SOME DEGENERATE WEAK CATEGORIES

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Abstract. We consider weak higher categories which have only a single cell in some lower dimensions. We show that bicategories with one 0-cell are monoidal categories, and tricategories with one 0-cell and one 1-cell are braided monoidal categories. We then show an analogue to this classical result for Batanin weak 3-categories.

Contents

1. Introduction.

In this paper, we will examine various degenerate categories. We show using analogues of the Eckmann-Hilton argument that bicategories in the sense of [4] with one 0-cell have the structure of monoidal categories, and those with one 0-cell and one 1-cell have the structure of commutative monoids. Tricategories in the sense of [9] with one 0-cell and one 1-cell have the structure of braided monoidal categories. These results have been summarized in [6],[7].

The problem is next considered in the case of a Batanin weak ω-category. The main theorem of this paper states that a Batanin weak 3-category with one 0-cell and one 1-cell has a braided monoidal product.

2. Categories.

We begin with the example of an ordinary category to illustrate how we will proceed in higher dimensions.

Definition 2.1. A *category* $\mathcal{C}$ consists of data:
(1C.1) A set of 0-cells denoted $|\mathcal{C}|$.
(1C.2) For each pair of objects C, D in $|\mathcal{C}|$, a set of 1-cells denoted $[C, D]$.
(1C.3) For each object A in $|\mathcal{C}|$, a function

$$1_A : \mathbb{1} \to [A, A]$$

Date: DEADLINE AUGUST 26, 2011.

where $\mathbb{1}$ denotes the terminal set.
(1C.4) For each triple of objects A, B, C in $|\mathcal{C}|$, a function

$$\circ : [B, C] \times [A, B] \to [A, C].$$

(1C.5) (Associativity) For each quadruple of objects A, B, C, D in $|\mathcal{C}|$, the following diagram commutes.

$$\begin{tikzcd}
{[C,D] \times [B,C] \times [A,B]} \arrow[r, "\circ \times id"] \arrow[d, "id \times \circ"] & {[B,D] \times [A,B]} \arrow[d, "\circ"] \\
{[C,D] \times [A,C]} \arrow[r, "\circ"] & {[A,D]}
\end{tikzcd}$$

(1C.6) (Identity) For each pair of objects A, B in $|\mathcal{C}|$, the following diagram commutes.

$$\begin{tikzcd}
{[B,B] \times [A,B]} \arrow[dr, "\circ"'] & {[A,B]} \arrow[l, "1_B \times id"'] \arrow[r, "id \times 1_A"'] \arrow[d, equal] & {[A,B] \times [A,A]} \arrow[dl, "\circ"] \\
& {[A,B]} &
\end{tikzcd}$$

Definition 2.2. A *monoid* $\mathcal{M}$ consists of data:
(m.1) A set $|M|$.
(m.2) A function

$$1 : \mathbb{1} \to M.$$

(m.3) A function

$$\bullet : M \times M \to M.$$

(m.4) (Associativity) The following diagram commutes.

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\bullet \times id"] \arrow[d, "id \times \bullet"] & M \times M \arrow[d, "\bullet"] \\
M \times M \arrow[r, "\bullet"] & M
\end{tikzcd}$$

(m.5) (Identity) The following diagram commutes.

$$\begin{tikzcd}
M \times M \arrow[dr, "\bullet"'] & M \arrow[l, "1 \times id"'] \arrow[r, "id \times 1"] \arrow[d, equal] & M \times M \arrow[dl, "\bullet"] \\
& M &
\end{tikzcd}$$

Theorem 2.3. *The 1-cells of a category with a single 0-cell form a monoid.*

Proof. Observe that the data (1C.2), (1C.3), (1C.4) for such a category reduce to the data (m.1), (m.2), (m.3) respectively, and the category axioms (1C.5), (1C.6) reduce to the monoid axioms (m.5), (m.6) respectively. □

3. Bicategories.

Now we consider degenerate bicategories. The definition of a bicategory here is due to [4].

Definition 3.1. A *bicategory* $\mathcal{C}$ consists of data:
(2C.1) A set of 0-cells denoted $|\mathcal{C}|$.
(2C.2) For each pair of objects A, B in $|\mathcal{C}|$, a category $[A, B]$ whose n-cells are $(n+1)$-cells of $\mathcal{C}$.
(2C.3) For each object A in $|\mathcal{C}|$, a functor

$$1_A : \mathbb{1} \to [A, A]$$

where $\mathbb{1}$ denotes the terminal category.
(2C.4) For each triple of objects A, B, C in $|\mathcal{C}|$, a functor

$$\otimes : [B, C] \times [A, B] \to [A, C].$$

(2C.5) (Associativity) For each quadruple of objects A, B, C, D in $|\mathcal{C}|$, a natural isomorphism.

$$\begin{tikzcd}
{[C,D] \times [B,C] \times [A,B]} \arrow[r, "\otimes \times id"] \arrow[d, "id \times \otimes"] \arrow[dr, phantom, "\Swarrow \alpha"] & {[B,D] \times [A,B]} \arrow[d, "\otimes"] \\
{[C,D] \times [A,C]} \arrow[r, "\otimes"] & {[A,D]}
\end{tikzcd}$$

(2C.6) (Identity) For each pair of objects A, B in $|\mathcal{C}|$, natural isomorphisms.

$$\begin{tikzcd}
{[B,B] \times [A,B]} \arrow[dr, "\otimes"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}" near end] & {[A,B]} \arrow[l, "1_B \times id"'] \arrow[r, "id \times 1_A"] \arrow[d, equal] & {[A,B] \times [A,A]} \arrow[dl, "\otimes"] \arrow[dl, phantom, "\overset{\rho}{\Rightarrow}" near start] \\
& {[A,B]} &
\end{tikzcd}$$

(2C.7) (Associativity coherence) The following diagram commutes.

$$\begin{tikzcd}
((- \otimes -) \otimes -) \otimes - \arrow[r, "\alpha"] \arrow[d, "\alpha \otimes id"] & (- \otimes -) \otimes (- \otimes -) \arrow[dd, "\alpha"] \\
(- \otimes (- \otimes -)) \otimes - \arrow[d, "\alpha"] & \\
- \otimes ((- \otimes -) \otimes -) \arrow[r, "id \otimes \alpha"] & - \otimes (- \otimes (- \otimes -))
\end{tikzcd}$$

(2C.8) (Identity coherence) The following diagram commutes.

$$\begin{tikzcd}
(- \otimes 1) \otimes - \arrow[r, "\lambda \otimes id"] \arrow[dr, "\alpha"] & - \otimes - \arrow[d, "id \otimes \rho"] \\
& - \otimes (1 \otimes -)
\end{tikzcd}$$

Definition 3.2. Let $\mathcal{M}$ be a monoid. Define a function $t : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ by $(m, n) \mapsto (n, m)$. We say that $\mathcal{M}$ is *commutative* if $\mathcal{M}$ satisfies the axiom:
(m.3) The following diagram commutes.

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M} \arrow[r, "t"] \arrow[dr, "\bullet"] & \mathcal{M} \times \mathcal{M} \arrow[d, "\bullet"] \\
 & \mathcal{M}
\end{tikzcd}$$

Theorem 3.3. *The 2-cells of a bicategory with a single 0-cell and a single 1-cell form a commutative monoid with a distinguished invertible element.*

Proof. Let $\mathcal{C}$ be a bicategory with a single 0-cell C and a single 1-cell c. Let $[\mathcal{C}]$ denote the category $[C, C]$, and let $\llbracket \mathcal{C} \rrbracket$ denote the set $[c, c]$. Let $\circ$ denote the operation $\llbracket \mathcal{C} \rrbracket$ inherits from composition in $[\mathcal{C}]$ (vertical composition), and let $\otimes$ denote the operation $\llbracket \mathcal{C} \rrbracket$ inherits from composition in $\mathcal{C}$ (horizontal composition). Theorem 2.3 says that $\llbracket \mathcal{C} \rrbracket$ is a monoid under $\circ$.

Since the natural isomorphisms α, λ, ρ each consist of a single element of $\llbracket \mathcal{C} \rrbracket$, we let α, λ, ρ denote these elements respectively. Let 1 denote the identity on c in $[\mathcal{C}]$, i.e., the identity for $\circ$.

We show that $\alpha = 1, \lambda = \rho^{-1}$, and 1 is an identity for $\otimes$. Then by the Eckmann-Hilton argument, $\otimes = \circ$ makes $\llbracket \mathcal{C} \rrbracket$ a commutative monoid.

By the naturality of λ, we see that the following diagrams commute.

$$\begin{tikzcd}
c \otimes c \arrow[r, "\lambda"] \arrow[d, "1 \otimes 1"] & c \arrow[d, "1"] \\
c \otimes c \arrow[r, "\lambda"] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
c \otimes c \arrow[r, "\lambda"] \arrow[d, "1 \otimes \lambda"] & c \arrow[d, "\lambda"] \\
c \otimes c \arrow[r, "\lambda"] & c
\end{tikzcd}$$

Since λ is invertible, we see that $1 \otimes 1 = 1$ and $1 \otimes \lambda = \lambda$. Thus we also see that

$$(1 \otimes 1) \otimes \lambda = \lambda = 1 \otimes (1 \otimes \lambda).$$

By the naturality of α and λ, the following diagram commutes. Recall that since the only 1-cell in $\mathcal{C}$ is c, each of the objects in the following diagram is equal to c.

$$\begin{tikzcd}
c \otimes ((c \otimes c) \otimes c) \arrow[r, "1 \otimes \alpha"] \arrow[d, "\lambda"] & c \otimes (c \otimes (c \otimes c)) \arrow[d, "\lambda"] \\
(c \otimes c) \otimes c \arrow[r, "\alpha"] & c \otimes (c \otimes c) \\
(c \otimes c) \otimes c \arrow[r, "\alpha"] \arrow[u, "\lambda = (1 \otimes 1) \otimes \lambda"] & c \otimes (c \otimes c) \arrow[u, "1 \otimes (1 \otimes \lambda) = \lambda"]
\end{tikzcd}$$

Since λ is invertible, we read from the outside of the above diagram that

$$1 \otimes \alpha = \alpha.$$

Similarly,

$$\alpha \otimes 1 = \alpha.$$

By axiom (2C.7), the following diagram commutes,

$$\begin{tikzcd}
((c \otimes c) \otimes c) \otimes c \arrow[r, "\alpha"] \arrow[d, "\alpha \otimes 1 = \alpha"] & (c \otimes c) \otimes (c \otimes c) \arrow[dd, "\alpha"] \\
(c \otimes (c \otimes c)) \otimes c \arrow[d, "\alpha"] & \\
c \otimes ((c \otimes c) \otimes c) \arrow[r, "1 \otimes \alpha = \alpha"] & c \otimes (c \otimes (c \otimes c))
\end{tikzcd}$$

and since α is invertible, we see that

$$\alpha = 1.$$

For any element x of $[\![\mathcal{C}]\!]$, by the naturality of α and λ, the following diagram commutes.

$$\begin{tikzcd}
c \arrow[d, "x"] & c \otimes c \arrow[l, "\lambda"'] \arrow[r, equal] \arrow[d, "1 \otimes x"] & (c \otimes c) \otimes c \arrow[r, "\alpha = 1"] \arrow[d, "(1 \otimes 1) \otimes x"] & c \otimes (c \otimes c) \arrow[r, "\lambda"] \arrow[d, "1 \otimes (1 \otimes x)"] & c \otimes c \arrow[d, "1 \otimes x"] \\
c & c \otimes c \arrow[l, "\lambda"] \arrow[r, equal] & (c \otimes c) \otimes c \arrow[r, "\alpha = 1"] & c \otimes (c \otimes c) \arrow[r, "\lambda"'] & c \otimes c
\end{tikzcd}$$

Since λ is invertible, we can read from the outside of the above diagram that

$$1 \otimes x = x,$$

and similarly,

$$x \otimes 1 = x.$$

Now, following the Eckmann-Hilton argument, by the functoriality of $\otimes$, we see that for any x, y in $[\![\mathcal{C}]\!]$,

$$\begin{aligned} x \circ y &= (x \otimes 1) \circ (1 \otimes y) \\ &= (x \circ 1) \otimes (1 \circ y) \\ &= x \otimes y, \end{aligned}$$

and

$$\begin{aligned} x \circ y &= (1 \otimes x) \circ (y \otimes 1) \\ &= (1 \circ y) \otimes (x \circ 1) \\ &= y \otimes x \\ &= y \circ x. \end{aligned}$$

Thus, $\circ = \otimes$ makes $[\![\mathcal{C}]\!]$ a commutative monoid.

