof. Since $\mathbb{R}^2$ is path connected, we can assume that the initial point in the topological definition is the set of integers $\{1, 2, 3, ..., n\}$. Consider the space $\mathbb{R}^3$ as $\mathbb{C}^1 \times \mathbb{R}^1$. Now, every loop in the fundamental group can be thought of as the movement in the complex plane of these n points, varying with the real variable time t.

As we vary t, each point will trace a curve in space that starts and ends at an integer in the set. This gives a geometric realization of the braid. In a similar way, every geometric braid can be considered as a loop in the unordered n-configuration space, and therefore an element of the fundamental group.

Two topological braids are isotopic if one can be continuously deformed into the other, which is true if and only if their corresponding geometric braids are isotopic. Thus we have shown the equivalence of the definitions. □

5. Knots to Braids

As previously stated, the closure of a braid is always a knot or link. However, this closure is not unique. For example, consider conjugation by σ_i. When the braid is closed, the top and bottom of the braid connect, and thus σ_i will cancel with σ_i^{-1}. However, many braids are not preserved under conjugation. This example shows that two distinct braids can yield the same knot. So how do we know in general when two braids correspond to the same knot? Markov provided an answer to this question in the following theorem.

Theorem 5.1. *(Markov) The closures of two braids B and B' represent the knot or link L if and only if one braid can be deformed into the other by a finite number of Markov moves or their inverses, as defined below.*

There are two types of Markov moves.

Definition 5.2. A *type one Markov move* takes an n-strand braid to another n-strand braid via conjugation by σ_i for some $i \in 1, 2, ..., n-1$.

Definition 5.3. A *type two Markov move* takes an n-strand braid to an $n+1$-strand braid by adding σ_n or σ_n^{-1} to the end. In other words, an n-strand braid B becomes $B\sigma_n$ or $B\sigma_n^{-1}$.

As explained above, conjugation by σ_i is the same as multiplication by $\sigma_i\sigma_i^{-1}$ once the braid is closed, and thus we still get the same knot. Figure 11 illustrates how adding σ_n to the end of an n-strand braid gives us the same knot with an extra twist. In other words, we are doing a type one Reidemeister move. For a proof that these are the only two moves necessary, see [4].

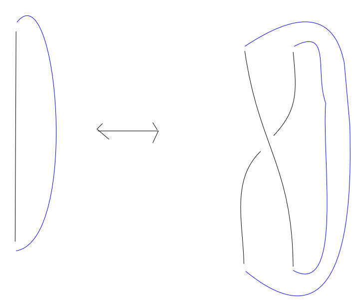

FIGURE 11. Both of these knots are projections of the unknot.

The final question we wish to answer is which knots can be written as braids. Remarkably, it turns out that they *all* can.

Theorem 5.4. *(Alexander) Every knot or link is isotopic to a closed braid.*

Proof. Suppose we have a link L and its projection. We orient the components of L. We then choose a point P on the projection such that P does not intersect the knot. Though P is a point in the projection, it is helpful to think of P as an axis extending through the projection plane. The goal will be to manipulate L so that every component is oriented in a particular direction around this axis.

Fix an orientation about P. We consider a finite number of subarcs of L such that each subarc is either oriented in our orientation or in the reverse orientation, not both. (i.e. at every point where the orientation about P changes, we put a point to separate the subarcs.) If there are no subarcs oriented in the reverse direction, then we're done. Otherwise, choose some subarc S which is oriented opposite our chosen orientation. We divide up S further into a finite number of subarcs $\{S_i\}$, such that each subarc contains at most one crossing.

Now consider an arbitrary S_i. Keeping the endpoints of the subarc fixed, we can pull the subarc across our axis P to give it the correct orientation. We pull it over all other parts of the knot, except possibly the single crossing on S_i. (Figure 12). Note that we can avoid adding another crossing to any part of S which is still oriented incorrectly, which ensures that S is re-oriented in a finite number of steps.

To every subarc oriented in the reverse direction, we can apply the same procedure, splitting it up into subarcs with at most one crossing and then pulling each piece over the knot and across the axis. We can continue doing this until we have a link which is all oriented around the axis.

But this is exactly what we want, because a link that is entirely oriented around an axis is braided around that axis. If we take a half-plane through the axis going out to infinity, this half-plane necessarily passes through every strand, and can be regarded as the plane of the braid closure. (Figure 13).

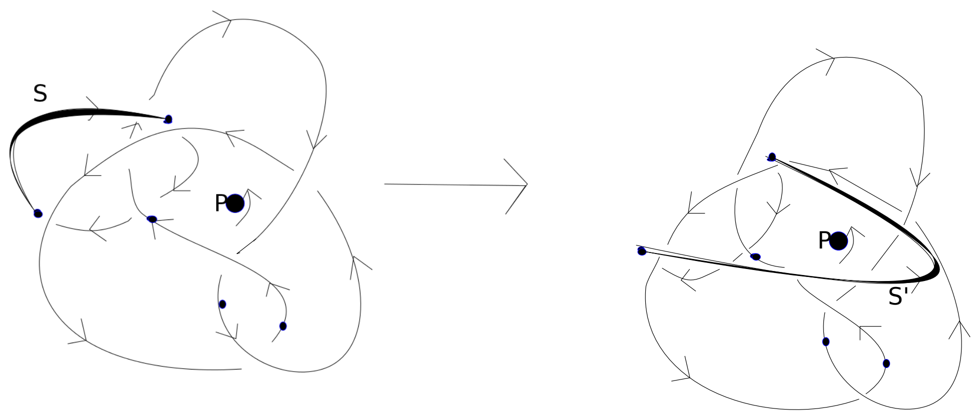

FIGURE 12. While at first S is going the opposite orientation from that of P, on the right we see S' with the correct orientation. This motion is an isotopy of $\mathbb{R}^3$. If instead S had an undercrossing, S would simply go under the other arc and then be pulled across P.

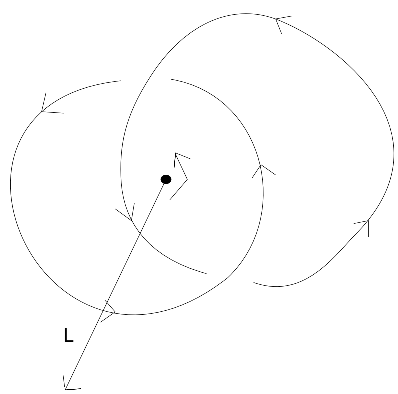

FIGURE 13. If we regard the line L as the plane of the braid closure, with the given orientation as "down", then we see that this picture gives us the 2-strand braid $\sigma_1\sigma_1$.

This completes the proof. □

Acknowledgments. It is a pleasure to thank my mentors, William Lopes and Katie Mann, for all their advice and guidance on this project. They answered all the random questions that I had and gave me lots of things to think about in the future. I couldn't have asked for more helpful and enthusiastic mentors, and they made the whole experience very enjoyable.

References

[1] Colin Adams. The Knot Book. W.H. Freeman and Company, 1994.

[2] Vassily Manturov. Knot Theory. Chapman and Hall/CRC, 2004.

[3] J. W. Alexander. A Lemma on Systems of Knotted Curves. Proc. Nat. Acad. Sci. USA, Vol. 9, 1923.

[4] Kunio Murasugi and Bohdan I. Kurpita. A Study of Braids. Kluwer Academic Publishers, 1999.

[5] W. B. Raymond Lickorish. An Introduction to Knot Theory, Graduate Texts in Mathematics. Springer-Verlag, 1997.