THEOREM NUMBER PRIME THE: A BACKWARDS PROOF

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ABSTRACT. The Prime Number Theorem describes the distribution of the primes in terms of an asymptotic relation; namely, that the number of primes less than or equal to x is asymptotically equivalent to the function $\log x/x$. We will prove this relation, and in doing so discuss many fundamental topics in the field of analytic number theory, including Dirichlet convolutions and the zeta function.

CONTENTS

1. INTRODUCTION

The Prime Number Theorem is the statement that the number of primes less than or equal to a given positive, real number x is approximately $x/\log(x)$. The theorem originated as a pattern that mathematicians noticed as they were looking at tables of prime numbers. The theorem remained an open question until the end of the 19th century, when it was proven by Hadamard and de la Valle Poussin (for a more detailed discussion of the history of The Prime Number Theorem, see chapter five of [1]).

The modern proof of the Prime Number Theorem unites many topics in real analysis, complex analysis, and number theory. In this paper, the author will assume the reader is familiar with basic identities, formulas, and definitions from these fields. For readers who are not familiar with arithmetic functions, an appendix is provided. The formulas from complex analysis are taken as givens, because an in-depth discussion of the subject is prohibited by length constraints. Additionally, the technical details of the proofs of these formulas are outside the scope of this project. The author instead refers interested readers to [3], where chapter three offers a detailed look at the tools used in this paper, and chapters seven and eight deal with the Prime Number Theorem from the vantage point of complex analysis specifically.

The author acknowledges a great debt to A.J. Hildebrand's online notes, *An Introduction to Analytic Number Theory* ([1]). These notes guided my research and provided a wonderful example of a proof of the Prime Number Theorem. The proof within is in substance similar to the proof presented in chapter five of [1] and chapter eight of [3].

What differentiates this paper from a rehashing of the proof is its form. A typical proof begins with a series of complicated estimates on the zeta function. These early sections seem to have little relevance, and it is difficult to see the big picture. To combat this problem, the proof in this paper is presented in reverse. First, the Prime Number Theorem is reduced and simplified, until it becomes a question of getting certain bounds on the zeta function.

We begin in Section 2 by turning the question of the Prime Number Theorem into a question of an integral of a related function $\psi(x)$. In Section 3, we deduce what estimate of the integral we would need to prove the Prime Number Theorem, and show that this estimate will be valid if certain estimates on the zeta function are valid. In Section 4 we finally prove the bounds on the zeta function, completing the theorem.

2. Simplifying the problem

At first glance, the Prime Number Theorem seems almost impossible to attack. For one thing, there is not an obvious closed form of $\pi(x)$. Our first step is to reduce the problem to a function that does have a closed form.

2.1. π, ψ, θ, and Chebyshev Estimates.

We begin by defining two auxiliary functions, ψ and θ. These functions are related to $\pi(x)$ but are easier to work with.

Definition 2.1. $\pi(x)$ *is the function that returns the number of primes less than or equal to a given number* x*. It is defined for real, positive* x*.*

Definition 2.2. *The von Mangold function is defined as* $\Lambda(n) = \log(p)$*, if* $n = p^m$*, and as* $\Lambda(n) = 0$ *for other values of* n*.*

The identity $(\Lambda * 1)(n) = \log(n)$ can easily be verified using the definitions of Dirichlet convolutions and the definitions of the functions involved.

Definition 2.3. $\psi(x) = \sum_{n \leq x} \Lambda(n)$. $\psi(x)$ *is defined for real, positive* x*.*

Definition 2.4. $\theta(x) = \sum_{p \leq x} \log p$ *for real, positive* x*.*

These functions are useful in finding the Chebyshev estimate for $\pi(x)$. The Chebyshev estimate provides a weaker version of the Prime Number Theorem, namely that $\pi(x) \asymp \frac{x}{\log x}$. This notation says that for given functions $f(x)$ and $g(x)$, if $f(x) \asymp g(x)$, there exist constants c_1 and c_2 such that for $x \geq 2$ the inequality $c_1 g(x) \leq f(x) \leq c_2 g(x)$ holds. This result is derived from the Chebyshev type estimates for $\psi(x)$ and $\theta(x)$. Note that the Prime Number Theorem is equivalent to stating that c_1 and c_2 above are equal to 1. In the interest of space, we will only sketch the proofs. For a more detailed discussion of Chebyshev type estimates, including complete proofs, consult Chapter 3 of [1].

