FORCING

ROWAN JACOBS

Abstract. ZFC, the axiom system of set theory, is not a complete theory. That is, there are statements such that ZFC $+ \varphi$ and ZFC $+ \neg\varphi$ are both consistent. The Continuum Hypothesis—the statement that the size of the real numbers is the first uncountable cardinal $\aleph_1$—is a famous example. This paper will investigate why ZFC is incomplete and what structures satisfy its axioms. The proof technique of forcing is a powerful way to construct structures satisfying ZFC $+ \varphi$ for a wide range of φ. This technique will be developed and its applications to the Continuum Hypothesis and the Axiom of Choice will be demonstrated.

1. Ideas Behind Forcing

1.1. **Models of ZFC.** During the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ century, set theory played the role of a solid foundation underneath other fields, such as algebra, analysis, and topology. However, without an axiomatization, set theory opened itself up to paradoxes, such as Russell's Paradox. If every predicate defines a set, then we can define a set S of all sets which do not contain themselves. Does S contain itself? If so, then it must not contain itself, since it is an element of S, and all elements of S are sets that do not contain themselves. If not, then it must contain itself, since it is a set that does not contain itself, and all sets that do not contain themselves are in S. With this and similar paradoxes, it became clear that the naive form of set theory, in which any predicate over the entire universe of sets can define a set, could not stand as a foundation for other fields of mathematics. Thus several axiomatizations of set theory were introduced in order to formalize the field of set theory and eliminate its contradictions and ambiguities. The only one to have any traction outside the world of logicians was ZFC, or Zermelo-Fraenkel with the Axiom of Choice. The universe of ZFC is often implicitly taken to be the background of many areas of mathematics, for it allows us to define and rigorously work with the structures of algebra, analysis, and topology, among others.

Like any other set of axioms, e.g. the axioms satisfied by a group, a field, or a topological space, the axioms of ZFC describe a particular type of structure—in this case, a universe of set theory. There is of course more than one isomorphism class of groups, fields, or rings. If we wish to work inside 'the universe of ZFC,' then the question arises, is the same true of universes of set theory, or do the axioms of ZFC describe a unique (or essentially unique) object?

In order to answer this question, we will restrict ourselves to first-order logic, that is, logic in which predicates and functions cannot be quantified over or taken as arguments for other predicates or functions. It is worth noting that ZFC is a first-order axiomatization of set theory, and does not contain any higher-order axioms. We study first-order axioms because models of first-order logic are much more amenable to study than those of higher-order logics. First-order logic gives us several tools such as the theorems of Compactness, Godel's Completeness Theorem, and the Lowenheim-Skolem Theorem, which are not available in higher-order logic, making the study of higher-order models much more difficult.

Given any set of first-order axioms over some *language* (i.e. some set of function, relation, and constant symbols), called a *theory*, we call a collection of objects with interpretations for each symbol that satisfy all of the axioms a *model* of our theory. (If $\mathcal{M}$ is a model of T, we write $\mathcal{M} \models T$.) Thus, our question is equivalent to the question of whether there is more than one model of ZFC, up to isomorphism.

The answer turns out to be that there is. It is a theorem of model theory, known as the Lowenheim-Skolem Theorem, that any theory with infinite models has infinite models of *any* cardinality above the cardinality of the language. The language of set theory consists only of the relations of $=$ (equality) and $\in$ (containment), so every infinite cardinality is above the cardinality of our language. So there must exist models of ZFC that are very, very small. In particular, there exists some *countable* model of ZFC. How can this happen, when it is a theorem of ZFC that there exists an uncountable set, and all its elements must also belong to the model? If we view our countable model $\mathcal{M}$ as a subset of the canonical model of ZFC, and we let X be a set such that $\mathcal{M} \models X$ is uncountable, then there is a bijection $f : \omega \to X$ in V, but no such bijection is an element of $\mathcal{M}$. Thus, it is consistent both that the universe of $\mathcal{M}$ is a countable set, and that it models the statement "there exists an uncountable set."

Not only does ZFC have models of different cardinalities, but it also has models which satisfy different statements. This follows from Godel's First Incompleteness Theorem, which states that any axiom system sufficient to express elementary arithmetic cannot be both consistent and complete. ZFC is one such axiom system, and if it is consistent, then it cannot be a *complete* theory, meaning that there are statements φ such that ZFC must be consistent with both φ and $\neg\varphi$. Thus there is a model of ZFC $\mathcal{M} \models \varphi$, and another model $\mathcal{N} \models \neg\varphi$, so these models are not isomorphic. This paper will show how to prove that ZFC is incomplete, using the famous example of the Continuum Hypothesis (CH): the statement that $|\mathbb{R}| = \aleph_1$. The proof technique displayed here, known as *forcing*, is the most common way to show that a given statement is consistent with ZFC, and it can be used for many statements other than CH. This paper will primarily follow Thomas Jech's presentation of forcing in chapters 14 and 15 of *Set Theory: The Third Millenium Edition Revised and Expanded.*

