

CLASSICAL MECHANICS: THE THREE-BODY PROBLEM

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ABSTRACT. The Three-Body Problem is one of the oldest unsolved problems of classical mechanics. It arose as a natural extension of the Two-Body Problem that Newton solved in his *Principia* in 1687. Since then, the problem has motivated the development of numerous techniques in classical mechanics as well as dynamical systems. Understanding the Three-Body Problem is thus an excellent objective for our introduction to classical mechanics. This paper introduces Lagrangian and Hamiltonian mechanics before presenting two important theorems which describe the phase flows that are the subject of Hamiltonian mechanics- Liouville's Theorem and the Poincaré Recurrence Theorem. It concludes with a discussion about the analytical unsolvability of the Three-Body Problem.

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THE BIG PICTURE

Mechanics is a branch of physics concerned with the behavior of physical objects in space. At its core lies the question: *If the state of a dynamic system is known initially, and something is done to it, how will the state change in time as a response?* [1].

Mechanics can be split into two subfields with different characteristics:

Classical mechanics: Deals with macroscopic phenomenon, variables that are continuous, and is deterministic in nature.

Quantum mechanics: Deals with phenomenon at the subatomic scale, variables that are discrete and is probabilistic in nature.

Classical mechanics may be further divided into three formulations in order of increasing abstraction:

Newtonian mechanics: Based on a vectorial formulation in Cartesian space.

In contrast, analytical formulations such as Lagrangian and Hamiltonian mechanics deal with scalar properties of motion like kinetic and potential energies.

Lagrangian mechanics: Operates in a generalized coordinate space.

Hamiltonian mechanics: Operates in a phase space.

Formulations that are more abstract are often more powerful. In order to exploit symmetries of a system, it is often convenient to transform coordinate systems. However, when dealing in vectors, these transformations are often tedious and messy. Dealing in scalars allows us to track the evolution of the system from afar without having to keep tabs on each of the vectors in the system.

Section 1. In order to trace the derivation of the central equation of Hamiltonian mechanics from its beginnings in Newtonian mechanics, we review Lagrangian mechanics; first, deriving the Euler Lagrange equation using the calculus of variations; then, showing that Newton's Second Law of Motion, the Euler Lagrange equation, and Hamilton's Principle of Least Action are equivalent in determining a system's motion. Finally, by abstracting the Euler Lagrange equation into generalized coordinates, we arrive at a more powerful formulation which will allow us to solve the Two-Body Problem.

Sections 2, 3, 4. In the second half, we derive the Hamiltonian from the Lagrangian using Legendre transformations before introducing a couple of well-known theorems in dynamical systems: Liouville's theorem and the Poincare Recurrence theorem. Finally, we conclude by examining the unsolvability of the Three-Body Problem.

1. LAGRANGIAN MECHANICS

The endpoint of this section is the derivation of Hamilton's Principle of Least Action *ie.* the path that a mechanical system takes is one in which the Euler Lagrange equation is satisfied at every point along the path. This result is remarkable because the Euler Lagrange equation is a result of the calculus of variations rather than physical observation. That physical systems adhere to this relation seems to be nothing more than good fortune.

1.1. The Euler Lagrange Equation.

Definition 1.1. Let X be a Banach space. A curve is a continuous map $\gamma : [t_0, t_1] \rightarrow X$. A functional Φ is a mapping from the space of curves Γ (ie. paths) in a space X to the space of real numbers ie. $\Phi : \Gamma \rightarrow \mathbb{R}$.

Definition 1.2. Given a curve γ , another curve $\gamma + h$ which varies from γ by h , as well as a function L which maps from a space X to the real numbers, the derivative $F(h)$ of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$ is given by

$$F(h) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] h dt + \left(\frac{\partial L}{\partial \dot{x}} h \right) \Big|_{t_0}^{t_1}.$$

For an explanation of why this equation describes the derivative of a functional, refer to Arnol'd [2].

Definition 1.3. An extremal of a differential functional $\Phi(\gamma)$ is a curve γ such that $F(h) = 0$ for all h .

Lemma 1.4. Fundamental Lemma of Variations

If a continuous function $f(t)$, $t_0 \leq t \leq t_1$, satisfies $\int_{t_0}^{t_1} f(t) h(t) dt = 0$ for any continuous function $h(t)$ with $h(t_0) = h(t_1) = 0$ then $f(t) = 0$

This is a proof by contradiction that uses the fact that h is a continuous function. By constructing h such that it is non-zero at some point, its continuity implies that it has a non-zero integral over some interval around that point. As such, for the integral of $f(t)h(t)$ to be 0, $f(t)$ must be 0.[3]

Proof. Assume we have a continuous function f such that $f(t^*) > 0$ for all t^* , $t_0 \leq t^* \leq t_1$, (so that $f(t^*) \neq 0$). Since f is continuous, $f(t) > c$ where c is some constant, in some neighborhood Δ centered about t^* .