Finally, by axiom (2C.8), we see that the following diagram commutes,

$$\begin{tikzcd}
(c \otimes 1) \otimes c \arrow[r, "\lambda \otimes 1 = \lambda"] \arrow[dr, "\alpha = 1"'] & c \otimes c \arrow[d, "1 \otimes \rho = \rho"'] \\
& c \otimes (1 \otimes c)
\end{tikzcd}$$

and thus

$$\lambda = \rho^{-1}$$

is an invertible element in $[\![\mathcal{C}]\!]$. $\square$

Definition 3.4. A monoidal category $\mathcal{M}$ consists of data:
(M.1) A category $\mathcal{M}$.
(M.2) A functor

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}.$$

(M.3) A functor

$$1 : \mathbb{1} \to \mathcal{M}.$$

(M.4) A natural isomorphism.

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M} \times \mathcal{M} \arrow[r, "id\times\otimes"] \arrow[d, "\otimes\times id"] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{M} \times \mathcal{M} \arrow[d, "\otimes"] \\
\mathcal{M} \times \mathcal{M} \arrow[r, "\otimes"] & \mathcal{M}
\end{tikzcd}$$

(M.5) Natural isomorphisms.

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M} \arrow[ddrr, "\otimes"'] & & \mathcal{M} \arrow[ll, "1\times id"'] \arrow[rr, "id\times 1"] \arrow[dd, equal] & & \mathcal{M} \times \mathcal{M} \arrow[ddll, "\otimes"] \\
& \overset{\lambda}{\Rightarrow} & & \overset{\rho}{\Rightarrow} & \\
& & \mathcal{M} & &
\end{tikzcd}$$

(M.6) (Associativity coherence) The following diagram commutes.

$$\begin{tikzcd}
((-\otimes -)\otimes -)\otimes - \arrow[r, "\alpha(\otimes\times id\times id)"] \arrow[d, "\alpha\otimes id"] & (-\otimes -)\otimes(-\otimes -) \arrow[dd, "\alpha(id\times id\times\otimes)"] \\
(-\otimes(-\otimes -))\otimes - \arrow[d, "\alpha(id\times\otimes\times id)"] & \\
-\otimes((-\otimes -)\otimes -) \arrow[r, "id\otimes\alpha"] & -\otimes(-\otimes(-\otimes -))
\end{tikzcd}$$

(M.7) (Identity coherence) The following diagram commutes.

$$\begin{tikzcd}
(-\otimes 1)\otimes - \arrow[d, "\alpha"] \arrow[dr, "\lambda\otimes id"] & \\
-\otimes(1\otimes -) \arrow[r, "id\otimes\rho"'] & -\otimes -
\end{tikzcd}$$

Theorem 3.5. *The 1-cells and 2-cells of a bicategory with a single 0-cell form a monoidal category.*

Proof. Observe that the data (2C.2), (2C.3), (2C.4), (2C.5), (2C.6), (2C.7), (2C.8) for such a bicategory reduce to the data (M.1), (M.2), (M.3), (M.4), (M.5), (M.6), (M.7), respectively, for a monoidal category. □

4. TRICATEGORIES.

Now we examine tricategories which have only one 0-cell and one 1-cell. Here we use the definition of tricategory given in [9]. However, since the definition is unwieldy, we only state the definition in the degenerate case in which we are interested.

Definition 4.1. A *tricategory* $\mathcal{C}$ with one 0-cell and one 1-cell consists of data:
(3C.1) A singleton set denoted $|\mathcal{C}|$;
(3C.2) A bicategory $[\mathcal{C}]$ whose n-cells are $(n+1)$-cells of $\mathcal{C}$. Since $[\mathcal{C}]$ has a single 0-cell (the single 1-cell of $\mathcal{C}$), we use $[\![\mathcal{C}]\!]$ to denote its underlying category of 1-cells and 2-cells.
(3C.3) A pseudo-functor

$$* : [\mathcal{C}]^2 \to [\mathcal{C}].$$

That is,
(3C.3.1) A functor

$$* : [\![\mathcal{C}]\!]^2 \to [\![\mathcal{C}]\!].$$

(3C.3.2) A natural isomorphism

$$\gamma_*^{\otimes} : (- * -) \otimes (- * -) \implies (- \otimes -) * (- \otimes -)$$

where $\gamma_{\otimes}^{*}$ denotes the inverse of $\gamma_*^{\otimes}$.
(3C.3.3) An invertible 3-cell

$$\delta^* : 1 \to 1 * 1.$$

(3C.3.4) (Composition) For all $A, B, C, D, E, F \in [\![\mathcal{C}]\!]$, the following diagram commutes.

$$\begin{tikzcd}
((B * A) \otimes (D * C)) \otimes (F * E) \arrow[r, "\gamma_*^{\otimes} \otimes id"] \arrow[d, "\alpha"] & ((B \otimes D) * (A \otimes C)) \otimes (F * E) \arrow[d, "\gamma_*^{\otimes}"] \\
(B * A) \otimes ((D * C) \otimes (F * E)) \arrow[d, "id \otimes \gamma_*^{\otimes}"] & ((B \otimes D) \otimes F) * ((A \otimes C) \otimes E) \arrow[d, "\alpha * \alpha"] \\
(B * A) \otimes ((D \otimes F) * (C \otimes E)) \arrow[r, "\gamma_*^{\otimes}"] & (B \otimes (D \otimes F)) * (A \otimes (C \otimes E))
\end{tikzcd}$$

(3C.3.5) (Unit) For all $A, B \in [\mathcal{C}]$ the following diagram commutes

$$\begin{tikzcd}
1 \otimes (B * A) \arrow[r, "\lambda^{\otimes}"] \arrow[d, "\delta^* \otimes id"] & B * A \arrow[r, "\rho^{\otimes}"] \arrow[dd, equal] & (B * A) \otimes 1 \arrow[d, "id \otimes \delta^*"] \\
(1 * 1) \otimes (B * A) \arrow[d, "\gamma_*^{\otimes}"] & & (B * A) \otimes (1 * 1) \arrow[d, "\gamma_*^{\otimes}"] \\
(1 \otimes B) * (1 \otimes A) \arrow[r, "\lambda^{\otimes} * \lambda^{\otimes}"] & B * A \arrow[r, "\rho^{\otimes} * \rho^{\otimes}"] & (B \otimes 1) * (A \otimes 1)
\end{tikzcd}$$

(3C.4) A pseudo-functor

$$I : \mathbb{1} \to [\mathcal{C}].$$

That is,
(3C.4.1) A 2-cell i in $\mathcal{C}$.
(3C.4.2) An invertible 3-cell

$$\delta^I : 1 \implies i.$$

(3C.5) (Associativity) A pseudo-natural equivalence.

$$\begin{tikzcd}
{[\mathcal{C}]^3} \arrow[r, "*\times id"] \arrow[d, "id\times *"'] \arrow[dr, phantom, "\Swarrow_a"] & {[\mathcal{C}]^2} \arrow[d, "*"] \\
{[\mathcal{C}]^2} \arrow[r, "*"] & {[\mathcal{C}]}
\end{tikzcd}$$

That is,

(3C.5.1) A weakly invertible 2-cell a of $\mathcal{C}$.

(3C.5.2) A natural isomorphism

$$\tau : [- * (- * -)] \otimes a \implies a \otimes [(- * -) * -].$$

(3C.5.3) The following diagram commutes.

$$\begin{tikzcd}
1 \otimes a \arrow[rr, "\rho^{\otimes}\lambda^{\otimes}"] \arrow[d, "\delta^{*}\otimes id"] & & a \otimes 1 \arrow[d, "id\otimes\delta^{*}"] \\
(1 * 1) \otimes a \arrow[d, "(\delta^{*} * id)\otimes id"] & & a \otimes (1 * 1) \arrow[d, "id\otimes(id*\delta^{*})"] \\
((1 * 1) * 1) \otimes a \arrow[rr, "\tau"] & & a \otimes (1 * (1 * 1))
\end{tikzcd}$$

(3C.5.4) For every $A, B, C, D, E, F \in \llbracket \mathcal{C} \rrbracket$ the following diagram commutes.

$$\begin{tikzcd}
(((F * E) * D) \otimes ((C * B) * A)) \otimes a \arrow[r, "\gamma^{\otimes}_{*}\otimes id"] \arrow[d, "\alpha"] & (((F * E) \otimes (C * B)) * (D \otimes A)) \otimes a \arrow[d, "(\gamma^{\otimes}_{*} * id)\otimes id"] \\
((F * E) * D) \otimes (((C * B) * A) \otimes a) \arrow[d, "id\otimes\tau"] & (((F \otimes C) * (E \otimes B)) * (D \otimes A)) \otimes a \arrow[d, "\tau"] \\
((F * E) * D) \otimes (a \otimes (C * (B * A))) \arrow[d, "\alpha^{id1}"] & a \otimes ((F \otimes C) * ((E \otimes B) * (D \otimes A))) \\
(((F * E) * D) \otimes a) \otimes (C * (B * A)) \arrow[d, "\tau\otimes id"] & a \otimes ((F \otimes C) * ((E \otimes D) \otimes (B * A))) \arrow[u, "id\otimes(id*\gamma^{\otimes}_{*})"] \\
(a \otimes (F * (E * D))) \otimes (C * (B * A)) \arrow[r, "\alpha"] & a \otimes ((F * (E * D)) \otimes (C * (B * A))) \arrow[u, "id\otimes\gamma^{\otimes}_{*}"]
\end{tikzcd}$$

(3C.6) (Identity) Pseudo-natural equivalences.

$$\begin{tikzcd}
{[\mathcal{C}]^2} \arrow[dr, "*"'] \arrow[dr, phantom, "\overset{\ell}{\Rightarrow}", bend left] & {[\mathcal{C}]} \arrow[l, "1\times id"'] \arrow[r, "id\times 1"] \arrow[d, equal] \arrow[d, phantom, "\overset{r}{\Rightarrow}", shift left=3ex] & {[\mathcal{C}]^2} \arrow[dl, "*"] \\
& {[\mathcal{C}]} &
\end{tikzcd}$$

That is:

(3C.6.1) Weakly invertible 2-cells ℓ, r of $\mathcal{C}$.