Theorem 2.5.

$$\psi(x) \asymp x.$$

Sketch of Proof. We first note that it suffices to prove the statement for $x \geq x_0$. Then we apply Euler's Partial Summation formula (for a proof and formal statement see chapter 2 in [1]) to $S(x) = \sum_{n \leq x} \log n$, and bound the trailing terms. We define a new function $D(x) = S(x) - 2S(x/2)$. Then we find another estimate of $S(x)$ using the identity $\log n = (\Lambda * 1)(n) = \sum_{d|n} \Lambda(d)$. We write $D(x)$ in terms of these two estimates of $S(x)$. These estimates yield the inequalities $\psi(x) \geq D(x)$ and $\psi(x) \leq D(x) + \psi(x/2)$. We can use these relations to apply the bounds on $D(x)$ to $\psi(x)$, and achieve the desired result. □

Corollary 2.6.

$$\theta(x) \asymp x.$$

Sketch of Proof. We begin by writing

$$\begin{aligned}
\psi(x) - \theta(x) &= \sum_{p^m \leq x} \log p - \sum_{p \leq x} \log p \\
&= \sum_{p \leq \sqrt{x}} \log p \sum_{2 \leq m \leq \frac{\log x}{\log p}} 1 \\
&\leq \sum_{p \leq \sqrt{x}} (\log p) \frac{\log x}{\log p} = \sum_{p \leq \sqrt{x}} \log x \\
&\leq \sqrt{x} \log x.
\end{aligned}$$

As $\sqrt{x} \log x$ is positive for $x \geq 1$, it is clear that $\theta(x) \leq \psi(x)$. Thus the upper bound established in Theorem 2.5 holds for $\theta(x)$ as well, with the same values of c_2 and x_0. Algebraic manipulation shows that $\theta(x) \geq \psi(x) - \sqrt{x} \log x$. Therefore the lower bound can be $c_1/2$, though x_0 must also be increased correspondingly. □

Corollary 2.7.

$$\pi(x) \asymp \frac{x}{\log x}.$$

Sketch of Proof. The lower bound in this case comes almost directly from Corollary 2.6. The upper bound follows from the inequality,

$$\pi(x) \leq \pi(\sqrt{x}) + \frac{1}{\sqrt{\log x}} \sum_{\sqrt{x} < p \leq x} \log p \leq \frac{2}{\log x} \theta(x),$$

and the upper bound in Corollary 2.6. □

2.2. **Relationships among π, ψ, and θ.** In this subsection, we will establish relationships among our three functions π, ψ, and θ. This will later enable us to rewrite our problem in terms of $\psi(x)$.

Lemma 2.8. $\theta(x) = \psi(x) + O(\sqrt{x})$.

Proof. Begin by noting that

$$\psi(x) - \theta(x) = \sum_{p^m \leq x} \log p - \sum_{p \leq x} \log p.$$

The first sum provides $m \log p$ where m is the maximum integer such that $1 \leq m \leq \frac{\log x}{\log p}$. Because the second sum subtracts one of each of these m copies, we can rewrite the sum as

$$\begin{aligned} \sum_{p^m \leq x} \log p - \sum_{p \leq x} \log p &= \sum_{p \leq \sqrt{x}} \log p - \sum_{2 \leq m \leq \frac{\log x}{\log p}} 1 \\ &\leq \sum_{p \leq \sqrt{x}} (\log p) \frac{\log x}{\log p} \\ &= \sum_{p \leq \sqrt{x}} \log x. \end{aligned}$$

From here, apply the Chebyshev estimate for $\pi(x)$ (Corollary 2.7), to get

$$\sum_{p \leq \sqrt{x}} \log x \leq \pi(\sqrt{x}) \log x = O\left(\frac{\sqrt{x}}{\log(\sqrt{x})} \log x \right) = O(\sqrt{x}).$$

This completes the proof. □

The above lemma allows us to rewrite the easy estimate of $\pi(x)$ in terms of $\theta(x)$ as the more useful estimate of $\pi(x)$ in terms of $\psi(x)$.