We may also construct a continuous function $h(t)$ such that

$$h(t) = \begin{cases} 0, & t \text{ outside } \Delta \quad (\text{so } h(t_0) = h(t_1) = 0 \text{ is satisfied}), \\ 1, & t^* - \frac{\Delta}{4} < t < t^* + \frac{\Delta}{4}, \\ > 0, & t^* - \frac{\Delta}{2} < t < t^* + \frac{\Delta}{2}. \end{cases}$$

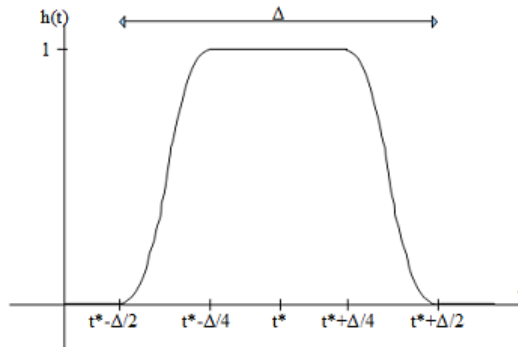


FIGURE 1. h

As such,

$$\begin{aligned} \int_{t_0}^{t_1} f(t) h(t) dt &\geq c \cdot \Delta \cdot \frac{\Delta}{2} \\ &> 0 \end{aligned}$$

which contradicts the hypothesis. Therefore, $f(t) = 0$ for all t such that $t_0 \leq t \leq t_1$. \square

Lemma 1.5. *The curve $\gamma : x = x(t)$ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$ on the space of curves passing through the points $x(t_0) = x_0, x(t_1) = x_1$ if and only if $\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$ along the curve $x(t)$.*

Proof. From Def 1.3, $F(h)$, the derivative of the extremal of the functional $\Phi(\gamma)$, is 0.

$$F(h) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] h dt + \left(\frac{\partial L}{\partial \dot{x}} h \right) \Big|_{t_0}^{t_1} = 0.$$

We may eliminate the last term since $\left(\frac{\partial L}{\partial \dot{x}} h \right) = 0$ because $h(t_0) = h(t_1) = 0$.

The remaining term now fits the form $\int_{t_0}^{t_1} f(t) h(t) dt = 0$. Applying the Fundamental Lemma of Variations, we conclude

$$f(t) = \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] = 0.$$

Conversely, if $f(t) = 0$ then $F(h) = 0$. This implies that the curve is an extremal. \square

Definition 1.6.

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \quad \text{is known as the Euler Lagrange equation.}$$

Remark 1.7. Lemma 2.5 may be restated: Curve γ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t) dt$ on the space of curves joining (t_0, x_0) and (t_1, x_1) if and only if the Euler Lagrange equation is satisfied along γ .

1.2. Hamilton's Principle of Least Action.

Definition 1.8. The Lagrangian of a system \mathcal{L} is defined as the difference between the kinetic energy T and potential energy V of the system. *i.e.* $\mathcal{L} \equiv T - V$.

Definition 1.9. The action of a system S is the integral of the Lagrangian \mathcal{L} of the system. *i.e.* $S = \int_{t_0}^{t_1} \mathcal{L} dt$.

Theorem 1.10. Hamilton's Principle of Least Action

Motion of the mechanical system coincides with extremals of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(x, \dot{x}, t) dt$ where \mathcal{L} is the Lagrangian.

Proof. By Lemma 1.5, a curve that is an extremal of a functional, is identical to a curve for which the Euler Lagrange equation is satisfied at all points. As such, all we need to show is that the motion of a mechanical systems, as governed by Newton's second law $F_{x_i} = \dot{p}_{x_i}$, satisfies the Euler Lagrange equation at all points.

For a mechanical system, the potential and kinetic energies can be expressed as $V = V(x)$ and $T = \frac{1}{2} \sum m_i \dot{x}_i^2$ respectively. Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = m \dot{x}_i = p_{x_i} \quad \frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_{x_i}$$

$$\Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{d}{dt} p_{x_i} = F_{x_i} = -\frac{\partial V}{\partial x_i} = \frac{\partial \mathcal{L}}{\partial x_i},$$

the Euler Lagrange equation is indeed satisfied. \square

Remark 1.11. That Hamilton's principle is named the Principle of *Least Action* is simply due to historical reasons. In fact, physical systems follow the path for which action is at a stationary point *ie.* $F(h) = 0$.

Remark 1.12. From Theorem 1.10, we have now proved that for a mechanical system, the following three statements are equivalent:

- (1) The system's path is determined by Newton's second law $F = \dot{p}$.
- (2) The path is determined by the Euler Lagrange equation.
- (3) The path is determined by Hamilton's Principle of Least Action.

1.3. Generalized Coordinates. Up to this point, we have been deliberately vague about the coordinate system which we have adopted. In this section, we shall learn that the Lagrangian is invariant under coordinate transformations. This means that we have the luxury of choosing a coordinate system that is convenient for our purposes. In the following section, we shall observe how the ability to switch coordinate systems allows us to solve the Two-Body Problem easily.

Since no single coordinate system is intrinsically better than another, we may introduce a set of generalized coordinates and associated descriptors to represent any particular system we might be using.

Definition 1.13.