2. The Machinery of Forcing

2.1. **Canonical Models.** The *Von Neumann universe*, denoted V. V is constructed hierarchically, as follows. V_0 is the empty set. If β is an ordinal for which V_β is defined, then $V_{\beta+1}$ is the power set of V_β, i.e. $V_{\beta+1} = \mathcal{P}(V_\beta)$. If λ is a limit ordinal, then V_λ is the union of all V_α for $\alpha < \lambda$. V is then defined as the union of V_α for all ordinals α. (It is worth noting that in ZFC, defining sets by quantifying over all ordinals is not allowed. So V is not a set itself, but is called a *proper class*. This prevents Russell's Paradox, among other problems.) In V, all sets can be classified by *rank*. The rank of a set S is the minimum α such that $S \in V_\alpha$.

The construction of V is not the only way to hierarchically build a model. There is a very important submodel of V called *the constructible universe* and denoted L. We define $L_0 = \emptyset$ and for λ a limit ordinal, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$, as before. However, instead of letting $L_{\beta+1}$ be the power set of L_β, we define $L_{\beta+1}$ as only the subsets of L_β which are definable by a formula in the language $\{=, \in\}$ of set theory with parameters and quantifiers restricted to L_β (that is, we are not allowed to refer to sets of rank higher than β in constructing $L_{\{}\beta + 1\}$). Like V, L is a model of ZFC. In fact, the two may be equal, a hypothesis referred to as $V = L$. If $V \neq L$, then V and L may model many different statements. For instance, L always models the continuum hypothesis, whereas (as will be shown later) it is consistent that $V \not\models$ CH. It is worth noting that L is the minimal proper class model of ZFC which contains all the ordinals of V.

Both L and V are proper classes, but models can also be sets. Let $\mathcal{M}$ be a set that models ZFC. Then $\mathcal{M}$ is called a *transitive model* if its underlying set M is a transitive set, i.e. if for any element x, $x \in M \Rightarrow x \subset M$. The property of transitivity of a model is necessary for substructuress and superstructures of that model to behave properly.[1]

[1]More technically, bounded formulas are only guaranteed to be absolute relative to a class of models if those models are transitive.

There are some sets that every transitive model of ZFC will contain. For instance, the Axiom of Infinity[2] ensures that the natural numbers $\mathbb{N}$ (alternately called ω when viewed as an ordinal number) will be in every transitive model. In particular, not only does every model $\mathcal{M}$ have some interpretation of $\mathbb{N}$, but all of these interpretations are necessarily the same set. By contrast, every transitive model of ZFC must contain some interpretation of $\mathbb{R}$, by taking the power set of $\mathbb{N}$. However, what subsets of $\mathbb{N}$ exist will vary from model to model, so the interpretations of $\mathbb{R}$ will not necessarily be identical. In general this will be true of all power set constructions—different models may disagree on which subsets of some given set exist, so the power sets of that set may be different.

In order to build a model containing new sets, we will first build a poset that 'approximates' the set we are trying to construct. Generally this will be a finite or countable (relative to $\mathcal{M}$) approximation of a final set that will be uncountable. By taking a filter on this poset, and then taking the union of the elements of the filter, the approximated set can be constructed. The actual construction of the new model requires building a boolean algebra B from our poset, then passing to a B-valued model of ZFC in which our new set exists. Then, our filter on the poset forms an ultrafilter on the boolean algebra, with which we convert our B-valued model into a $\{0, 1\}$-valued model containing our desired set.

2.2. **Notions of Forcing (Forcing Posets).** Let $\mathcal{M}$ be a transitive model of ZFC. (It is occasionally required that $\mathcal{M}$ is a countable transitive model.) Then we call $\mathcal{M}$ the *ground model*. (It is often convenient, but not required, to assume that $\mathcal{M}$ is the canonical universe V of set theory.) If X is a definable element of V, then $X^{\mathcal{M}}$ is the corresponding set in $\mathcal{M}$, i.e. $A \in M$ such that $\mathcal{M} \models X = A$.

A nonempty poset $(P, <)$ in $\mathcal{M}$ is called a *notion of forcing*, whose elements are *forcing conditions*. In general, if $p, q \in P$ and $p < q$, we say p is *stronger* than q. That is, p represents a stronger condition than q. It is customary for the poset to have a largest element, denoted $\mathbf{1}$, such that all elements of P are stronger than $\mathbf{1}$. p and q are called *compatible* if there exists r stronger than both p and q. Otherwise they are *incompatible*. A set of incompatible elements is called an *antichain*. One ubiquitous example of a notion of forcing is the poset of finite partial functions from $\kappa \times \lambda$ to $\{0, 1\}$, for (usually regular) cardinals κ and λ. We say $f \leq g$ if f extends g. In posets of this form, the largest element $\mathbf{1}$ is the empty set. This counterintuitive notation occurs because every nonempty condition is stronger than the empty set, i.e. imposes more restrictions on a function extending it. For instance if $\alpha \in \kappa$ and $\beta \in \lambda$, then $\{((\alpha, \beta), 1)\} \leq \mathbf{1}$ because $\{((\alpha, \beta), 1)\}$ sets the restriction that any f extending it must have $f(\alpha, \beta) = 1$, whereas $\mathbf{1}$ sets no such restriction.