Lemma 2.9. $\pi(x) = \frac{\psi(x)}{\log x} + O(\frac{x}{\log^2 x})$.

Proof. We begin with the definition of $\pi(x)$:

$$\pi(x) = \sum_{p \leq x} 1 = \sum_{p \leq x} \log p \frac{1}{\log p}.$$

An application of Euler's summation formula (see chapter 2 in [1]) gives us

$$\frac{\theta(x)}{\log x} - \int_2^x \theta(t) \left(-\frac{1}{t(\log t)^2} \right) dt = \frac{\theta(x)}{x} + \int_2^x \theta(t) \left(\frac{1}{t(\log t)^2} \right) dt.$$

Using Chebyshev's estimate that $\theta(x) \asymp x$ (Corollary 2.6), we get

$$\begin{aligned}\int_2^x \theta(t)\left(\frac{1}{t(\log t)^2}\right) dt &= O\left(\int_2^x \frac{1}{(\log t)^2} dt\right) \\ &\leq O\left(\int_2^{\sqrt{x}} \frac{1}{(\log 2)^2} dt + \int_{\sqrt{x}}^x \frac{1}{(\log \sqrt{t})^2} dt\right) \\ &= O\left(\sqrt{x} + \frac{x}{(\log \sqrt{x})^2}\right) \\ &= O\left(\frac{x}{(\log \sqrt{x})^2}\right) \\ &= O\left(\frac{x}{(\log x)^2}\right).\end{aligned}$$

This estimate shows that

$$\pi(x) = \frac{\theta(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Lemma 2.8 allows $\theta(x)$ to be replaced with $\psi(x)$, giving us

$$\pi(x) = \frac{\psi(x) + O(\sqrt{x})}{\log x} + O\left(\frac{x}{(\log x)^2}\right) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\sqrt{x}\log x}\right) + O\left(\frac{x}{(\log x)^2}\right).$$

But, as $\sqrt{x} \geq \log x$ for large x,

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

This completes the proof. □

2.3. **A statement equivalent to the Prime Number Theorem.** The above statement of $\pi(x)$ in terms of $\psi(x)$ allows us to generate an equivalent statement of the Prime Number Theorem. This statement is easier to manipulate, and easier to prove. From here on we will use the notation $\sim$ to indicate that for functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.

Theorem 2.10. *The statement $\psi(x) \sim x$ implies the Prime Number Theorem.*

Proof. Suppose $\psi(x) \sim x$, or $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$. Note that by Lemma 2.9, we have

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Divide both sides of the above equation by $\frac{x}{\log x}$, and take the limit to get

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = \lim_{x\to\infty} \frac{\psi(x)}{x} + O\left(\frac{1}{\log x}\right) = 1$$

or $\pi(x) \sim \frac{x}{\log x}$, which is the Prime Number Theorem. □

Although this statement is easier to work with than our original statement, it is still rather difficult to estimate $\psi(x)$. Rather than estimating $\psi(x)$, we will estimate $\psi_1(x) = \int_1^x \psi(y)dy = \sum_{n\leq x} \Lambda(n)(x-n)$. It is useful to estimate this value, because we can apply Perron's Formula. This formula relates a complex integral of a Dirichlet series of a function to the integral of the partial sums of the original function, in this case $\Lambda(x)$ (recall that $\psi(x)$ equals the partial sums of $\Lambda(y)$). Perron's Formula (Theorem A.4) gives us the following relation:

$$\psi_1(x) = \int_1^x \psi(y)dy = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}ds.$$

$F(s)$ is the Dirichlet series associated with the von Mangold function. Lemma A.14 gives us $F(s) = -\zeta'(s)/\zeta(s)$. This complex integral allows us to use the tools of complex analysis to complete the proof.

2.4. **Changing the estimate from $\psi(x)$ to $\int_1^x \psi(y)dy$.** It is relatively straightforward to turn an estimate of $\int_1^x \psi(y)dy$ into an estimate of $\psi(x)$. The following lemma shows that given an estimate of $\int_1^x \psi(y)dy$ with a suitable remainder term, we can achieve the desired estimate of ψ.