Generalized coordinates: $\{q_1, q_2, \dots, q_n\}$

Generalized velocities: $\{\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n\}$

Generalized forces: $\frac{\partial \mathcal{L}}{\partial q_j}$

Generalized momenta: $\frac{\partial \mathcal{L}}{\partial \dot{q}_j}$

Proposition 1.14. *A system of N particles, originally described in terms of cartesian coordinates and perhaps time, can be expressed in terms of generalized coordinates and time, and vice versa:*

$$(1.15) \quad x_i = x_i(q_1, q_2, \dots, q_n, t) \quad \text{for } i \in [1, 3N]$$

$$(1.16) \quad q_j = q_j(x_1, x_2, \dots, x_{3N}, t) \quad \text{for } j \in [1, n].$$

Proposition 1.17. *If the system is allowed to vary, the change in cartesian coordinates can also be expressed in terms of generalized coordinates, and vice versa:*

$$(1.18) \quad \delta x_i = \frac{\partial x_i}{\partial q_1} \delta q_1 + \frac{\partial x_i}{\partial q_2} \delta q_2 + \dots + \frac{\partial x_i}{\partial q_n} \delta q_n = \sum_{j=1}^n \frac{\partial x_i}{\partial q_j} \delta q_j$$

$$(1.19) \quad \delta q_j = \frac{\partial q_j}{\partial x_1} \delta x_1 + \frac{\partial q_j}{\partial x_2} \delta x_2 + \dots + \frac{\partial q_j}{\partial x_{3N}} \delta x_{3N} = \sum_{x=1}^{3N} \frac{\partial q_j}{\partial x_i} \delta x_i.$$

By taking the time derivative, velocities can be expressed in terms of generalized coordinates:

$$(1.20) \quad \dot{x}_i = \frac{\partial x_i}{\partial t} = \frac{\partial x_i}{\partial q_1} \dot{q}_1 + \frac{\partial x_i}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x_i}{\partial q_n} \dot{q}_n = \sum_{j=1}^n \frac{\partial x_i}{\partial q_j} \dot{q}_j$$

$$(1.21) \quad \dot{q}_j = \frac{\partial q_j}{\partial t} = \frac{\partial q_j}{\partial x_1} \dot{x}_1 + \frac{\partial q_j}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial q_j}{\partial x_{3N}} \dot{x}_{3N} = \sum_{x=1}^{3N} \frac{\partial q_j}{\partial x_x} \dot{x}_x.$$

Theorem 1.22. *The Lagrangian \mathfrak{L} is invariant under coordinate transformation.*

This means that coordinate transformation gives a Lagrangian with the same *functional dependence* of the old and new variables, thus allowing it to describe the same equation of motions; namely, the Euler Lagrange equations.

Proof. To prove this, we need to show that the Euler Lagrange equation is satisfied despite transformation to generalized coordinates, ie.

$$\mathfrak{L}(x_1, x_2, \dots, x_{3N}, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_{3N}) \rightarrow \mathfrak{L}(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n).$$

Since the generalized position q is a function of position x and velocities \dot{x} , ie. $\mathfrak{L}(x, \dot{x})$,

$$(1.23) \quad \frac{\partial \mathfrak{L}}{\partial q_j} = \sum_{i=1}^{3N} \left[\frac{\partial \mathfrak{L}}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_j} + \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{\partial \dot{x}_i}{\partial q_j} \right].$$

According to equation (1.21), unlike the generalized position q , the generalized velocity \dot{q} is only a function of velocity \dot{x} .

$$(1.24) \quad \frac{\partial \mathfrak{L}}{\partial \dot{q}_j} = \sum_{i=1}^{3N} \left[\frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \right]$$

From equation (1.20),

$$(1.25) \quad \dot{x}_i = \sum_{j=1}^n \frac{\partial x_i}{\partial q_j} \dot{q}_j \Rightarrow \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}.$$

Substituting this result into equation (1.24), taking the total time derivative and applying the product rule,

$$(1.26) \quad \frac{d}{dt} \frac{\partial \mathfrak{L}}{\partial \dot{q}_j} = \frac{d}{dt} \left[\sum_{i=1}^{3N} \left[\frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{\partial \dot{x}_i}{\partial q_j} \right] \right]$$

$$(1.27) \quad = \sum_{i=1}^{3N} \left[\frac{d}{dt} \left(\frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \right) \cdot \frac{\partial x_i}{\partial q_j} + \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_j} \right) \right].$$

The second term can be simplified using equation (1.20) again:

$$(1.28) \quad \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_j} \right) = \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \sum_{k=1}^n \frac{\partial}{\partial q_k} \frac{\partial x_i}{\partial q_j} \frac{\partial q_k}{\partial t} + \frac{\partial}{\partial t} \frac{\partial x_i}{\partial q_j}$$

$$(1.29) \quad = \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{\partial}{\partial q_j} \left(\sum_{k=1}^n \frac{\partial x_i}{\partial q_k} \dot{q}_k + \frac{\partial x_i}{\partial t} \right)$$

$$(1.30) \quad = \frac{\partial \mathfrak{L}}{\partial \dot{x}_i} \cdot \frac{\partial}{\partial q_j} (\dot{x}_i).$$

Substituting this result back into equation (1.27) and reintroducing the term from equation (1.23),

$$(1.31) \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \sum_{i=1}^{3N} \left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) \cdot \frac{\partial x_i}{\partial q_j} + \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \cdot \frac{\partial \dot{x}_i}{\partial q_j} \right]$$

$$(1.32) \quad - \sum_{i=1}^{3N} \left[\frac{\partial \mathcal{L}}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_j} + \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \cdot \frac{\partial \dot{x}_i}{\partial q_j} \right]$$

$$(1.33) \quad = \frac{\partial x_i}{\partial q_j} \cdot \sum_{i=1}^{3N} \left[\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} \right].$$

Therefore, the Lagrangian in generalized coordinates satisfies the Euler Lagrange equation if and only if it satisfies the equation in its original coordinates. \square

1.4. The Two-Body Problem. [3] We may now apply the Euler Lagrange equation to tackle the Two-Body Problem:

Consider two objects with masses m_1 and m_2 located at r_1 and r_2 in cartesian coordinates respectively. The only forces in the system are the mutual forces of attraction F_{1on2} and F_{2on1} . These two forces are assumed to be conservative and can thus be regarded as gradients of a potential $U(r_1, r_2)$. We assume that these objects are small enough to be considered point masses.