(3C.6.2) Natural isomorphisms.

$$\sigma : \ell^{-1} \otimes - \otimes \ell \implies i * -,$$
$$\tau : - * i \implies r \otimes - \otimes r^{-1}.$$

(3C.6.3) The following diagrams commute.

$$\begin{tikzcd}
1 \arrow[r, "\delta^*"] \arrow[dr, "\sigma"] & 1 * 1 \arrow[d, "\delta^I * 1"] & 1 \arrow[r, "\delta^*"] & 1 * 1 \arrow[d, "1 * \delta^I"] \\
 & i * 1 & & 1 * i \arrow[ul, "\tau"]
\end{tikzcd}$$

(3C.6.4) For every $A, B \in \llbracket \mathcal{C} \rrbracket$, the following diagrams commute.

$$\begin{tikzcd}
(B \otimes A) \otimes \ell \arrow[r, "\sigma"] \arrow[d, "\alpha"] & \ell \otimes (i * (B \otimes A)) \\
B \otimes (A \otimes \ell) \arrow[d, "id \otimes \sigma"] & \ell \otimes ((i \otimes i) * (B \otimes A)) \arrow[u, "id \otimes ((id \otimes (\delta^I)^{-1}) * id)"'] \\
B \otimes (\ell \otimes (i * A)) \arrow[d, "\alpha^{-1}"] & \ell \otimes ((i * B) \otimes (i * A)) \arrow[u, "id \otimes \gamma^{\otimes}_{*}"'] \\
(B \otimes \ell) \otimes (i * A) \arrow[r, "\sigma \otimes id"] & (\ell \otimes (i * B)) \otimes (i * A) \arrow[u, "\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
((B \otimes A) * i) \otimes r \arrow[r, "\tau"] \arrow[d, "(id * \delta^I) \otimes id"] & r \otimes (B \otimes A) \\
((B \otimes A) * (i \otimes i)) \otimes r \arrow[d, "(\gamma^{*}_{\otimes}) \otimes id"] & (r \otimes B) \otimes A \arrow[u, "\alpha"'] \\
((B * i) \otimes (A * i)) \otimes r \arrow[d, "\alpha"] & ((B * i) \otimes r) \otimes A \arrow[u, "\tau \otimes id"'] \\
(B * i) \otimes ((A * i) \otimes r) \arrow[r, "id \otimes \tau"] & (B * i) \otimes (r \otimes A) \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

(3C.7) (Associativity coherence) An invertible modification

$$\pi : (- * a) \otimes a \otimes (a * -) \Longrightarrow a \otimes a.$$

(3C.8) (Identity coherence) Invertible modifications

$$\mu : (1 * \ell) \otimes a \otimes (r * 1) \Longrightarrow 1$$

$$\nu : (\ell * 1) \Longrightarrow \ell \otimes a$$

$$\xi : (1 * \rho) \Longrightarrow a \otimes r \ .$$

Satisfying certain coherence conditions.

Definition 4.2. Let $\mathcal{M}$ be a monoidal category as in Definition 3.4. We say that $\mathcal{M}$ is *braided* if it has additional data
(M.8) A natural isomorphism

$$\beta : \otimes \implies \otimes\, t$$

where $t : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is the functor induced by $(m, n) \mapsto (n, m)$.
(M.9) The following diagrams commute for all objects A, B, C in $\mathcal{M}$.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha"] \arrow[d, "\beta\otimes id"] & A \otimes (B \otimes C) \arrow[r, "\beta"] & (B \otimes C) \otimes A \arrow[d, "\alpha"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[r, "id\otimes\beta"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha^{-1}"] \arrow[d, "id\otimes\beta"] & (A \otimes B) \otimes C \arrow[r, "\beta"] & C \otimes (A \otimes B) \arrow[d, "\alpha^{-1}"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}"] & (A \otimes C) \otimes B \arrow[r, "\beta\otimes id"] & (C \otimes A) \otimes B
\end{tikzcd}$$

Remark 4.3. From these axioms, it can be shown that:
(M.10) The following diagrams commute for all objects A in $\mathcal{M}$.

$$\begin{tikzcd}
1 \otimes A \arrow[r, "\lambda"] \arrow[dr, "\beta_{1,A}"] & A \arrow[d, "\rho"] \\
& A \otimes 1
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes A & A \arrow[l, "\lambda^{-1}"'] \\
& A \otimes 1 \arrow[u, "\rho^{-1}"'] \arrow[ul, "\beta_{A,1}"]
\end{tikzcd}$$

(See [10].)

Theorem 4.4. *The* 2*-cells and* 3*-cells of a tricategory with a single* 0*-cell and a single* 1*-cell form a braided monoidal category.*

Proof. Let $\mathcal{C}$ be such a tricategory. Let C denote the single 0-cell of $\mathcal{C}$ and c denote the single 1-cell of $\mathcal{C}$. The datum (3C.2) of a tricategory tells us that $[C, C]$ is a bicategory. Theorem 3.5 then tells us that $[c, c]$ is a monoidal category under $\otimes$. Let 1 denote the identity of $\otimes$.

Consider ℓ, r, σ, τ from (3C.6) and δ^I from (3C.4) of the definition. Recall that ℓ and r are weakly invertible which means that there are 2-cells ℓ^{-1} and r^{-1} of $\mathcal{C}$ such that

$$\ell \otimes l^{-1} \cong 1,$$

$$r \otimes r^{-1} \cong 1.$$

Let σ' and τ' denote the composites

$$\ell^{-1} \otimes - \otimes \ell \xRightarrow[\cong]{\sigma} i * - \xRightarrow[\cong]{\delta^I * -} 1 * -$$

$$r \otimes - \otimes r^{-1} \xRightarrow[\cong]{\tau} - * i \xRightarrow[\cong]{- * \delta^I} - * 1$$

respectively (where we write x for x^{-1} where the meaning is clear). Note that both σ', τ' are natural isomorphisms. The composites

$$\ell \xRightarrow[\cong]{} \ell^{-1} \otimes \ell \otimes \ell \xRightarrow[\cong]{\sigma'_\ell} 1 * \ell$$

$$r \xRightarrow[\cong]{} r^{-1} \otimes r \otimes r \xRightarrow[\cong]{\tau'_r} 1 * r$$

are natural isomorphisms which we call σ'_ℓ and τ'_r in what follows.

Consider ν and μ from (3C.8) of the definition. The composite

$$r \otimes \ell \otimes r^{-1} \otimes r \xrightarrow[\cong]{\tau'_\ell \otimes 1_r} (\ell * 1) \otimes r \xrightarrow[\cong]{\nu \otimes 1_r} \ell \otimes a \otimes r \xrightarrow[\cong]{\sigma'_\ell \otimes 1_a \otimes \tau'_r} (1 * \ell) \otimes a \otimes (r * 1) \xrightarrow[\cong]{\mu} 1,$$

shows that $r \cong \ell^{-1}$, and thus the composite

$$\ell \otimes a \otimes r \xrightarrow[\cong]{\sigma'_\ell \otimes 1_a \otimes \tau'_r} (1 * \ell) \otimes a \otimes (r * 1) \xrightarrow[\cong]{\mu} 1,$$

shows that $a \cong 1$. Thus, there is a natural isomorphism

$$\tau : - * (- * -) \implies (- * -) * -$$

from (3C.5.2).

For all 2-cells A in $\mathcal{C}$, the composite

$$\ell^{-1} \otimes (1 * A) \otimes \ell \xrightarrow[\cong]{\sigma'} 1 * (1 * A) \xrightarrow[\cong]{\tau} (1 * 1) * A \xrightarrow[\cong]{\delta^* * 1_A} 1 * A \xrightarrow[\cong]{\sigma'} \ell^{-1} \otimes A \otimes \ell.$$

shows that $1 * A$ is naturally isomorphic to A. Similarly, $A * 1$ is naturally isomorphic to A. Call these isomorphisms $\lambda^* : 1 * A \to A$ and $\rho^* : A \to A * 1$ respectively.

Let $A, B \in [\![\mathcal{C}]\!]$, and define $\beta_{AB} : A \otimes B \to B \otimes A$ to be the composite:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\lambda^* \otimes \rho^*"] & (1 * A) \otimes (B * 1) \arrow[r, "\gamma^{\otimes}_{*}"] & (1 \otimes B) * (A \otimes 1) \arrow[d, "(\rho^{\otimes} \lambda^{\otimes}) * (\lambda^{\otimes} \rho^{\otimes})"] \\
B \otimes A & (B * 1) \otimes (1 * A) \arrow[l, "\rho^* \otimes \lambda^*"'] & (B \otimes 1) * (1 \otimes A) \arrow[l, "\gamma^{*}_{\otimes}"']
\end{tikzcd}$$

This is an isomorphism natural in A and B.

Now we check (M.10) for a braided monoidal category. Axiom (M.9) is proven in [7].

The following diagram commutes because of the naturality of σ'.

$$\begin{tikzcd}
l^{-1} \otimes (1 * 1) \otimes l \arrow[r, "\sigma'"] \arrow[d, "1_{l^{-1}} \otimes \delta^* \otimes 1_l"] & 1 * (1 * 1) \arrow[r, "1_1 * \delta^*"] & 1 * 1 \arrow[r, "\sigma'"] & l^{-1} \otimes 1 \otimes l \\
l^{-1} \otimes 1 \otimes l \arrow[urr, bend right, "\sigma'"] \arrow[urrr, bend right, equal] & & &
\end{tikzcd}$$

Thus, $\lambda_1^* = \delta^*$. Similarly, $\rho_1^* = \delta^*$.