It is important to note that the poset and its ordering must be elements of $\mathcal{M}$. However, we are going to use the poset to construct sets that are not already in $\mathcal{M}$, in order to obtain a new model. By adjoining a *generic set* G, we will form the new model $\mathcal{M}[G]$.

A *dense* subset of P is a set D such that for all $p \in P$, there is $q \in D$ stronger than p. A filter on a poset P is a nonempty subset F of P such that: (a) if $p \in F$ and $p \leq q$, then $q \in F$ and (b) if $p, q \in F$ there is $r \in F$ extending both p and q. If this filter intersects every dense set D in the ground model $\mathcal{M}$, it is called *$\mathcal{M}$-generic*, or more commonly just generic. Dense sets thus allow us to place restrictions on our generic set. If $\mathcal{D}$ is a collection of dense sets, and G intersects all sets in $\mathcal{D}$, then G is called $\mathcal{D}$-generic.

Lemma: Let $(P, <)$ be a poset and $\mathcal{D}$ a countable collection of dense subsets of P. Then there is a $\mathcal{D}$-generic filter on P. In fact, for any $p \in P$, there exists such a filter containing p.[3]

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[2]The Axiom of Infinity, in one of its forms, states that there exists a set S such that $\emptyset \in S$ and if $x \in S$ then $x \cup \{x\} \in S$. Assuming no other sets are added to S, this is the usual set-theoretic construction of the natural numbers.

[3]Jech p. 203.

Proof: Let D_i enumerate the sets in $\mathcal{D}$. Let $p_0 = p$ as specified in the theorem, and for all n let p_n be stronger than p_{n-1} such that $p_n \in D_n$. Then the set $G = \{q \in P \mid \exists n \in \omega, q \geq p_n\}$ is a $\mathcal{D}$-generic filter on P and $p \in G$.

2.3. **Examples.** [4]

2.3.1. *Cohen Generic Reals.* Let P be a poset (in some ground model $\mathcal{M}$) whose elements are finite sequences of 0s and 1s, so that $p \in P$ has the form $\langle p(0), \ldots, p(n-1)\rangle$. Let $p < q$ if p extends q. If G is $\mathcal{M}$-generic, let $f = \bigcup G$, i.e. the union $\bigcup_{p \in G} p$ of all elements of G. Then f is a function from ω to $\{0,1\}$—i.e. a characteristic function of a subset of $\mathbb{N}$. The sets $D_n = \{p \in P \mid n \in \mathrm{dom}(p)\}$ are dense in P for all natural numbers n.[5] Thus, $f : \omega \to \{0,1\}$ is a total function. It is the characteristic function of some $A \subset \omega$, and A is called a *Cohen generic real*. f is not in $\mathcal{M}$, because for all $g : \omega \to \{0,1\}$, the set $D_g = \{p \in P \mid p \not\subset g\}$ is dense. (This is once again in $\mathcal{M}$ by Separation.) Thus, D_g intersects G, so $f \neq g$.

2.3.2. *Collapsing a Cardinal.* Elements of P are finite sequences of countable ordinals, and $p < q$ if p extends q. For all $\alpha < \omega_1^{\mathcal{M}}$ (i.e., the set A in $\mathcal{M}$ such that $\mathcal{M} \models A$ is the first uncountable ordinal), the set $E_\alpha = \{p \in P \mid \alpha \in \mathrm{range}(p)\}$ is dense. Thus, if G is generic, then it intersects all E_α. So the sequence $f = \bigcup G$ has all ordinals $\alpha < \omega_1^{\mathcal{M}}$ in its range, making it a surjective function from ω to $\omega_1^{\mathcal{M}}$. Thus, in $\mathcal{M}[G]$, $\omega_1^{\mathcal{M}}$ is countable rather than uncountable (and there is some other set $\omega_1^{\mathcal{M}[G]}$, such that $\mathcal{M}[G] \models \omega_1^{\mathcal{M}[G]}$ is uncountable). We say we have *collapsed* ω_1 when we produce this type of result. A similar process can be used to collapse any regular cardinal κ (by replacing $\omega_1^{\mathcal{M}}$ in the definition of our poset with $\kappa^{\mathcal{M}}$).