In general, to describe a system of n particles fully, $6n$ variables must be specified. For each point particle, there are three components of position in space and three components of momentum. As such, to solve a two body system, we need 12 independent equations. The crux to solving the Two-Body Problem is reducing the problem to one particle in one dimension. This simplification is done by exploiting conservation laws and invariances in the system.

Change of Variables. We observe that the potential energy of the system does not depend on the absolute position of the objects, but the position of the objects relative to each other *ie.* $U(r_1, r_2) = U(|r_1 - r_2|)$. To exploit the translational invariance of the system, we can introduce a new variable r where

$$(1.34) \quad r = r_1 - r_2.$$

Since rotation typically takes place about the centre of mass of a system, we allow rotational symmetries to be exploited by setting the other variable R to be the position of the centre of mass:

$$(1.35) \quad R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m_1 r_1 + m_2 r_2}{M}$$

where $M = m_1 + m_2$ is the total mass of the system.

Expressing the new variables (R, r) in terms of the old ones (r_1, r_2) , we have

$$(1.36) \quad r_1 = R + \frac{m_2}{M} r \quad \text{and} \quad r_2 = R - \frac{m_1}{M} r.$$

The kinetic energy of the system is therefore given by

$$\begin{aligned}
 (1.37) \quad T &= \frac{1}{2}(m_1\dot{r}_1^2 + m_2\dot{r}_2^2) \\
 &= \frac{1}{2}\left(m_1\left[R + \frac{m_2}{M}r\right]^2 + m_2\left[r_2 = R - \frac{m_1}{M}r\right]^2\right) \\
 &= \frac{1}{2}\left(M\dot{R}^2 + \frac{m_1m_2}{M}\dot{r}^2\right) \\
 &= \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2
 \end{aligned}$$

where $\mu = \frac{m_1m_2}{M}$ is the reduced mass.

Changing Inertial Frame. Translational invariance can be exploited so that only the positions of the particles relative to each other *ie.* variable r , need be described. To do this, we decompose the Lagrangian into two components, one describing the centre of mass and the other describing the relative position:

$$\begin{aligned}
 (1.38) \quad \mathcal{L} &= T - U \\
 &= \left[\frac{1}{2}M\dot{R}^2\right] + \left[\frac{1}{2}\mu\dot{r}^2 - U(r)\right] \\
 &= \mathcal{L}_{cm} + \mathcal{L}_{rel}.
 \end{aligned}$$

Analyzing \mathcal{L}_{cm} using the Euler Lagrange equation, we note that since \mathcal{L}_{cm} is independent of R , \dot{R} must be constant. This simply means that total momentum is conserved. However, we may also imagine the system to be a free particle of mass M and position R which moves with constant velocity. Using translational invariance, we may assume that we are in an inertial frame moving with velocity \dot{R} . As such, the velocity of the centre of mass in this frame is 0.

Since R is out of the picture, all that remains is solving for r and \dot{r} in three dimensions. Analyzing \mathcal{L}_{rel} , the Euler Lagrange equation produces the relation:

$$(1.39) \quad \mu\ddot{r} = -\nabla U(r)$$

which can be similarly interpreted as an equation governing a single particle of mass μ , position r and potential energy $U(r)$. Solving for this ‘‘particle’’ is equivalent to solving the Two-Body Problem.

Reducing 3D to 2D via Conservation of Angular Momentum. We can express the total angular momentum in terms of variable r using the transformations given in equation (1.35):

$$\begin{aligned}
 (1.40) \quad L &= r_1 \times p_1 + r_2 \times p_2 \\
 &= r_1 \times m_1\dot{r}_1 + r_2 \times m_2\dot{r}_2 \\
 &= \frac{m_1m_2}{M^2}(m_2r \times \dot{r} + m_1r \times \dot{r}) \\
 &= r \times \mu\dot{r}.
 \end{aligned}$$

We know that this quantity is conserved, and more importantly, that the vectors produced by the cross product $r \times \dot{r}$ will point in the same direction. This means that the two vectors r and \dot{r} will remain in a fixed plane. As such, we can describe r in two dimensions rather than three.

The natural coordinate system to pick to describe r in 2D is polar coordinates. The Lagrangian from equation (1.38), with the first term removed, is thus

$$(1.41) \quad \mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

Reducing to 1D. Since this Lagrangian is independent of ϕ , the Euler Lagrange equation implies the conservation of angular momentum l with respect to ϕ

$$(1.42) \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{constant} = l.$$

We can use this relation to eliminate ϕ and restate the problem solely in terms of r . By applying the Euler Lagrange equation we obtain

$$(1.43) \quad -\frac{dU}{dr} = \mu \ddot{r} - \mu r \dot{\phi}^2.$$

This is simply an expression of $F = ma$ in the radial direction where mass is now μ and $\mu r \dot{\phi}^2$ is the fictitious centrifugal force.