By the naturality of $\lambda^{\otimes}$, $\rho^{\otimes}$, and axiom (3C.3.5) of the definition, we see that the following diagram commutes.

$$\begin{tikzcd}
1 \otimes A \arrow[r, "\lambda_A^{\otimes}"] \arrow[d, "id \otimes \lambda^*"] & A \arrow[r, "\rho_A^{\otimes}"] \arrow[d, "\lambda^*"] & A \otimes 1 \arrow[d, "\lambda^* \otimes id"] \\
1 \otimes (1 * A) \arrow[r, "\lambda_{1*A}^{\otimes}"] \arrow[d, "\delta^* \otimes id"] & 1 * A \arrow[r, "\rho_{1*A}^{\otimes}"] \arrow[dd, equal] & (1 * A) \otimes 1 \arrow[d, "id \otimes \delta^*"] \\
(1 * 1) \otimes (1 * A) \arrow[d, "\gamma_*^{\otimes}"] & & (1 * A) \otimes (1 * 1) \arrow[d, "\gamma_*^{\otimes}"] \\
(1 \otimes 1) * (1 \otimes A) \arrow[r, "\lambda_A^{\otimes} * \lambda_A^{\otimes}"] & 1 * A \arrow[r, "\rho_A^{\otimes} * \rho_A^{\otimes}"] & (1 \otimes 1) * (A \otimes 1)
\end{tikzcd}$$

Thus, $\beta_{A1} = \rho_A^{\otimes} \lambda_A^{\otimes}$ as desired. Similarly, $\beta_{1A} = (\lambda_A^{\otimes})^{-1} (\rho_A^{\otimes})^{-1}$. □

5. BATANIN ω-CATEGORIES.

In the previous sections, weak n-categories are defined in terms of weak $(n-1)$-categories and the morphisms between them in a fairly regular way. It seems natural to extend this to definitions of n-categories for all finite n and then for $n = \omega$. Batanin introduced one approach in [1]. We briefly review the construction and then consider degenerate forms as we did above. The main result in this section is Theorem 5.16 which states that such a weak 3-category also has a braided monoidal structure. We note that weak Batanin categories include weak Trimble categories ([5]) and Martin-Löf complexes ([8]) so these results apply there as well.

We now summarize the description of Batanin weak ω-categories given by Leinster in [11]. Any statement which is not proven here is proven in [11].

Let $\mathbb{G}$ be the category generated by the graph

$$\begin{tikzcd}
0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 3 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \cdots \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & n \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \cdots
\end{tikzcd}$$

such that $ss = ts$ and $st = tt$ for all such composable pairs of s, t. Let the *category of globular sets* be the functor category $\hat{\mathbb{G}} = [\mathbb{G}^{\mathrm{op}}, \mathcal{S}et]$. Objects of $\hat{\mathbb{G}}$ are accordingly called *globular sets*. Let X be a globular set and $n < \omega$. Then let $X(n)$ denote the set which is the image of $n \in \mathbb{G}$ under the functor X, and let $s, t : X(n) \to X(n-1)$ denote $X(s : n-1 \to n), X(t : n-1 \to n)$ respectively. A morphism of globular sets $f : X \to Y$ is then a collection of functions $\{f_n : X(n) \to Y(n)\}_{n<\omega}$ such that $f_{n-1}s = sf_n$ and $f_{n-1}t = tf_n$ for all such s, t.

Any *strict* ω-category has a set of n-cells and source and target functions from its set of $(n+1)$-cells to its set of n-cells for all $n < \omega$. The forgetful functor from the category of strict ω-categories to $\hat{\mathbb{G}}$ is in fact monadic, and the resulting monad is cartesian. Let $(\mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}})$ denote this monad on $\hat{\mathbb{G}}$.

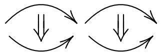
Let $x, y \in X(n)$ and $w \in X(n+1)$. We say that x, y are *parallel* if $n = 0$ or if $s(x) = s(y)$ and $t(x) = t(y)$. We write $w : x \to y$ to mean that $s(w) = x$ and $t(w) = y$.

We regard $\mathcal{T}$ as the free strict ω-category monad. Let 1 be the terminal object in $\hat{\mathbb{G}}$. We describe $\mathcal{T}1$ as the globular set of *pasting diagrams*. In a sense, $\mathcal{T}1(n)$ consists of all ways to compose a finite number of cells of dimension less than or equal to n. Each such composition itself looks like a finite globular set. For all $\tau \in \mathcal{T}1$, let $\hat{\tau}$ denote its representation in $\hat{\mathbb{G}}$. For each $\tau \in \mathcal{T}1(n)$, there is $1_\tau : \tau \to \tau \in \mathcal{T}1(n+1)$ such that $\hat{\tau} = \hat{1}_\tau$ which we regard as the identity of τ. We will later use the fact that for all $n < \omega$,

$$\mathcal{T}X(n) = \coprod_{\tau \in \mathcal{T}1(n)} \hat{\mathbb{G}}(\hat{\tau}, X).$$

Thus we can think of an element of $\mathcal{T}X$ as a labeling of a pasting diagram of $\mathcal{T}1$ with compatible cells of X.

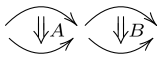
Example 5.1. Let $\tau \in \mathcal{T}1(2)$ denote the following pasting diagram.



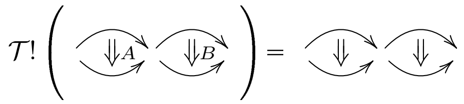
Let $A, B \in X(2)$ be such that $ts(A) = tt(A) = ss(B) = st(B)$. Then the following diagram represents the evident element of $\hat{\mathbb{G}}(\hat{\tau}, C) \subset \mathcal{T}X(2)$

$$\begin{tikzcd}
ss(A) \arrow[r, bend left, "s(A)"{name=U1}] \arrow[r, bend right, "t(A)"'{name=D1}] & ts(A) \arrow[r, bend left, "s(B)"{name=U2}] \arrow[r, bend right, "t(B)"'{name=D2}] & ts(B)
\arrow[from=U1, to=D1, Rightarrow, "A"]
\arrow[from=U2, to=D2, Rightarrow, "B"]
\end{tikzcd}$$

which we denote by the simpler diagram



since all information omitted here is implied by information which is included. Then $\mathcal{T}!$ acts on such a diagram by removing labels.



Note that $\hat{\mathbb{G}}(\hat{1}_\tau, X) = \hat{\mathbb{G}}(\hat{\tau}, X)$ since $\hat{1}_\tau = \hat{\tau}$. Thus, for each $\sigma \in \mathcal{T}X(n)$, there is $1_\sigma : \sigma \to \sigma \in \mathcal{T}X(n+1)$ which we regard as the identity of σ.

We consider $\mathcal{T}$-operads. Recall that a $\mathcal{T}$-operad is a cartesian monad morphism $\pi : \mathcal{P} \implies \mathcal{T}$. In particular, that means that the following diagram is a pullback for all $X \in \hat{\mathbb{G}}$, $n \leqslant \omega$ where $! : X \to 1$ denotes the unique terminal map.

$$\begin{tikzcd}
\mathcal{P}X(n) \arrow[r, "\pi_X"] \arrow[d, "\mathcal{P}!"] & \mathcal{T}X(n) \arrow[d, "\mathcal{T}!"] \\
\mathcal{P}1(n) \arrow[r, "\pi_1"] & \mathcal{T}1(n)
\end{tikzcd}$$

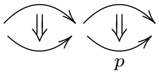
For any $\tau \in \mathcal{T}1(n)$, let $\mathcal{P}\tau$ denote $\pi_1^{-1}\tau \subset \mathcal{P}1(n)$. Thus,

$$\begin{aligned}\mathcal{P}X(n) &= \coprod_{\tau \in \mathcal{T}1(n)} \mathcal{P}\tau \times \hat{\mathbb{G}}(\hat{\tau}, X) \\ &= \{\langle p, t\rangle \mid p \in \mathcal{P}(n), t \in \mathcal{T}X(n), \pi_1(p) = \mathcal{T}!(t)\}\end{aligned}$$

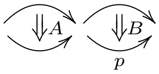
We will use the notation $\langle p, t\rangle$ extensively in what follows. Note that $\mathcal{P}!\langle p, t\rangle = p$ and $\pi\langle p, t\rangle = t$

Thus, we can think of $\mathcal{P}1$ as consisting of a subset $\mathcal{P}\tau$ for each pasting diagram $\tau \in \mathcal{T}1$. We can think of $\mathcal{P}X$ as consisting of a subset $\mathcal{P}\tau \times \{t\}$ for each labeled pasting diagram $t \in \mathcal{T}X(n)$ with $\mathcal{T}!(t) = \tau$.

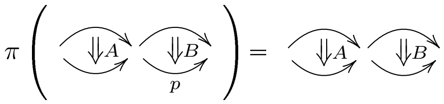
Example 5.2. Let $\tau \in \mathcal{T}1(2)$, $t \in \mathcal{T}X(2)$ be those described in Example 5.1. Let $p \in \mathcal{P}\tau$. We let the following diagram denote p,

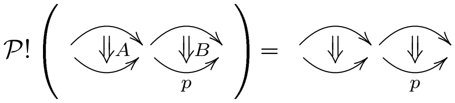


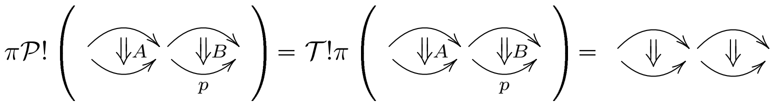
and we let the following diagram denote $\langle p, t\rangle \in \mathcal{P}X(2)$.



That is, to denote an element $\langle p, t\rangle \in \mathcal{P}X$, we draw the diagram $\mathcal{T}!(t)$, label the diagram with the necessary arrows to depict t, and label the entire diagram p. Note that the π acts on such a diagram by removing the label p, $\mathcal{P}!$ acts by removing the arrow labels, and $\pi\mathcal{P}! = \mathcal{T}!\pi$ acts by removing all labels.







Definition 5.3. Let $\pi : \mathcal{P} \implies \mathcal{T}$ be a $\mathcal{T}$-operad. We say that $\mathcal{P}$ is *normal* if π_1 restricted to $\mathcal{P}1(0)$ is a bijection.

Definition 5.4. Let $\pi : \mathcal{P} \implies \mathcal{T}$ be a $\mathcal{T}$-operad. We say that $\mathcal{P}$ is *contractible* if for each $n < \omega$, parallel pair of cells $x, y \in P1(n)$, and cell $z \in T1(n+1)$ with $z : \pi(x) \to \pi(y)$, there exists a cell $w \in P1(n+1)$ with $w : x \to y$ and $\pi(w) = z$. We say that $\mathcal{P}$ *has a contraction* if $\mathcal{P}$ is equipped with such a function $(n, x, y, z) \mapsto w$.

Definition 5.5. A *globular operad* is a normal $\mathcal{T}$-operad with a contraction.

Lemma 5.6. *Let $\pi : \mathcal{P} \implies \mathcal{T}$ be a globular operad, let $X \in \hat{\mathbb{G}}$, and let $n < \omega$. Let $x, y \in \mathcal{P}X(n)$ be parallel with $\pi(x) = \pi(y)$. Then there exists $w \in \mathcal{P}X(n+1)$ such that $w : x \to y$ and $\pi(w) = 1_{\pi(x)} = 1_{\pi(y)}$.*

Proof. Write $x = \langle x', z\rangle$ and $y = \langle y', z\rangle$ where $\mathcal{P}!\langle x', z\rangle = x', \mathcal{P}!\langle y', z\rangle = y'$, and $\pi\langle x', z\rangle = \pi\langle y', z\rangle = z$.