2.4. **Boolean-Valued Models.**

A *Boolean-valued model* $\mathcal{A}$, given a complete boolean algebra B, is a model in which the relations of equality and containment have values in B rather than always being strictly true or false. Formally, $\mathcal{A}$ consists of a *Boolean universe* $\mathcal{A}$ and a B-valued function on formulas $\varphi(a_1, \ldots, a_n)$, denoted $||\varphi(a_1, \ldots, a_n)||$. This function represents a truth assignment, where truth values are taken from B. It is defined inductively from the functions $||\cdot = \cdot||, ||\cdot \in \cdot|| : A \times A \to B$, which satisfy the following constraints:

- $||x = x|| = 1$
- $||x = y|| = ||y = x||$
- $||x = y|| \cdot ||y = z|| \leq ||x = z||$
- $||x \in y|| \cdot ||v = x|| \cdot ||w = y|| \leq ||v \in w||$

If $||\varphi||$ (the truth value of φ) is 1, we say φ is *valid* in $\mathcal{A}$. We say $\mathcal{A}$ is a Boolean-valued model of ZFC if all the axioms of ZFC are valid in $\mathcal{A}$.

Taking ultrafilters (which can be viewed as Boolean algebra homomorphisms $U : B \to \{0,1\}$) provides a way to ensure the existence of generic filters in forcing posets.

2.4.1. *The Boolean-valued Von Neumann Universe.* Just as we have a canonical $\{0,1\}$-valued model V of ZFC, we have a canonical B-valued model, V^B for any complete Boolean algebra B. Like V, V^B is a proper class, rather than a set model. It is constructed by transfinite induction, in a manner analoguous to the construction of V in section 2.1:

- $V_0^B = \emptyset$
- $V_{\alpha+1}^B$ is the set of all B-valued functions x with $\mathrm{dom}(x) \subset V_\alpha^B$
- $V_\lambda^B = \bigcup_{\beta<\lambda} V_\beta^B$ if λ is a limit cardinal
- V^B is the union of V_α^B for all $\alpha \in Ord$

[4] These examples are from Jech, p. 202.

[5] These sets are in $\mathcal{M}$ because $\mathcal{M} \models ZFC$ (so any set formed by the axioms of ZFC from a set already in $\mathcal{M}$ is in $\mathcal{M}$) and D_n are formed from P by the Axiom of Separation. In general we will assume that dense subsets of P formed by restricting P with a predicate are in $\mathcal{M}$, as the proof is a one-step application of Separation.

That is, the sets of rank α are Boolean-valued functions of the sets of lower rank, just as the sets of rank α in V can be viewed (via their characteristic functions) as $\{0,1\}$-valued functions of the sets of lower rank. Every set x of V has a canonical *name* in V^B, i.e. a function $\breve{x}$ in V^B corresponding canonically to x. These names are defined such that $\breve{\emptyset} = \emptyset$, and $\breve{x}$ is the function whose domain is $\{\breve{y} \mid y \in x\}$ such that for all $y \in x$, $\breve{x}(\breve{y}) = 1$. This definition of names can be extended to the Boolean-valued analogue $\mathcal{M}^B$ of any ground model $\mathcal{M}$. When we construct $\mathcal{M}[G]$ from $\mathcal{M}^B$, we will need to use names in order to specify properties of elements of $\mathcal{M}^B$—and thus $\mathcal{M}[G]$—from within the ground model $\mathcal{M}$.

Names preserve the property of being an ordinal. That is, it can be proven that for all $x \in V^B$,

$$||x \text{ is an ordinal}|| = \sum_{\alpha \in Ord} ||x = \breve{\alpha}||$$

The following theorem can also be proven, although the proof is long, tedious, and unenlightening:
Theorem: Every axiom of ZFC is valid in V^B.

2.4.2. *Generic Ultrafilters.* If we have an ultrafilter $G \subset B$ which is generic over $\mathcal{M}$ (i.e., the set G intersects all dense subsets of B^+ that are elements of $\mathcal{M}$—this is analoguous to the definition of $\mathcal{M}$-generic for posets), we can produce a $\{0,1\}$-valued model $\mathcal{M}[G]$ through the following process. For all $x \in M^B$, we define x^G by induction on rank: $\emptyset^G = \emptyset$, and $x^G = \{y^G \mid x(y) \in G\}$ (that is, values of B that are in G are taken to be true, and other values are taken to be false). Then, we define $M[G] = \{x^G \mid x \in M^B\}$. This allows us to prove the following lemma:
Lemma: $\mathcal{M}[G] \models \varphi(x_1^G, \dots, x_n^G)$ iff $||\varphi(x_1, \dots, x_n)|| \in G$.
Proof: If φ is an atomic formula, then it has the form $x_i = x_j$ or $x_i \in x_j$. By double induction on the rank of x_i and x_j, it can be shown that $x_i^G = x_j^G$ iff $||x_i = x_j|| \in G$, and similarly $x_i^G \in x_j^G$ iff $||x_i \in x_j|| \in G$. (See Jech Lemma 14.28 for details.)