Substituting equation (1.42) into the fictitious centrifugal force of equation (1.43), we obtain the relation

$$(1.44) \quad \begin{aligned} F_{centrifugal} &= \mu r \dot{\phi}^2 \\ &= \mu r \left[\frac{l}{\mu r^2} \right]^2 \\ &= \frac{l^2}{\mu r^3}. \end{aligned}$$

This allows us to express the force in terms of a centrifugal potential energy

$$(1.45) \quad F_{centrifugal} = -\frac{d}{dr} \left(\frac{l^2}{2\mu r^2} \right) = -\frac{dU_{centrifugal}}{dr},$$

and thus specify an effective potential energy

$$(1.46) \quad U_{eff}(r) = U(r) + U_{centrifugal}(r) = U(r) + \frac{l^2}{2\mu r^2}.$$

We can now rewrite the radial equation (1.43) as

$$(1.47) \quad \mu \ddot{r} = -\frac{d}{dr} [U(r) + U_{centrifugal}(r)] = -\frac{d}{dr} U_{eff}(r).$$

This, together with the expression for conservation of energy

$$(1.48) \quad E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r),$$

allows us to completely describe the system as if we were dealing with one particle with the reduced mass μ in one dimensional r . It is possible to go on from here to solve explicitly for the possible orbits of the system *ie.* r as a function of ϕ , and obtain Kepler's laws of planetary motion.

Making Sense of the 3D Two-Body Problem. After simplifying the problem, r had become treated as some radial distance of a particle relative to some arbitrary axis of rotation. However, in the 3D Two-Body Problem, r is the relative distance between the two bodies as they rotate about their centre of mass. The position of the centre of mass can be determined from initial conditions because it moves with constant velocity. The rotation about the centre of mass can also be determined from initial conditions due to conservation of angular momentum. We evaluate how r and \dot{r} evolve in time based on equations (1.47) and (1.48). This is thus a complete description of the system.

2. HAMILTONIAN MECHANICS

In this section, we derive the equation for the Hamiltonian by applying the Legendre transform to the Lagrangian. With a definition of the Hamiltonian, we then show that the Hamiltonian is analogous to energy.

2.1. Legendre Transform.

The Legendre transform works no differently from other transforms like the Fourier transform and the Laplace transform: it converts a function into an alternate form that may be more convenient to deal with. More specifically, the Legendre transform converts a real-valued function into another real-valued function that is re-parameterized by its derivative. Criteria for applying the Legendre transform to a function $F(x)$ are:

- (1) $F(x)$ is strictly convex *ie.* its second derivative is always positive.
- (2) $F(x)$ is smooth *ie.* its derivative can be taken as many times as needed.

In this section, I provide a geometric interpretation of the Legendre transform before introducing it formally.

Definition 2.1. An epigraph of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is the set of points lying above its graph. *ie.*

$$\text{Epigraph of } f = \{(x, y) : x, y \in \mathbb{R}, y \geq f(x)\}.$$

Proposition 2.2. *The epigraph of a convex function is a convex set.*

Definition 2.3. The supporting hyperplane of a convex set in \mathbb{R}^2 is a line (or hyperplane if we are working in higher dimensions) that:

- (1) intersects the set; and
- (2) has the set on just one side of it.

Remark 2.4. Consider a convex function $y = x^2$. The supporting hyperplanes take the form of $y = mx + c$. For a given non-zero gradient m , notice the following about the y-intercept c :

- c is a large negative number:** The hyperplane does not intersect the set.
- c is a small negative number:** The hyperplane intersects the set at multiple points, implying that there are elements of the convex set on both sides of the hyperplane.
- c is some value in between:** The hyperplane is at a tangent to the curve, making it a supporting hyperplane for the epigraph of $y = x^2$.

Where $m = 0$, the unique y-intercept lies at $c = 0$. In general, given a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, for each slope m there is at most one value of c making