Since π is contractible, from $x', y' \in \mathcal{P}1(n)$ and $\mathcal{T}!(1_z) \in \mathcal{T}1(n+1)$, there is $v : x' \to y' \in \mathcal{P}1(n+1)$ such that $\pi(v) = \mathcal{T}!(1_z)$.

Thus there is an element $w = \langle v, 1_z\rangle \in \mathcal{P}X(n+1)$ as desired. □

In particular, if $\pi : \mathcal{P} \to \mathcal{T}$ is a globular operad, for each $x \in \mathcal{P}X(n)$, there exists a specified 'weak identity' $1_x \in \mathcal{P}X(n+1)$ with $1_x : x \to x$, and $\pi(1_x) = 1_{\pi x}$.

Let $\pi : \mathcal{P} \to \mathcal{T}$ be a globular operad. If we have a sequence $x_1, ..., x_n \in \mathcal{P}X(n)$ such that $s(x_i) = t(x_{i+1})$ for all $1 \leqslant i < n$, there is a clear composition $\pi x_1 \cdots \pi x_n$ in $\mathcal{T}X(n)$. Let $\tau = \mathcal{T}!(\pi x_1 \cdots \pi x_n)$ in $\mathcal{T}1$. Since $\pi\mathcal{P}!s(x_1) = s\tau$ and $\pi\mathcal{P}!t(x_n) = t\tau$, by contraction there is a specified $c : \mathcal{P}!s(x_1) \to \mathcal{P}!t(x_1) \in \mathcal{P}\tau$ such that $\pi p = \tau$. Thus, contraction implies that there is at least one composition $\langle c, \pi x_1 \cdots \pi x_n\rangle$ of $x_1, ..., x_n$ in $\mathcal{P}X$.

We now would like a notion of equivalence between any two parallel cells. We might like to say that $x, y \in \mathcal{P}X(n)$ are equivalent if they are parallel and there exist $f, g \in \mathcal{P}X(n+1)$ such that $gf, 1_x$ are equivalent and $fg, 1_y$ are equivalent.

Definition 5.7. Let $\pi : \mathcal{P} \to \mathcal{T}$ be a globular operad, let $X \in \hat{\mathbb{G}}$, and let $n < \omega$. We say that $x \in \mathcal{P}X(n)$ is an *equivalence* if $\pi(x) = 1_{\pi s(x)}$. We say that $y, z \in \mathcal{P}X(n-1)$ are *equivalent* if there exists an equivalence $x : y \to z$.

Let $x : y \to z \in \mathcal{P}X(n)$ be an equivalence. Then since $\pi t(x) = t\pi(x) = t(1_{\pi s(x)}) = \pi s(x)$,

$$\pi(y) = \pi(z)$$

By Lemma 5.6, there is $x' : z \to y$ such that $\pi(x') = 1_{\pi y}$. We know how to compose in $\mathcal{T}$, so let $\pi(x)\pi(x')$ denote the unique composition in $\mathcal{T}$. Since π is a monad morphism, for any composite $xx' = \langle c, \pi(x)\pi(x')\rangle$ in $\mathcal{P}X$, $\pi(xx') = \pi(x)\pi(x')$. Thus $\pi(xx') = \pi(x'x) = 1_{\pi y}1_{\pi y} = 1_{\pi y}$. Since $\pi(1_y) = \pi(1_z) = 1_{\pi y}$, by Lemma 5.6, there are equivalences $1_y \to x'x$ and $1_z \to xx'$. Thus, our definition of equivalence is in accordance with the suggested definition of equivalence above.

We would now like to understand what it might mean for an n-cell in $\mathcal{P}X$ to be natural. Let $a : b \to c \in X(n+1)$. Let $\beta, \gamma \in \mathcal{T}X(n)$ be elements of $\hat{\mathbb{G}}(\hat{\tau}, X)$ such that $\beta(i) = b$, $\gamma(i) = c$ for some $i \in \hat{\tau}(n)$ and β, γ agree everywhere else.

Define $\alpha \in \hat{\mathbb{G}}(\hat{\sigma}, X)$ so that α is the union of β and γ in dimensions less than or equal to n and is the function $\{(* \mapsto a)\}$ restricted to dimension $n+1$ so that $\alpha : \beta \to \gamma \in \mathcal{T}X(n+1)$. Call α the *extension* of β along a.

Now, for any $p, q \in P\tau$, there are equivalences $\langle e, 1_\beta\rangle : \langle p, \beta\rangle \to \langle q, \beta\rangle$ and $\langle e, 1_\gamma\rangle : \langle p, \gamma\rangle \to \langle q, \gamma\rangle$ in $\mathcal{P}X(n+1)$ by Lemma 5.6. Let $\overline{p}, \overline{q} \in \mathcal{P}1(n+1)$ be such that $s(\overline{p}) = t(\overline{p}) = p$ and $s(\overline{q}) = t(\overline{q}) = q$. Consider the following diagram in $\mathcal{P}X$.

$$\begin{tikzcd}
\langle p, \beta\rangle \arrow[r, "{\langle e, 1_\beta\rangle}"] \arrow[d, "{\langle \overline{p}, \alpha\rangle}"] & \langle q, \beta\rangle \arrow[d, "{\langle \overline{q}, \alpha\rangle}"] \\
\langle p, \gamma\rangle \arrow[r, "{\langle e, 1_\gamma\rangle}"] & \langle q, \gamma\rangle
\end{tikzcd}$$

Under π, this square maps to

$$\begin{tikzcd}
\beta \arrow[r, "1_\beta"] \arrow[d, "\alpha"'] & \beta \arrow[d, "\alpha"] \\
\gamma \arrow[r, "1_\gamma"'] & \gamma
\end{tikzcd}$$

where $1_\gamma \alpha = \alpha = \alpha 1_\beta$. By Lemma 5.6, there is an equivalence in $\mathcal{P}X(n+2)$ which fills the top square. Thus this square commutes up to a higher equivalence. In this sense, $\langle e, 1_\beta \rangle$ is natural in b. This motivates the following definition and proves the following lemma.

Definition 5.8. Let $\pi : \mathcal{P} \to \mathcal{T}$ be a globular operad and let $e : x \to y$ be an equivalence in $\mathcal{P}X(n+1)$. We say that e is *natural* in $b \in X(n)$ if b is in the image of $\pi(e)$ where $\pi(e)$ is regarded as an element of $\hat{\mathbb{G}}(\hat{\tau}, X)$.

Lemma 5.9. *Let $\langle e, 1_\beta \rangle : \langle p, \beta \rangle \to \langle q, \beta \rangle$ be an equivalence in $\mathcal{P}X(n+1)$ natural in $b \in X(n)$. Let $a : b \to c \in X(n+1)$, and let $\alpha : \beta \to \gamma \in \mathcal{T}X(n+1)$ be an extension of β along a. Let $\overline{p}, \overline{q} \in \mathcal{P}1(n+1)$ be such that $s(\overline{p}) = t(\overline{p}) = p$ and $s(\overline{q}) = t(\overline{q}) = q$. Then there is an equivalence $\langle e, 1_\gamma \rangle : \langle p, \gamma \rangle \to \langle q, \gamma \rangle \in \mathcal{P}X(n+1)$ natural in c, and $\langle \overline{q}, \alpha \rangle \langle e, 1_\beta \rangle$ is equivalent to $\langle e, 1_\gamma \rangle \langle \overline{p}, \alpha \rangle$ in $\mathcal{P}X$.*

Now we move on to defining Batanin weak ω-categories. Let $\mathcal{M}on(\hat{\mathbb{G}})$ denote the category of cartesian monads and cartesian monad morphisms on $\hat{\mathbb{G}}$, and consider the slice category $\mathcal{M}on(\hat{\mathbb{G}})/\mathcal{T}$. Consider the category whose objects are $\langle \mathcal{P}, c \rangle$ where $\mathcal{P}$ is a normal $\mathcal{T}$-operad from $\mathcal{M}on(\hat{\mathbb{G}})/\mathcal{T}$ with contraction c and whose arrows $f : \langle \mathcal{P}, c \rangle \to \langle \mathcal{Q}, d \rangle$ are arrows $f : \mathcal{P} \to \mathcal{Q}$ of $\mathcal{M}on(\hat{\mathbb{G}})/\mathcal{T}$ such that $fc = df$. Call this the *category of globular operads*. This category has an initial object. For the rest of this paper, let $\pi : (\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}}) \implies (\mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}})$ denote the initial object of $\mathcal{C}$.

Definition 5.10. A *Batanin weak ω-category* is a $\mathcal{P}$-algebra $\lambda : \mathcal{P}X \to X$ where $\mathcal{P}$ is regarded as a monad.

Definition 5.11. A *Batanin weak n-category* is a Batanin weak ω-category $\lambda : \mathcal{P}X \to X$ such that for all $m > n$, $s, t : X(m) \to X(m-1)$ are bijections and $s = t$.

Let $\langle \lambda : \mathcal{P}X \to X, \pi : \mathcal{P} \to \mathcal{T} \rangle$ be a Batanin ω-category. We call $e \in X(n)$ an *equivalence* if it is a composition $\lambda(\lambda\tilde{e}_1 \cdots \lambda\tilde{e}_m)$ in X where $e_1, ..., e_m$ are equivalences in $\mathcal{P}X$. We say an equivalence $e \in X(n)$ is *natural* in $A \in X(n-1)$ if it is a composition $\lambda(\lambda\tilde{e}_1 \cdots \lambda\tilde{e}_m)$ in X where $e_1, ..., e_m$ are equivalences natural in $\tilde{A}$ in $\mathcal{P}X$.

Now we unravel some of the monad axioms implicit in the definition of Batanin ω-categories.

Lemma 5.12. *Let $\langle \lambda : \mathcal{P}X \to X, \pi : \mathcal{P} \to \mathcal{T} \rangle$ be a Batanin ω-category. Then for any $\langle p, t \rangle$ in $\mathcal{P}^2X$,*

$$\mu^{\mathcal{P}} \langle p, t \rangle = \langle \mu^{\mathcal{P}} \langle p, \mathcal{P}! \circ t \rangle, \mu^{\mathcal{T}}(\pi \circ t) \rangle$$

where $\pi \circ t$, $\mathcal{P}! \circ t$ denote the composition of $\pi, \mathcal{P}!$, respectively, with t regarded as an element of $\hat{\mathbb{G}}(\widehat{\mathcal{T}!(t)}, \mathcal{P}X)$.