Assume $\varphi = \psi \wedge \theta$ where by inductive hypothesis $\mathcal{M}[G] \models \psi$ iff $||\psi|| \in G$ and similarly for θ. $\mathcal{M}[G] \models \varphi$ iff $\mathcal{M}[G] \models \psi$ and $\mathcal{M}[G] \models \theta$. $||\varphi|| = ||\psi|| \cdot ||\theta||$ and by properties of ultrafilters, finite sums and products of elements of G are still in G. So $||\varphi|| \in G$ if $\mathcal{M}[G] \models \varphi$. Now assume that $||\varphi|| \in G$. Then $||\psi|| \geq ||\varphi||$ and $||\theta|| \geq ||\varphi||$, so $||\psi||, ||\theta|| \in G$ by properties of ultrafilters. Thus $\mathcal{M}[G]$ models both ψ and θ. (The proof for $\psi \vee \theta$ proceeds similarly.)

Now assume $\varphi = \neg\psi$ where $\mathcal{M}[G] \models \psi$ iff $||\psi|| \in G$, and that $\mathcal{M}[G] \models \varphi$, so $\mathcal{M}[G] \not\models \psi$. So $||\psi|| \notin G$. Because G is an ultrafilter, either $||\psi||$ or its complement, $||\varphi||$, must be in G, so $||\varphi|| \in G$. For the other direction, if $||\varphi|| \in G$, then $||\psi|| \notin G$ because a set and its complement can't both be in an ultrafilter. So $\mathcal{M}[G] \not\models \psi$, so $\mathcal{M}[G] \models \varphi$ as desired.

Now assume $\varphi = \exists x\, \psi(x)$, ψ as above. $\mathcal{M}[G] \models \varphi$ iff there is $x \in M[G]$ such that $\mathcal{M}[G] \models \psi(x)$. Thus, there is $x \in M^B$ such that $\mathcal{M}[G] \models \psi(x^G)$, which is equivalent to $\exists x \in M^B\, ||\psi(x)|| \in G$ by hypothesis. So $\sum_{x \in M^B} ||\psi(x)|| \in G$, so $||\exists x\, \psi(x)|| \in G$. (The proof for $\forall x\, \psi(x)$ proceeds similarly.)

We may now state the following key theorem:
Generic Model Theorem: Let $\mathcal{M}$ be a transitive model of ZFC, and $(P, <)$ a notion of forcing in $\mathcal{M}$. If $G \subset P$ is generic over P, then there is a transitive model $\mathcal{M}[G]$ with the following properties:

- $\mathcal{M}[G] \models$ ZFC
- $M \subset M[G]$ (where $M, M[G]$ are the universes of $\mathcal{M}, \mathcal{M}[G]$)
- $Ord^{\mathcal{M}[G]} = Ord^{\mathcal{M}}$
- if $\mathcal{N}$ is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.
- $G \in M[G]$

To prove this, we must define a relation that links statements about $\mathcal{M}[G]$ to elements of the notion of forcing P.

2.5. **The Forcing Relation.**

2.5.1. *Required Lemma.* We denote a relation $\Vdash$ (read: *forces*) between elements of a forcing notion $p \in P$ and statements φ with parameters $\dot{a_1}, \ldots, \dot{a_n}$ that are names of elements of $\mathcal{M}$. In order to define this, we will use the following lemma, which intuitively states that notions of forcing may be converted into Boolean algebras, in order to form Boolean-valued models.[6]

Lemma: For every partially ordered set $(P, <)$ there is a complete boolean algebra $B(P)$ and a map $e : P \to B(P)^+$ such that the following conditions hold:

- e is order-preserving $(p \leq q \Rightarrow e(p) \leq e(q))$
- p, q are compatible iff $e(p) \cdot e(q)$ is nonzero
- the image $e(P)$ is dense in $B(P)$.

Proof: First we will consider the case where $(P, <)$ is *separative*, i.e. for all $p \not\leq q \in P$, there exists $r \leq p$ incompatible with q. In this case, define a *cut* (similar to a Dedekind cut) of P as a set $U \subset P$ such that if $p \leq q$ and $q \in U$, then $p \in U$. U_p will denote the cut generated by p, i.e. $\{x \mid x \leq p\}$. A cut is *regular* if for all $p \notin U$, there is q stronger than p such that U_q does not intersect U. As a result of separativity, all U_p are regular.

Let B be the set of all regular cuts of P. The intersection of every collection of regular cuts is a regular cut, as is the union of every collection of regular cuts, so B has meets and joins. The complement of an element $u \in B$ is the set $\{p \mid U_p \cap u = \emptyset\}$. $\emptyset$ and P serve as 0 and 1. The verification that this satisfies the complete Boolean algebra axioms is left as an exercise. The function e, defined as $e(p) = U_p$, can be verified to satisfy the required axioms.