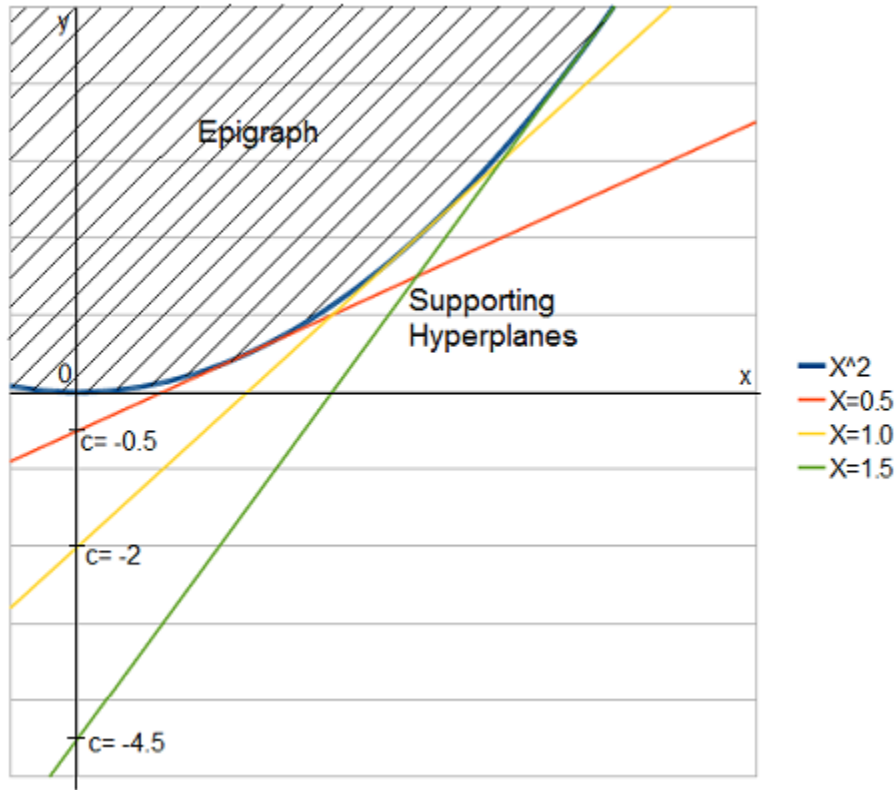


FIGURE 2

$y = mx + c$ a supporting hyperplane of the epigraph of f . Using this fact, a convex set can be defined by its supporting hyperplanes.

Remark 2.5. Since f is a convex function, its gradient is always increasing. As such, every point on the curve has a tangent with unique slope m . As we have observed, m has a bijective relationship with c that is determined by the curve. It is thus possible to associate any convex function f with a function h such that $y = mx + c$ is a supporting hyperplane of the epigraph of f if and only if $h(m) = -c$.

We define $h(m)$ to be the Legendre transform.

Theorem 2.6. *The Legendre transform of function $f(x)$ is:*

$$(2.7) \quad h(m) = \sup_x \{mx - f(x)\}.$$

Proof. In remark (2.4), we noted that supporting hyperplanes intersect $f(x)$ tangentially. We can use this fact to find supporting hyperplanes by picking the infimum of the set of all values of c for which hyperplanes (of the form $y = mx + c$) intersect the epigraph of $f(x)$. This set can be found by fixing m and choosing the values of

c for which:

$$\begin{aligned}
 (2.8) \quad & \exists x_0 \mid mx_0 + c \geq f(x_0). \\
 & \Rightarrow c \geq -mx_0 + f(x_0) \\
 & \Rightarrow \inf(c) = \inf_x \{-mx + f(x)\} \\
 & \Rightarrow h(m) = -c \\
 (2.9) \quad & = \sup_x \{mx - f(x)\}.
 \end{aligned}$$

□

2.2. Hamilton's Equations.

Definition 2.10. The Hamiltonian $H(p, q, t) = p\dot{q} - \mathcal{L}(q, \dot{q}, t)$ is the Legendre transform of the Lagrangian function viewed as a function of \dot{q} (p is the generalized momenta).

Proof. Substituting the appropriate variables into equation (2.9),

$$(2.11) \quad f(x) = \mathcal{L}(q, \dot{q}, t)$$

$$(2.12) \quad x = \dot{q}$$

$$(2.13) \quad m = \frac{\partial f}{\partial x} = \frac{\partial \mathcal{L}}{\partial \dot{q}} = p,$$

we conclude:

$$(2.14) \quad h(m) = mx - f(x) \rightarrow p\dot{q} - \mathcal{L}(q, \dot{q}, t) = H(p, q, t).$$

□

Theorem 2.15. *The system of Lagrange's equations is equivalent to the system of $2n$ first-order equations we call Hamilton's equations:*

$$(2.16) \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$(2.17) \quad \dot{q} = \frac{\partial H}{\partial p}.$$

Proof. The total differential of the Hamiltonian is:

$$(2.18) \quad dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt.$$

This is equal to the total differential of $p\dot{q} - \mathcal{L}(q, \dot{q}, t)$:

$$(2.19) \quad dH = \dot{q}dp - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial t} dt.$$

Comparing coefficients, we obtain Hamilton's equations:

$$(2.20) \quad \frac{\partial H}{\partial p} = \dot{q}$$

$$(2.21) \quad -\frac{\partial H}{\partial q} = \frac{\partial \mathcal{L}}{\partial q}$$

$$= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \quad (\text{Using the Euler Lagrange equation})$$

$$= \frac{d}{dt} p = \dot{p}$$

$$(2.22) \quad -\frac{\partial H}{\partial t} = \frac{\partial \mathcal{L}}{\partial t}.$$

The converse is proved by substituting Hamilton's equations into the differential expansion of H (equation (2.18)). Substituting equations (2.20) and (2.21),

$$(2.23) \quad dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt$$

$$= \dot{q} dp - \dot{p} dq - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= \dot{q} dp - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial t} dt.$$

Integrating on both sides, we conclude:

$$(2.24) \quad H = \int dH = \int \dot{q} dp - \int \frac{\partial \mathcal{L}}{\partial q} dq - \int \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= p\dot{q} - \mathcal{L}(q, \dot{q}, t).$$

□

Theorem 2.25. *Provided the relation between the generalized coordinates and Cartesians is time-independent, ie. $x_i = x_i(q_1, \dots, q_n)$, H is the total energy of the system:*

$$(2.26) \quad H = T + V.$$

Proof. We first express the total kinetic energy $T = \frac{1}{2} \sum_i m_i x_i^2$ in terms of generalized coordinates q_1, \dots, q_n .