Proof. Recall that

$$\mathcal{P}X(n) = \coprod_{\tau \in \mathcal{T}1(n)} \mathcal{P}\tau \times \hat{\mathbb{G}}(\hat{\tau}, X),$$

$$\mathcal{P}^2X(n) = \coprod_{\tau \in \mathcal{T}1(n)} \mathcal{P}\tau \times \hat{\mathbb{G}}(\hat{\tau}, \mathcal{P}X).$$

Thus, for any transformation α with domain $\mathcal{P}$, the induced transformation $\mathcal{P}\alpha$ with domain $\mathcal{P}^2$ is given by

$$(\mathcal{P}\alpha)(n) = \coprod_{\tau \in \mathcal{T}1(n)} \mathcal{P}\tau \times \hat{\mathbb{G}}(\hat{\tau}, \alpha).$$

That is, for any $\langle p, t\rangle \in \mathcal{P}^2X$,

$$\mathcal{P}\alpha\langle p, t\rangle = \langle p, \alpha \circ t\rangle.$$

Since π is a monad morphism, the following diagram commutes,

$$\begin{tikzcd}
\mathcal{P}^2X \arrow[r, "\mathcal{P}\pi"] \arrow[dr, "\mu^{\mathcal{P}}"'] & \mathcal{P}\mathcal{T}X \arrow[r, "\pi"] & \mathcal{T}^2X \arrow[d, "\mu^{\mathcal{T}}"] \\
& \mathcal{P}X \arrow[r, "\pi"] & \mathcal{T}X
\end{tikzcd}$$

and thus

$$\pi\mu^{\mathcal{P}}\langle p, t\rangle = \mu^{\mathcal{T}}\pi(\mathcal{P}\pi)\langle p, t\rangle = \mu^{\mathcal{T}}(\pi \circ t).$$

Since $\mathcal{P}$ is a cartesian monad, the following diagram is a pullback,

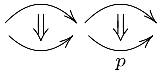
$$\begin{tikzcd}
\mathcal{P}^2X \arrow[r, "\mu^{\mathcal{P}}"] \arrow[d, "\mathcal{P}^2!"] & \mathcal{P}X \arrow[d, "\mathcal{P}!"] \\
\mathcal{P}^21 \arrow[r, "\mu^{\mathcal{P}}"] & \mathcal{P}1
\end{tikzcd}$$

and thus

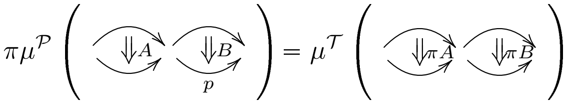
$$\mathcal{P}!\mu^{\mathcal{P}}\langle p, t\rangle = \mu^{\mathcal{P}}(\mathcal{P}(\mathcal{P}!))\langle p, t\rangle = \mu^{\mathcal{P}}\langle p, \mathcal{P}! \circ t\rangle.$$

□

Example 5.13. Let p be an element of $\mathcal{P}1$ depicted by the following diagram.



Let $A, B \in \mathcal{P}X$ be such that $ts(A) = ss(B)$. Then Lemma 5.12 says that the following equality holds in $\mathcal{T}X$.



Lemma 5.14. *Let* $\langle \lambda : \mathcal{P}X \to X, \pi : \mathcal{P} \to \mathcal{T}\rangle$ *be a Batanin ω-category. Let* $\langle p, t\rangle, \langle p, s\rangle \in \mathcal{P}^2X(n)$ *be such that* $\lambda \circ t = \lambda \circ s$ *where* $\lambda \circ t$ *denotes the composite of* λ *with* t *regarded as an element of* $\hat{\mathbb{G}}(\widehat{\mathcal{T}!(t)}, \mathcal{P}X)$. *Then*

$$\lambda\mu^{\mathcal{P}}\langle p, t\rangle = \lambda\mu^{\mathcal{P}}\langle p, s\rangle.$$

In particular, if $\langle p,t\rangle \in \mathcal{P}X(n)$ *and* $\langle p,s\rangle \in \mathcal{P}^2X(n)$ *such that* $t = \lambda \circ s$*, then*

$$\lambda\langle p,t\rangle = \lambda\mu^{\mathcal{P}}\langle p,s\rangle.$$

Proof. Suppose that $\langle p,t\rangle, \langle p,s\rangle \in \mathcal{P}^2X(n)$ are such that $\lambda \circ t = \lambda \circ s$. Since

$$(\mathcal{P}\lambda)(n) = \coprod_{\tau\in\mathcal{T}1(n)} \mathcal{P}\tau \times \hat{\mathbb{G}}(\hat{\tau},\lambda),$$

we see that

$$(\mathcal{P}\lambda)\langle p,t\rangle = \langle p,\lambda\circ t\rangle = \langle p,\lambda\circ s\rangle = (\mathcal{P}\lambda)\langle p,s\rangle.$$

Since λ is a monad algebra, the following diagram commutes,

$$\begin{tikzcd}
\mathcal{P}^2X \arrow[r, "\mathcal{P}\lambda"] \arrow[d, "\mu"] & \mathcal{P}X \arrow[d, "\lambda"] \\
\mathcal{P}X \arrow[r, "\lambda"] & X
\end{tikzcd}$$

and thus

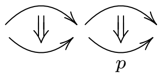
$$\lambda\mu^{\mathcal{P}}\langle p,t\rangle = \lambda(\mathcal{P}\lambda)\langle p,t\rangle = \lambda(\mathcal{P}\lambda)\langle p,s\rangle = \lambda\mu^{\mathcal{P}}\langle p,s\rangle.$$

Now suppose that $\langle p,t\rangle \in \mathcal{P}X(n)$ and $\langle p,s\rangle \in \mathcal{P}^2X(n)$ such that $t = \lambda\circ s$. Then

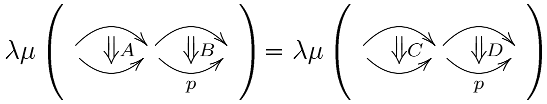
$$\lambda\langle p,t\rangle = \lambda\langle p,\lambda\circ s\rangle = \lambda(\mathcal{P}\lambda)\langle p,s\rangle = \lambda\mu^{\mathcal{P}}\langle p,s\rangle$$

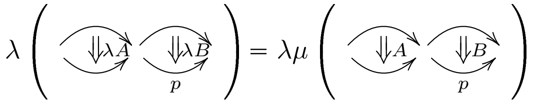
□

Example 5.15. Let p be an element of $\mathcal{P}1$ depicted by the following diagram.



Let $A,B,C,D \in \mathcal{P}X$ be such that $ts(A) = ss(B)$, $ts(C) = ss(D)$. Suppose that $\lambda(A) = \lambda(C)$, and $\lambda(B) = \lambda(D)$. Then by Lemma 5.14, the following equalities hold in X.





Let $\langle \lambda : \mathcal{P}X \to X, \pi : \mathcal{P} \to \mathcal{T}\rangle$ be a Batanin ω-category. For any $x \in X(n)$, we let $\tilde{x}$ denote both $\eta^{\mathcal{P}}(x) \in \mathcal{P}X(n)$ and $\eta^{\mathcal{T}}(x) \in \mathcal{T}X(n)$. Since π is a monad morphism, the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\eta^{\mathcal{P}}"] \arrow[dr, "\eta^{\mathcal{T}}"'] & \mathcal{P}X \arrow[d, "\pi"] \\
& \mathcal{T}X
\end{tikzcd}$$

and we see that

$$\pi(\tilde{x}) = \tilde{x}.$$

Since λ is a monad algebra, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta^{\mathcal{P}}"] \arrow[dr, equal] & \mathcal{P}X \arrow[d, "\lambda"] \\ & X \end{tikzcd}$$

and we see that

$$\lambda(\tilde{x}) = x.$$

Now we come to the main theorem.

Theorem 5.16. *A Batanin ω-category with one* 0*-cell and one* 1*-cell is braided monoidal.*

Let $\langle \lambda : \mathcal{P}X \to X, \pi : \mathcal{P}X \to \mathcal{T}X \rangle$ be such a category. Contraction specifies a $p_2 \in \mathcal{P}1(1)$ depicted by

$$\longrightarrow \; \xrightarrow[p_2]{}$$

and then it also specifies $r \in \mathcal{P}1(2)$ such that $s(r) = t(r) = p_2$ which is depicted by

$$\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, bend left] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow"] & {} \end{tikzcd}$$

Then let $A \otimes B$ denote the element of $X(2)$ depicted by the following diagram.

$$\lambda\left(\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & {} \arrow[r, bend left] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow B"] & {} \end{tikzcd}\right)$$

Thus X is equipped with a monoidal product $\otimes$. We say that this product is *braided monoidal* if for all $A, B \in X(2)$ there is an equivalence $\beta_{AB} : A \otimes B \to B \otimes A \in X(3)$ natural in A and B satisfying axioms analogous to (M8) and (M9).

Corollary 5.17. *A Batanin weak* 3*-category with one* 0*-cell and one* 1*-cell is braided monoidal.*

Now we prove Theorem 5.16. We do not show all the coherence laws for a braided monoidal category which are tedious but can be checked. We show that for any $A, B \in X(2)$, there is an equivalence $\beta_{AB} : A \otimes B \to B \otimes A \in X(3)$ natural in A and B such that β satisfies the analogue of axiom (M.10).

Proof. Let $\langle \lambda : \mathcal{P}X \to X, \pi : \mathcal{P} \to \mathcal{T} \rangle$ be a Batanin weak ω-category with one 0-cell and one 1-cell.

Since $\mathcal{P}$ is normal, $\mathcal{P}X(0) = \mathcal{T}X(0) = X(0)$. Thus, $\mathcal{P}X$ has a single 0-cell.

Let $0 \in \mathcal{T}X(1)$ denote the strict identity of the single 0-cell in $\mathcal{T}X$. By Lemma 5.6, there is a specified 1-cell in $\mathcal{P}X$ which we also denote $0 \in \mathcal{P}X(1)$ with source and target the unique 0-cell of $\mathcal{P}X$ such that $\pi(0) = 0$. Thus, we regard this as the weak identity on the 0-cell.

Let $1 \in X(1)$ denote the unique 1-cell of X, and let 1 also denote $\eta^{\mathcal{P}}1 \in \mathcal{P}X(1), \eta^{\mathcal{T}}1 \in \mathcal{T}X(1)$.

Let $p \in 1(1)$ denote the unique 1-cell of the terminal globular set 1. Thus there is $1_p : p \to p \in \mathcal{T}1(2)$, and contraction specifies $1_p : p \to p \in \mathcal{P}1(2)$. Then p and 1_p are denoted by the following diagrams

$$\xrightarrow[p]{} \qquad \xrightarrow[1_p]{}$$

so there are elements of $\mathcal{P}^2X(1)$ depicted by:

$$\xrightarrow[p]{0} \qquad \xrightarrow[p]{1}$$

and elements of $\mathcal{P}^2X(2)$ depicted by:

$$\xrightarrow[1_p]{0} \qquad \xrightarrow[1_p]{1}$$

Define

$$1_0 = \mu^{\mathcal{P}}\left(\xrightarrow[1_p]{0} \right), \quad 1_1 = \mu^{\mathcal{P}}\left(\xrightarrow[1_p]{1} \right)$$

to be elements in $\mathcal{P}X(2)$ which we regard as the weak identities of 0 and 1 in $\mathcal{P}X$. Now $\lambda 0 = 1 = \lambda 1$, so by Lemma 5.14, $\lambda 1_0 = \lambda 1_1$. Let

$$i = \lambda 1_0 = \lambda 1_1$$

in $X(2)$. By Lemma 5.12,

$$\pi 1_0 = \pi\mu^{\mathcal{P}}\left(\xrightarrow[1_p]{0} \right) = \mu^{\mathcal{T}}\left(\xrightarrow{\pi 0} \right) = \xrightarrow{0} = 1_0$$

in $\mathcal{T}X(2)$, and similarly

$$\pi 1_1 = 1_1.$$

in $\mathcal{T}X(2)$.