Now assume that P is not separative. Define an equivalence relation on P, $x \approx y$ iff for all z, x is compatible with z iff y is compatible with z. Then let $Q = P/\approx$ be the *separative quotient* of P, with a partial ordering given by $[x] \leq [y]$ iff for all $z \leq x$, z and y are compatible. Since Q is separative, we can define B as the complete Boolean algebra of all regular cuts of Q, as above, and so we have $e : Q \to B$. Composing this with the map $h(x) = [x]$ (which is $\leq$-preserving and preserves compatibility) gives $e \circ h : P \to B$, which is the desired function.

2.5.2. *The Forcing Relation.* Given e and $B(P)$, the forcing relation is defined as follows:

$$p \Vdash \varphi(\dot{a_1}, \ldots, \dot{a_n}) \quad \text{if and only if} \quad e(p) \leq ||\varphi(\dot{a_1}, \ldots, \dot{a_n})||$$

where $\dot{a_i}$ is an element of $M^{B(P)}$, also called a *P-name*.

Now we can prove the Generic Model Theorem.

Proof: Let $(P, <)$ be a notion of forcing in $\mathcal{M}$, with an $\mathcal{M}$-generic filter $G \subset P$. For every P-name $x \in M^P (= M^{B(P)})$, we define x^G inductively, such that $\emptyset^G = \emptyset$ and $x^G = \{y^G \mid \exists p \in G, e(p) \leq x(y)\}$. Then we let the universe of $\mathcal{M}[G]$ be $\{x^G \mid x \in M^P\}$. Now we use the generic filter G on P to define a generic ultrafilter H on B: $H = \{p \in B \mid \exists p \in G\, e(p) \leq u\}$—i.e., H is the set of elements above elements of $e(P)$. H is $\mathcal{M}$-generic because it's generated from the $\mathcal{M}$-generic filter G, and it can be easily checked that $x^G = x^H$ for all $x \in M^B$. Thus, $\mathcal{M}[G] = \mathcal{M}[H]$.

The first four conditions of the theorem follow. To verify that $G \in M[G]$, we can write G as $\{p \in P \mid e(p) \in H\}$, which gives $G \in M[H] = M[G]$, as desired.

The following result states that every statement that is true in the new model $\mathcal{M}[G]$ was forced by some element of the generic filter G.

The Forcing Theorem: Let $(P, <)$ be a notion of forcing in the ground model $\mathcal{M}$. If σ is a sentence in the language of set theory with P-names as parameters, then for all $G \subset P$ generic over $\mathcal{M}$, $\mathcal{M}[G] \models \sigma$ iff there is $p \in G$ such that $p \Vdash \sigma$.

Proof: Assume that $\mathcal{M}[G] \models \sigma$. Using the proof of the Generic Model Theorem, there is an ultrafilter H of $B(P)$ such that $\mathcal{M}[G] = \mathcal{M}[H]$. Then $\mathcal{M}[H] \models \sigma$ implies $||\sigma|| \in H$. By the

[6] The forcing relation can be defined without this lemma, as illustrated in Kunen. Kunen defines $p \Vdash \varphi$ iff for *all* $\mathcal{M}$-generic $G \subset P$, $p \in G$ implies that φ is true in $\mathcal{M}[G]$. However, this approach, which makes no reference to Boolean-valued models, makes proving the properties of the forcing relation more difficult.

construction of H, all elements of H are above some element of $e(P)$, so there is $p \in P$ such that $e(p) \leq ||\sigma||$. By the definition of the forcing relation, we get that $p \Vdash \sigma$. Now assume that there exists $p \in P$ such that $p \Vdash \sigma$. So $e(p) \leq ||\sigma||$ by definition, so $||\sigma|| \in H$, implying $\mathcal{M}[H] = \mathcal{M}[G] \models \sigma$ as desired.

3. Major Forcing Results

3.1. **Consistency of the Negation of the Continuum Hypothesis.** To construct a model of ZFC in which CH is false, we define the notion of forcing $P = \{p : \omega_2 \times \omega \to \{0,1\} \mid p \text{ is a finite partial function}\}$, where p is stronger than q iff p extends q. With G a generic filter on P, $f = \bigcup G$ will be a total function $f : \omega_2^{\mathcal{M}} \times \omega \to \{0,1\}$, defining $\omega_2^{\mathcal{M}}$ Cohen generic reals.

In order for $\mathcal{M}[G]$ to model $\neg$CH, we must make sure that $\omega_2^{\mathcal{M}} = \omega_2^{\mathcal{M}[G]}$. The easiest way to show this is to verify that P satisfies the *countable chain condition* (abbreviated *c.c.c.*), defined as the condition that every antichain in P is countable.