In section 2, we obtained an expression for velocities in terms of generalized coordinates (equation (1.20)):

$$\dot{x}_i = \sum_{j=1}^n \frac{\partial x_i}{\partial q_j} \dot{q}_j.$$

Taking the scalar product of this equation with itself,

$$\dot{x}_i^2 = \sum_{j=1}^n \left(\frac{\partial x_i}{\partial q_j} \dot{q}_j \right) \cdot \sum_{k=1}^n \left(\frac{\partial x_i}{\partial q_k} \dot{q}_k \right).$$

Kinetic energy is now described by a triple sum:

$$\begin{aligned}
 (2.27) \quad T &= \frac{1}{2} \sum_{i=1}^n m_i \dot{x}_i^2 \\
 &= \frac{1}{2} \sum_{i=1}^n m_i \sum_{j=1}^n \left(\frac{\partial x_i}{\partial q_j} \dot{q}_j \right) \sum_{k=1}^n \left(\frac{\partial x_i}{\partial q_k} \dot{q}_k \right) \\
 &= \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n A_{j,k} \dot{q}_j \dot{q}_k,
 \end{aligned}$$

where $A_{j,k}$ is the sum:

$$(2.28) \quad A_{j,k} = \sum_{i=1}^n m_i \cdot \frac{\partial x_i}{\partial q_j} \cdot \frac{\partial x_i}{\partial q_k}.$$

From equation (2.27), we may now take the derivative with respect to \dot{q} :

$$(2.29) \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^n A_{j,i} \dot{q}_j.$$

Using the definition of the Hamiltonian (definition (2.14)), we may now substitute our expression for generalized momentum as well as the Lagrangian:

$$\begin{aligned}
 (2.30) \quad H &= \sum_{i=1}^n p_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t) \\
 &= \sum_{i=1}^n \left(\sum_{j=1}^n A_{j,i} \dot{q}_j \right) \dot{q}_i - \mathcal{L}(q, \dot{q}, t) \\
 &= \sum_{j=1}^n \sum_{k=1}^n A_{j,k} \dot{q}_j \dot{q}_k - \mathcal{L}(q, \dot{q}, t) \\
 &= 2T - (T - V) \\
 &= T + V.
 \end{aligned}$$

□

3. PHASE SPACES

Definition 3.1. A phase space is the $2n$ -dimensional space with coordinates $q_1, \dots, q_n; p_1, \dots, p_n$, where q_i are generalized coordinates and p_i are generalized momentum.

Definition 3.2. The phase flow is the one-parameter group of transformations of phase space

$$(3.3) \quad g^t : (p(0), q(0)) \mapsto (p(t), q(t)),$$

where $p(t)$ and $q(t)$ are solutions of Hamilton's system of equations.

3.1. Liouville's Theorem.

Theorem 3.4. *Phase flows preserve volume: for any region D we have*

$$\text{volume of } g^t D = \text{volume of } D.$$

Proof. Let us first note that although we have used the terms “volume”, “area”, and “surface” as if we were working in three dimensions, the phase space is $2n$ dimensional. As such, a more accurate definition should be:

- Volume V : a $2n$ -dimensional region of the phase space.
- Area A : a $2n-1$ dimensional region.
- Surface S : an area bounding a volume.

We can express the change in the volume of region D as

$$(3.5) \quad \frac{dV}{dt} = \int_S n \cdot v \, dA.$$

where n is a vector normal to the surface S , v is the velocity of the flow in phase space, and dA is an infinitesimal area on the surface.

Applying the divergence theorem, we obtain

$$(3.6) \quad \frac{dV}{dt} = \int_V \nabla \cdot v \, dV.$$

Examining the velocity v more closely, we note that

$$(3.7) \quad v = (\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right).$$

Taking the divergence of v ,

$$(3.8) \quad \begin{aligned} \nabla \cdot v &= \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} \\ &= \frac{\partial}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial}{\partial p} \frac{\partial H}{\partial q} \\ &= 0 \end{aligned}$$

since the order of partial derivatives does not matter.

We conclude that

$$(3.9) \quad \frac{dV}{dt} = \int_V \nabla \cdot v \, dV = 0.$$

□

3.2. Poincaré Recurrence Theorem.

Theorem 3.10. *Let g be a volume-preserving continuous one-to-one mapping (such as a phase flow) which maps a finite volume region D of euclidean space onto itself: $gD = D$. Then in any neighborhood U of any point of D there is a point $x \in U$ which returns to U , ie. $g^n x \in U$ for some $n > 0$.*

Proof. Consider the images of the neighborhood U under successive mappings by g : U, gU, g^2U, \dots, g^nU . By Liouville's theorem, all of these images have the same

volume. If they never intersected, D would have an infinite volume. Therefore, for some $k \geq 0$ and $l \geq 0$ where $k > l$,

$$(3.11) \quad g^k U \cap g^l U \neq \emptyset.$$

Since the mapping g has the properties of a group,

$$(3.12) \quad g^{k-l} U \cap U \neq \emptyset.$$

If y is in this intersection, then it must be the result of n mappings from an element in U *ie.* $y = g^n x$ ($n = k - l$) with $x \in U$. Therefore, $x \in U$ and $g^n x \in U$ ($n = k - l$). \square

4. UNSOLVABILITY OF THE THREE-BODY PROBLEM

Much of the work we have described so far was developed in the 18th and 19th century in response to a central problem in celestial mechanics: the description of the motion of three bodies under mutual gravitation. Notable work was done by Euler (1760), Lagrange (1776), Laplace (1799), Hamilton (1834), Liouville (1836), Jacobi (1843), Poincaré (1889) [5].