Since the following diagrams commute,

$$\begin{tikzcd}
\mathcal{P}X \arrow[r, "\eta"] \arrow[d, "!"] & \mathcal{P}^2X \arrow[d, "\mathcal{P}!"] \\
1 \arrow[r, "\eta"] & \mathcal{P}1
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{P}X \arrow[r, "\eta"] \arrow[dr, "\eta"'] & \mathcal{P}^2X \arrow[d, "\pi"] \\
& \mathcal{T}\mathcal{P}X
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{P}X \arrow[r, "\eta"] \arrow[dr, equal] & \mathcal{P}^2X \arrow[d, "\mu"] \\
& \mathcal{T}\mathcal{P}X
\end{tikzcd}$$

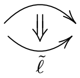
we see that for all $x \in \mathcal{P}X(2)$.

$$x = \mu^{\mathcal{P}}\eta^{\mathcal{P}}x = \mu^{\mathcal{P}}\left(\xrightarrow[p]{x} \right).$$

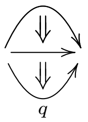
Therefore in particular, $s(1_0) = t(1_0) = 0$ and $s(1_1) = t(1_1) = 1$.

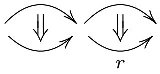
Since 1_0 is a weak identity of a weak identity of the unique object of X and 1_1 is a weak identity of the unique 1-cell of X, it will be appropriate to think of these as weak 'horizontal' and 'vertical' identities respectively. Thus, the horizontal and vertical identities do not coincide in $\mathcal{P}X$, but their images under λ do coincide in X. Note that in the proof of Theorem 4.3, we used 1 as a weak identity for both $*$ and $\otimes$, so the analogue to this held in traditional tricategories.

Let ℓ denote the unique 2-cell of the terminal globular set 1. Then there is $\tilde{\ell} = \eta^{\mathcal{P}}\ell \in \mathcal{P}1(2)$ depicted by the following diagram.



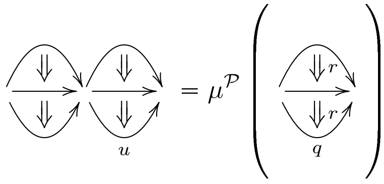
Now, contraction specifies $q, r \in \mathcal{P}1(2)$ which are depicted by

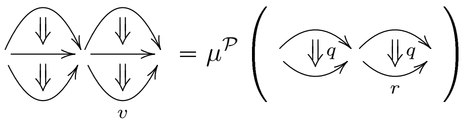


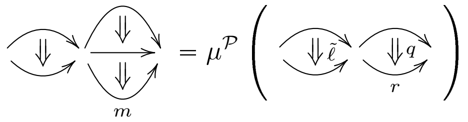


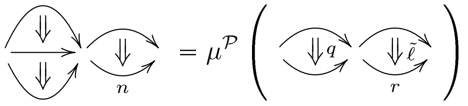
respectively, such that $s(q) = t(q) = p$ and $s(r) = t(r) = p_2$.

Define $u, v, m, n \in \mathcal{P}1(2)$ by the following diagrams.









Note that

$$s(u) = t(u) = \mu^{\mathcal{P}} \left(\xrightarrow[p]{p_2} \right) = p_2$$

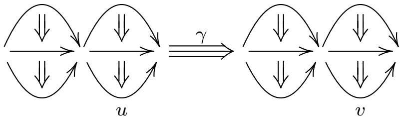
by the observation above, and since the following diagram commutes,

$$\begin{tikzcd}
\mathcal{P}1 \arrow[r, "\mathcal{P}\eta"] \arrow[dr, equal] & \mathcal{P}^2 1 \arrow[d, "\mu"] \\
& \mathcal{P}1
\end{tikzcd}$$

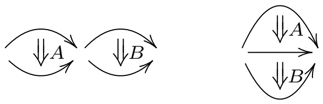
we see that

$$s(v) = t(v) = \mu^{\mathcal{P}} \left(\xrightarrow{p} \xrightarrow[p_2]{p} \right) = p_2.$$

Thus, since $s(u) = s(v)$, $t(u) = t(v)$, and $\pi u = \pi v$, there is an equivalence $\gamma \in \mathcal{P}1(3)$ depicted below.

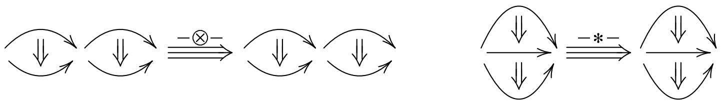


For any $A, B \in X(2)$, let $A \otimes B, A * B \in \mathcal{T}X(2)$ denote



respectively. Let $A \otimes B, A * B$ also denote $\langle r, A \otimes B\rangle, \langle q, A * B\rangle \in \mathcal{P}X(2)$ where the meaning is clear.

There are pasting diagrams in $\mathcal{T}1(3)$.



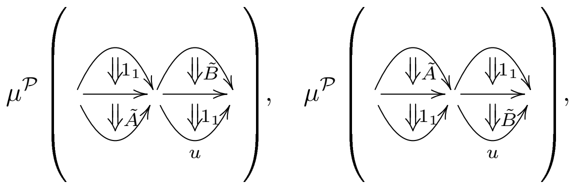
Let $x : A \to B, y : C \to D \in \mathcal{P}X(3)$ be equivalences so that $\pi x = 1_{\pi A}, \pi y = 1_{\pi C}$. Let $\overline{r}, \overline{q} \in \mathcal{P}1(3)$ be such that $s(\overline{r}) = t(\overline{r}) = r$, $\pi\overline{r} = - \otimes -$, $s(\overline{q}) = t(\overline{q}) = q$, and $\pi\overline{q} = - * -$ (which exist by contraction). Since by Lemma 5.12,

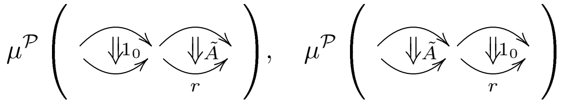
$$\pi\mu^{\mathcal{P}}\langle \overline{r}, x \otimes y\rangle = \mu^{\mathcal{T}}(\pi x \otimes \pi y) = \mu^{\mathcal{T}}(1_{\pi A} \otimes 1_{\pi B}) = 1_{A \otimes B},$$

$\mu^{\mathcal{P}}\langle \overline{r}, x \otimes y\rangle$ is an equivalence. Similarly, $\mu^{\mathcal{P}}\langle \overline{q}, x * y\rangle$ is an equivalence.

If $x : A \to B$ is equivalent to $x' : A \to B$ and $y : C \to D$ is equivalent to $y' : C \to D$ in $\mathcal{P}X$, we can similarly show that $\mu^{\mathcal{P}}\langle \overline{r}, x \otimes y\rangle$ is equivalent to $\mu^{\mathcal{P}}\langle \overline{r}, x' \otimes y'\rangle$ and $\mu^{\mathcal{P}}\langle \overline{q}, x * y\rangle$ is equivalent to $\mu^{\mathcal{P}}\langle \overline{q}, x' * y'\rangle$ in $\mathcal{P}X$.

Now, for all $A, B \in X(2)$, there are elements of $\mathcal{P}X$ given by the following diagrams.





Since $\pi\tilde{A} = \tilde{A}$ and $\pi 1_0 = 1_0$, $\pi 1_1 = 1_1$ are strict identities, we see by Lemma 5.12 that the following equalities hold in $\mathcal{T}X$.

$$\pi\mu^{\mathcal{P}}\left(\begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{A}"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{B}"] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \end{tikzcd}\right) = \mu^{\mathcal{T}}\left(\begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{A}"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{B}"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \end{tikzcd}\right) = \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow B"] \arrow[r, bend right=50] & {} \end{tikzcd} = \pi\left(\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow B"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd}\right),$$

$$\pi\mu^{\mathcal{P}}\left(\begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{A}"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{B}"] & {} \end{tikzcd}\right) = \mu^{\mathcal{T}}\left(\begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{A}"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{B}"] & {} \end{tikzcd}\right) = \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow B"] \arrow[r, bend right=50] & {} \end{tikzcd} = \pi\left(\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow B"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd}\right),$$

$$\pi\mu^{\mathcal{P}}\left(\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow 1_0"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow \tilde{A}"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd}\right) = \mu^{\mathcal{T}}\left(\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow 1_0"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow \tilde{A}"] \arrow[r, bend right=50] & {} \end{tikzcd}\right) = \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \end{tikzcd},$$

$$\pi\mu^{\mathcal{P}}\left(\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow \tilde{A}"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow 1_0"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd}\right) = \mu^{\mathcal{T}}\left(\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow \tilde{A}"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow 1_0"] \arrow[r, bend right=50] & {} \end{tikzcd}\right) = \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \end{tikzcd}.$$

Thus, by Lemma 5.6, there are equivalences in $\mathcal{P}X(3)$ natural in A, B:

$$\begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow B"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd} \overset{\langle x, 1_{A\otimes B}\rangle}{\Rrightarrow} \mu^{\mathcal{P}} \begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{A}"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{B}"] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \end{tikzcd}$$

$$\mu^{\mathcal{P}} \begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{A}"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{B}"] & {} \end{tikzcd} \overset{\langle y, 1_{A\otimes B}\rangle}{\Rrightarrow} \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow A"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow B"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd}$$

$$\mu^{\mathcal{P}} \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow 1_0"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow \tilde{A}"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd} \overset{\langle z, 1_A\rangle}{\Rrightarrow} \mu^{\mathcal{P}} \begin{tikzcd} {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow \tilde{A}"] \arrow[r, bend right=50] & {} \arrow[r, bend left=50] \arrow[r, phantom, "\Downarrow 1_0"] \arrow[r, bend right=50, "r"'] & {} \end{tikzcd}$$

Note that $\langle x, 1_{A\otimes B}\rangle, \langle y, 1_{A\otimes B}\rangle, \langle z, 1_A\rangle$ are the analogues of $\rho^*_A \otimes \lambda^*_B$, $\lambda^*_A \otimes \rho^*_B$, $\rho^{\otimes}_A \lambda^{\otimes}_A$, respectively, from the proof of Theorem 4.3.