Theorem: If P satisfies the countable chain condition and G is an $\mathcal{M}$-generic filter of P, then for all limit ordinals α, $\mathrm{cf}^{\mathcal{M}}(\alpha) = \mathrm{cf}^{\mathcal{M}[G]}(\alpha)$. As a result, for all cardinals κ, $\kappa^{\mathcal{M}} = \kappa^{\mathcal{M}[G]}$.
Proof: It is enough to show that if $\kappa^{\mathcal{M}}$ is a regular cardinal, so is $\kappa^{\mathcal{M}[G]}$. So assume $\kappa^{\mathcal{M}}$ (from here on just denoted κ) is regular, and let $\lambda < \kappa$. Let $\dot{f}$ be a name, and $p \in P$ such that $p \Vdash \dot{f}$ is a function from λ to κ. For every ordinal $\alpha < \lambda$ define the set A_α to be $\{\beta < \kappa \mid \exists q < p\, q \Vdash \dot{f}(\alpha) = \beta\}$, the set of β that some q extending p forces to be a value for $f(\alpha)$. The set of witnesses $\{q_\beta$ such that $q_\beta \Vdash \dot{f}(\alpha) = \beta \mid \beta \in A_\alpha\}$ is an antichain, because if $\beta \neq \gamma$, then q_β and q_γ are not compatible. By c.c.c, this set is countable. So A_α must also be countable, for all $\alpha < \lambda$. Since κ is regular, there is $\gamma < \kappa$ which is an upper bound to the set $\bigcup_{\alpha<\kappa} A_\alpha$. So for each $\alpha < \lambda$, $p \Vdash \dot{f}(\alpha) < \gamma$. So $p \Vdash \dot{f}$ is bounded below κ for all $\dot{f} \in M^P$ and $p \in P$. Thus in $\mathcal{M}[G]$, for every $\lambda < \kappa$, $\mathrm{cf}(\kappa) > \lambda$. So $\mathrm{cf}(\kappa) = \kappa$, so κ is regular in $\mathcal{M}[G]$ as desired.

It remains to show (a) that P is c.c.c. and (b) that the Cohen generic reals defined by P are distinct. In fact, we can show that a much broader class of notions of forcing is c.c.c., although the proof requires details of infinitary combinatorics (namely the Δ-Lemma) that are outside the scope of this paper.

Lemma: Let P be a set of finite functions on a countable set C, where $p \leq q$ iff p extends q. Assume that for all $p, q \in P$, if $p \cup q$ is a function then it is also in P. Then P has countable chain condition.
Proof: Let $F \subset P$ be uncountable, and W be the set of domains of elements of F. Since C is countable, $|W| = 2^{|C|}$ is uncountable and all its elements are finite. By the Δ-Lemma, there is an uncountable Δ-system $Z \subset W$. That is, there is a finite S such that $X \cup Y = S$ for all distinct $X, Y \in Z$. Let G be the set of all $p \in P$ with domain in Z. Again using the Δ-lemma, we get that there are uncountably many p with identical $p \restriction S$. If p, q have S as their intersection and are identical on that intersection, then they are compatible functions, and thus are compatible as elements of P. So F is not an antichain.

(To apply this proof, we assume that $\mathcal{M}$ is a *countable* transitive model, and thus $\omega_2^{\mathcal{M}} \times \omega$ is countable in V.)

For all $\alpha < \omega_2$, define $f_\alpha(n) = f(\alpha, n)$, so f_α is the α^{th} function defined by G. For all α and n, this is defined, as the sets $D_{\alpha,n} = \{p \in P \mid (\alpha, n) \in \mathrm{dom}(p)\}$ are dense in P. Assume $\alpha \neq \beta$. The set $D = \{p \in P \mid \exists n\, p(\alpha, n) \neq p(\beta, n)\}$ is dense in P, so it intersects G. So $f_\alpha \neq f_\beta$. We can conclude that $\mathcal{M}[G] \models$ there are ω_2 distinct real numbers.

3.2. **Consistency of the Continuum Hypothesis.** Forcing may be used not only to show that ZFC is consistent with ¬CH, but also that it is consistent with CH—i.e. that the continuum hypothesis is independent of ZFC.[7]

First we let P be the poset of countable (not finite!) partial functions from ω_1 to $\mathbb{R}$. The sets $D_x = \{p \in P \mid x \in \mathrm{dom}(p)\}$ and $D_r = \{p \in P \mid r \in \mathrm{range}(p)\}$ are dense, so G intersects all of them. Thus, if G is generic over P, then $\bigcup G$ is a surjection from ω_1 to $\mathbb{R}$, thus showing that $\mathcal{M}[G] \models |\mathbb{R}^{\mathcal{M}}| \leq \omega_1^{\mathcal{M}}$. We know that $\omega_1^{\mathcal{M}} \leq \omega_1^{\mathcal{M}[G]}$, since $M \subset M[G]$. It remains to be shown that $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}[G]}$.