The motion of two bodies had been solved by Kepler (1609) and Newton (1687) earlier on in the 17th century. However, it was apparent even then that a solution to the Three-Body Problem was not going to be as straightforward. As Newton prophesized, "[an exact solution for three bodies] exceeds, if I am not mistaken, the force of any human mind." Poincaré later accounted for the difficulty by pointing to the chaotic nature of the system:

...it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.

We round off this review of classical mechanics by demonstrating the solution to the Two-Body Problem and making a few remarks regarding the nature of the Three-Body Problem.

In essence, the Three-Body Problem cannot be solved because, unlike the Two-Body Problem, the 18 variables ($6n$) that we need to completely describe the system, cannot be reduced to a single variable that we can solve. What allowed the simplification of the Two-Body Problem were invariances and conserved quantities, which I will next formalize as 'first integrals'. There are insufficient first integrals to allow us to reduce the 18 variables needed to fully describe a Three-Body Problem.

Definition 4.1. A first integral of an equation is a non-constant continuously-differentiable function whose derivative vanishes identically on the solutions of that equation [4].

Example 4.2. For a scalar equation $y' = f(x, y)$, its first integral is a function $F(x, y)$ which takes the form of the general solution $F(x, y) = C$, where C is an arbitrary constant.

Therefore, $F(x, y)$ satisfies the linear equation

$$(4.3) \quad \frac{\partial F(x, y)}{\partial x} + \frac{\partial F(x, y)}{\partial y} f(x, y) = 0.$$

If a conservative mechanical system is invariant with respect to some variable q_i , then H is not an explicit function of q_i . Using Hamilton's equations, this means that p_i is a first integral. Since first integrals are functions that remain constant along any given solution of the system, they provide relations between variables of the system. This allows us to reduce the dimension of the Hamiltonian system by one by eliminating a coordinate q_i and the corresponding momentum p_i .

However, it was shown by Bruns in 1887 that there are no more than the following 10 independent integrals that are algebraic in the rectangular coordinates and time:

- 3 for center of mass
- 3 for linear momentum
- 3 for angular momentum
- 1 for energy

Poincaré later showed that the equations of motion do not admit any uniform transcendental integrals for values of the masses sufficiently small, other than these 10 integrals. As such, it was proven that it is impossible to reduce the 18 variables of the Three-Body Problem in order to produce an analytic solution.

4.1. Known Solutions. [5] In order to solve the problem numerically, it is important to identify configurations of the particles that allow us to introduce new forms of symmetries into the system. Although the equations of motion of the three body system have no rest points since the bodies attract each other and cannot just sit in space without moving, these symmetries allow us to find rest points of the differential equations *modulo symmetries*. These are solutions, which we call central configurations, that evolve purely by symmetry - by rotation or scaling - so that the ratios of the interbody distances remain constant.

For the Three-Body Problem, there are exactly five configurations. Three are collinear and were discovered by Euler. Two are shaped like equilateral triangles and were discovered by Lagrange.

Another class of solutions are periodic orbits. These are not identified by the symmetries that central configurations possess, instead, their defining characteristic is the stability of their orbits. For the Three-Body Problem, we do not know if a single stable periodic solution exists. However, Kolmogorov, Arnol'd, and Moser have devised a weaker form of stability called "KAM stability". The KAM theorem says that if a certain necessary condition -the KAM twist condition- is satisfied along a periodic orbit, then the orbit is "KAM stable". Building on the Liouville and Poincaré Recurrence theorems, KAM theorem implies that a great many orbits starting near our known orbit will stay near to it for all time.

In 2001, Montgomery and Chenciner discovered the only known KAM stable orbit. It is a figure-eight shaped orbit in which all three bodies move around a single figure-eight in the plane.

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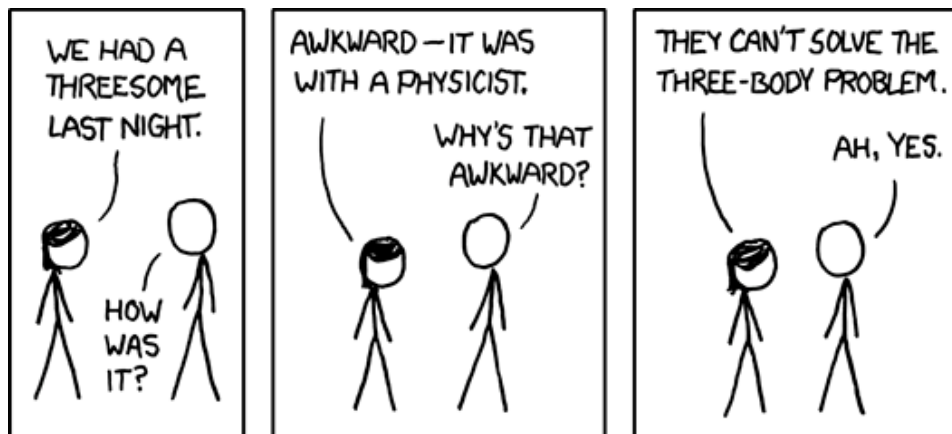


FIGURE 3. Threesome (<http://xkcd.com/613/>)

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