By Lemma 5.12, we see that since $\lambda 1_1 = \lambda i$, the following equalities hold.

$$\lambda\mu^{\mathcal{P}}\left(\begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{B}"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{A}"] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \end{tikzcd}\right) = \lambda\left(\begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow i"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow B"] & {} \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow A"] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift right=3ex, "\Downarrow i"] & {} \end{tikzcd}\right)$$

$$\lambda\mu^{\mathcal{P}}\left(\begin{tikzcd}[column sep=large] {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \tilde{A}"] \arrow[r, phantom, shift right=3ex, "\Downarrow 1_1"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow \tilde{B}"] & {} \end{tikzcd}\right) = \lambda\left(\begin{tikzcd}[column sep=large] {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow A"] \arrow[r, phantom, shift right=3ex, "\Downarrow i"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow i"] \arrow[r, phantom, shift right=3ex, "\Downarrow B"] & {} \end{tikzcd}\right)$$

Since $\lambda 1_0 = i, \lambda\tilde{B} = B$, by Lemma 5.14, we see that

$$s\lambda\langle z, 1_B\rangle = \lambda s\langle z, 1_B\rangle = \lambda\mu^{\mathcal{P}}\langle r, 1_{1_0\otimes\tilde{B}}\rangle = \lambda\langle r, 1_{i\otimes B}\rangle,$$

and similarly,

$$\begin{aligned} t\lambda\langle z, 1_A\rangle &= \lambda\langle r, 1_{A\otimes i}\rangle, \\ t\lambda\langle z, 1_B\rangle &= \lambda\langle r, 1_{B\otimes i}\rangle, \\ s\lambda\langle z, 1_A\rangle &= \lambda\langle r, 1_{i\otimes A}\rangle. \end{aligned}$$

Thus, by Lemma 5.14 again, we see that

$$\lambda\left(\begin{tikzcd}[column sep=huge] {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "q"'] \arrow[r, phantom, shift left=3ex, "s\lambda\langle z,1_B\rangle"] \arrow[r, phantom, shift right=3ex, "t\lambda\langle z,1_A\rangle"] & {} \end{tikzcd}\right) = \lambda\left(\begin{tikzcd}[column sep=large] {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow i"] \arrow[r, phantom, shift right=3ex, "\Downarrow A"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow B"] \arrow[r, phantom, shift right=3ex, "\Downarrow i"] & {} \end{tikzcd}\right)$$

$$\lambda\left(\begin{tikzcd}[column sep=huge] {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "q"'] \arrow[r, phantom, shift left=3ex, "t\lambda\langle z,1_B\rangle"] \arrow[r, phantom, shift right=3ex, "s\lambda\langle z,1_A\rangle"] & {} \end{tikzcd}\right) = \lambda\left(\begin{tikzcd}[column sep=large] {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow B"] \arrow[r, phantom, shift right=3ex, "\Downarrow i"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow i"] \arrow[r, phantom, shift right=3ex, "\Downarrow A"] & {} \end{tikzcd}\right)$$

Therefore,

$$\begin{aligned} t\lambda\langle x, 1_{A\otimes B}\rangle &= s\lambda(\lambda\langle z, 1_B\rangle * \lambda\langle z, 1_A\rangle^{-1}), \\ t\lambda(\lambda\langle z, 1_B\rangle * \lambda\langle z, 1_A\rangle^{-1}) &= s\lambda\langle y, 1_{B\otimes A}\rangle. \end{aligned}$$

Then we can make the following composite in X which we call β_{AB}.

$$\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & {} \arrow[r, bend left] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow B"] & {} \end{tikzcd} \overset{\lambda\langle x,1_{A\otimes B}\rangle}{\Rrightarrow} \begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow i"] \arrow[r, phantom, shift right=3ex, "\Downarrow A"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow B"] \arrow[r, phantom, shift right=3ex, "\Downarrow i"] & {} \end{tikzcd} \overset{\lambda(\lambda\langle z,1_B\rangle}{\underset{\lambda\langle z,1_A\rangle^{-1})}{\overset{*}{\Rrightarrow}}} \begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow B"] \arrow[r, phantom, shift right=3ex, "\Downarrow i"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow i"] \arrow[r, phantom, shift right=3ex, "\Downarrow A"] & {} \end{tikzcd} \overset{\lambda\langle y,1_{B\otimes A}\rangle}{\Rrightarrow} \begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & {} \arrow[r, bend left] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow A"] & {} \end{tikzcd}$$

(Note that we have omitted writing λ on each 2-cell in the interest of space though it is implied.)

Now we show that β_{Ai} is equivalent to $\lambda\langle z, 1_A\rangle^{-1}$ for all $A \in X(2)$. This is (M.10) of a braided monoidal category and was also proved for Theorem 4.3.

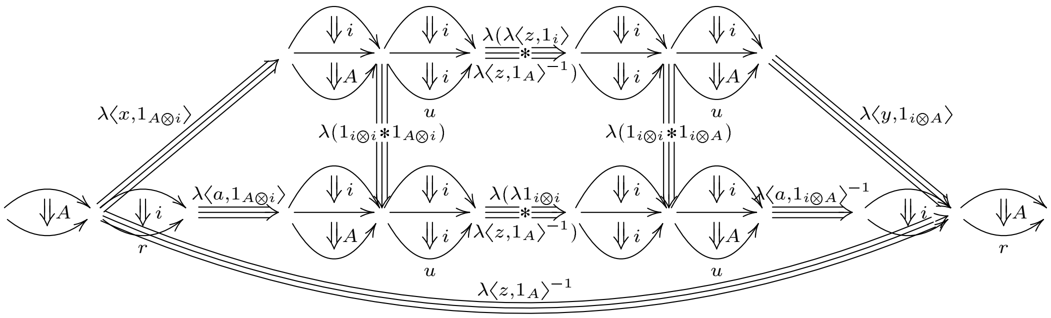
Let $\langle a, 1_{A\otimes B}\rangle$ denote the equivalence in $\mathcal{P}X$ below.

$$\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & {} \arrow[r, bend left] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow B"] & {} \end{tikzcd} \Rrightarrow \begin{tikzcd} {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow A"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60, "u"'] \arrow[r, phantom, shift left=3ex, "\Downarrow 1_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow B"] & {} \end{tikzcd}$$

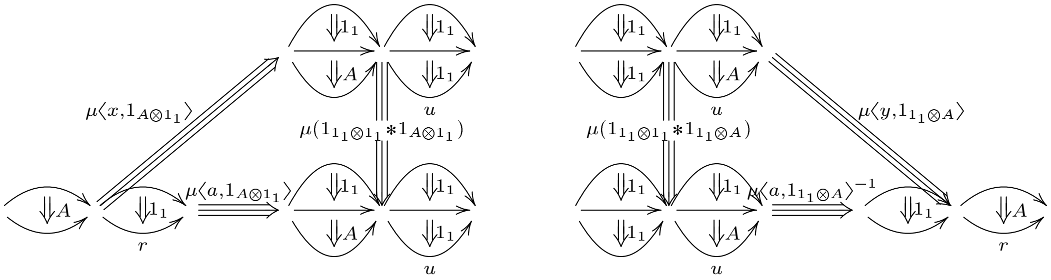
From now on, we omit writing $\mu^{\mathcal{P}}$ since all diagrams will be in $\mathcal{P}X$ or X.

We show that the following diagram commutes.

(5.18)

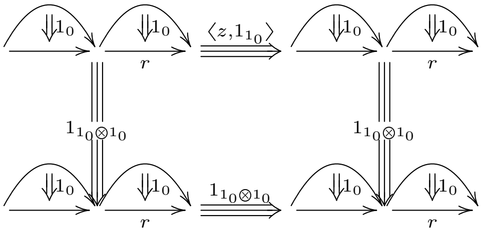


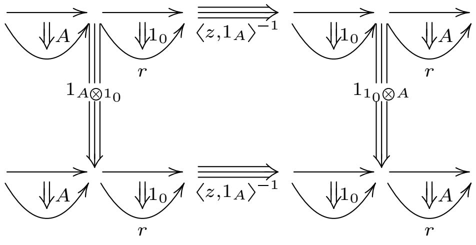
Since $\lambda 1_1 = i$, by Lemma 5.14, $\lambda\langle x, 1_{A\otimes i}\rangle = \lambda\mu\langle x, 1_{\tilde{A}\otimes 1_1}\rangle$, $\lambda(1_{i\otimes i} * 1_{A\otimes i}) = \lambda\mu(1_{1_1\otimes 1_1} * 1_{\tilde{A}\otimes 1_1})$, $\lambda(1_{i\otimes i} * 1_{i\otimes A}) = \lambda\mu(1_{1_1\otimes 1_1} * 1_{\tilde{1}_1\otimes A})$, and $\lambda\langle y, 1_{i\otimes A}\rangle = \lambda\mu\langle y, 1_{\tilde{1}_1\otimes A}\rangle$. Therefore, the two triangles in diagram (5.18) are images of



under λ. Since each 3-cell in the left triangle maps to $1_{A\otimes 1_1}$ under π, and each 3-cell in the right triangle maps to $1_{1_1\otimes A}$, there are equivalences filling these triangles, and thus there are equivalences also filling the corresponding triangles in Diagram (5.18).

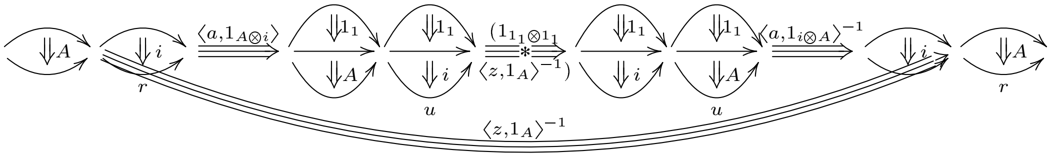
Since $\lambda 1_0 = i$, by Lemma 5.14, $\lambda\langle z, 1_i\rangle = \lambda\mu\langle z, 1_{1_0}\rangle$, $\lambda 1_{i\otimes i} = \lambda\mu 1_{1_0\otimes 1_0}$, $\lambda 1_{A\otimes i} = \lambda\mu 1_{\tilde{A}\otimes 1_0}$, and $\lambda 1_{i\otimes A} = \lambda\mu 1_{1_0\otimes\tilde{A}}$. The square in Diagram (5.18) is the composition of the images of





under λ. Since each 3-cell in the top square maps to 1_{1_0} under π, and each 3-cell in the bottom square maps to 1_A under π, there are equivalences filling these squares, and thus there is an equivalence also filling the corresponding square in Diagram (5.18). Note that we will use the fact that $\lambda\langle z, 1_i\rangle$ is equivalent to $\lambda 1_{1_0\otimes 1_0}$ is equivalent to $\lambda\langle z, 1_i\rangle^{-1}$ again later.

Since the equivalences $\langle a, 1_{A\otimes i}\rangle$ and $\langle a, 1_{i\otimes A}\rangle$ are natural in $A\otimes i, i\otimes A$ respectively, we see that there is an equivalence filling the following diagram, and thus there is an equivalence filling the corresponding part of Diagram (5.18).



Thus Diagram (5.18) commutes, and the composite β_{Ai} above is equivalent to $\lambda\langle z, 1_A\rangle^{-1}$ fulfilling axiom (M.10). □

Acknowledgments. It is a pleasure to thank my mentor, Daniel Schaeppi for his guidance and support.

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