Let f be an element of $\mathbb{R}^{\mathcal{M}[G]}$, and let $\dot{f}$ be a name for that element. If $p \in P$ forces that $\dot{f}$ is a function from ω to $\{0, 1\}$, then we can find a chain of p_n, for all $n \in \omega$, such that $p_n \leq p_{n-1} \leq \cdots \leq p_0 \leq p$ and p_n forces $f(n)$ to have a particular value. This is a countable descending sequence of countable functions, so their union q is also a countable function, and is thus in P. q is, or at least contains, a function from ω to $\{0, 1\}$ which is equal to f, so f is in $\mathcal{M}$. So $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}[G]}$ as desired. Thus, $\mathcal{M}[G] \models |\mathbb{R}| = \omega_1$.

3.3. **Independence of the Axiom of Choice.** If $\mathcal{M} \models \mathrm{AC}$, then so does $\mathcal{M}[G]$. However, $\mathcal{M}[G]$ may have submodels which model ¬AC. In order to construct one such submodel, we will add countably many Cohen generic reals to $\mathcal{M}$, then let $\mathcal{N}$ be the hereditarily ordinal-definable sets of $\mathcal{M}[G]$ and show that a well-ordering of our new reals is *not* hereditarily ordinal-definable, and thus not in $\mathcal{N}$. To make this easier, we assume that $\mathcal{M} \models (V = L)$.

Let P be the poset of finite partial functions from $\omega \times \omega$ to $\{0, 1\}$. If G is a generic filter on P, then let $a_i = \{n \in \omega \mid \exists p \in G\, p(i, n) = 1\}$, $A = \{a_i \mid i \in \omega\}$. $\dot{A}$ and $\dot{a}_i$ will be the canonical names in $\mathcal{M}[G]$ for A and its elements. For every pair $\dot{a}_i, \dot{a}_j$ and $p \in P$ there exists $q \supset p$ and $n \in \omega$ such that $q(i, n) = 1$ and $q(j, n) = 0$. So every $p \in P$ forces that all $\dot{a}_i$ are distinct.

Let $N \subset M[G]$ be the class of all sets in $\mathcal{M}[G]$ that are hereditarily ordinal-definable with parameters in A—that is, the transitive closure of each element of $\mathcal{N}$ must be ordinal-definable. (Intuitively, $\mathcal{N}$, all its elements, all the elements of its elements, and so on, must be ordinal-definable.) Thus, $\mathcal{N}$ is a submodel of $\mathcal{M}[G]$, and is in fact a transitive model of ZFC. It remains to be shown that there is no well-ordering of A in $\mathcal{N}$.

Assume that $f : A \to Ord$ is one-to-one and ordinal-definable with parameters in A. Then there is some finite sequence $s = \langle x_0, \ldots, x_k \rangle$ such that f is ordinal-definable with parameters $x_0, \ldots, x_k, A$. For any $a \in A$ which is not some x_i, there is some φ such that $\mathcal{M}[G] \models$ a is the unique set such that $\varphi(a, \alpha_1, \ldots, \alpha_n, s, A)$, where α_i are ordinals.

Let $p_0 \Vdash \varphi(\dot{a}, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A})$. We will show that there exists a name $\dot{b}$ and $q \in P$ extending p_0 such that $q \Vdash \dot{a} \neq \dot{b}$ and $q \Vdash \varphi(\dot{b}, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A})$. There are $i, i_0, \ldots, i_k$ and p_1 extending p_0 such that $p_1 \Vdash \dot{a} = \dot{a}_i$ and for all j, $p_1 \Vdash \dot{a}_{i_j} = \dot{x}_j$. Let j be a natural number distinct from i, such that for all m, $p_1(j, m)$ is not defined.

We define $\pi : \omega \to \omega$ as the transposition $(i\, j)$. This permutation induces an automorphism π of P, and thus of B and $\mathcal{M}^B$. $\pi(\dot{a}_i) = \dot{a}_j$ and vice versa, and for all other n, $\pi(\dot{a}_n) = \dot{a}_n$. We also have $(\pi p_1)(i, m)$ not defined for all m, but p_1 and πp_1 agreeing at all other points. So p_1 and πp_1 are compatible. Let q extend both of them. We get $q \Vdash \varphi(\dot{a}_i, \dots) \wedge \varphi(\dot{a}_j, \dots) \wedge \dot{a}_i \neq \dot{a}_j$, contradicting the hypothesis that f is ordinal-definable with parameters in A. Thus, $\mathcal{N} \models$ there exists a set that cannot be well-ordered. So, as desired $\mathcal{N} \models \mathrm{ZF} + \neg\mathrm{AC}$.

[7]The traditional method of showing that CH is consistent with ZFC does not rely on forcing, but instead uses the fact that any model of ZFC has a submodel that models $V = L$, and that $(V = L) \vdash \mathrm{CH}$. This method is generally considered easier, but is outside the scope of this paper.

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