Theorem 2.11. *Suppose we have an estimate of $\psi_1(x)$ such that for large enough x, $\psi_1(x) = \frac{x^2}{2} + O(f(x))$, where $\lim_{x\to\infty}\frac{f(x)}{x^2} = 0$. Then we have the Prime Number Theorem.*

Proof. Choose x large enough such that our estimate holds. Thus, because $\psi(x)$ is nondecreasing, for $0 < \delta < 1/2$ we have

$$\psi_1(x) - \psi_1(x(1-\delta)) = \int_{x(1-\delta)}^x \psi(y)dy \le \delta x\psi(x).$$

We then plug in our hypothetical estimate to generate

$$\begin{aligned}
\delta x\psi(x) &\ge \frac{x^2}{2} + O(f(x)) - \frac{(x(1-\delta))^2}{2} + O(f(x(1-\delta))) \implies \\
\delta x\psi(x) &\ge \frac{x^2 - x^2 + 2\delta x^2 - \delta^2}{2} + O(f(x) + f(x(1-\delta))) \implies \\
\psi(x) &\ge x - \frac{\delta}{2} + O\left(\frac{f(x) + f(x(1-\delta))}{x}\right) \\
&\ge x + O\left(\frac{f(x) + f(x(1-\delta))}{x}\right).
\end{aligned}$$

We can use a similar argument using x and $x(1+\delta)$ to generate the inequality

$$\psi(x) \le x + O\left(\frac{f(x) + f(x(1-\delta))}{x}\right).$$

These two inequalities together imply

$$\psi(x) = x + O\left(\frac{f(x) + f(x(1-\delta))}{x}\right).$$

Now it just remains to show that $\psi(x) \sim x$. We have shown

$$\lim_{x\to\infty}\frac{\psi(x)}{x} = 1 + O\left(\frac{f(x) + f(x(1-\delta))}{x^2}\right).$$

Note that by hypothesis $\lim_{x\to\infty}\frac{f(x)}{x^2} = 0$, which means the error term above must go to zero, implying that $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$. We can then use Theorem 2.10, which allows us to complete the proof of the Prime Number Theorem. □

3. An estimate for ψ_1

As I mentioned earlier, we transferred our estimate from $\psi(x)$ to $\psi_1(x)$ so that we could use the tools of complex analysis, more specifically contour integration, to generate our estimates. The estimates will be based on Perron's Formula. Perron's Formula is

$$\begin{aligned}\psi_1(x) &= \int_1^x \psi(y)dy \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}ds \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds, \qquad (1)\end{aligned}$$

for $a > \max(\sigma_0, 0)$ where $F(s)$ converges on the half-plane $\sigma > \sigma_0$. From this point we will use the notation $\sigma = Re(s)$ and $t = Im(s)$. We will now pause to give an overview of how we will obtain an estimate of this integral.

3.1. **Overview.** We begin by assuming $x \geq e$, choosing $a = 1 + 1/\log x$, setting $1/2 < 1 - c/(\log T)^9 = b < 1$, and restricting T such that $e \leq T \leq x$. We will later show that there exists a c such that there are no zeros in the path L_6, L_4, L_3, L_2. The fact that these assumptions are valid will be proven later. In this section we will use two hypothetical bounds on $|\zeta'(s)/\zeta(s)|$ to estimate the remainder of $\psi_1(x)$. These bounds will be stated as givens in this section, and proven later. We will then split the contour into 6 paths (see Figure 1):

$$\begin{array}{ll} L_1 = (a - i\infty, a - Ti] & L_2 = (b - Ti, a - Ti] \\ L_3 = (b + Ti, b - Ti] & L_4 = (a + Ti, b + Ti] \\ L_5 = (a + Ti, a + i\infty) & L_6 = (a - Ti, a + Ti]. \end{array}$$

We will later show that we can choose c small enough such that there is only the simple pole $s = 1$ in the interior of the path L_6, L_4, L_3, L_2. By the Cauchy Integration Formula, we can then evaluate (1) on L_6 simply by calculating the residue of the pole at $s = 1$, and estimating (1) on L_2, L_3, and L_4. We will then estimate (1) on L_1 and L_5.

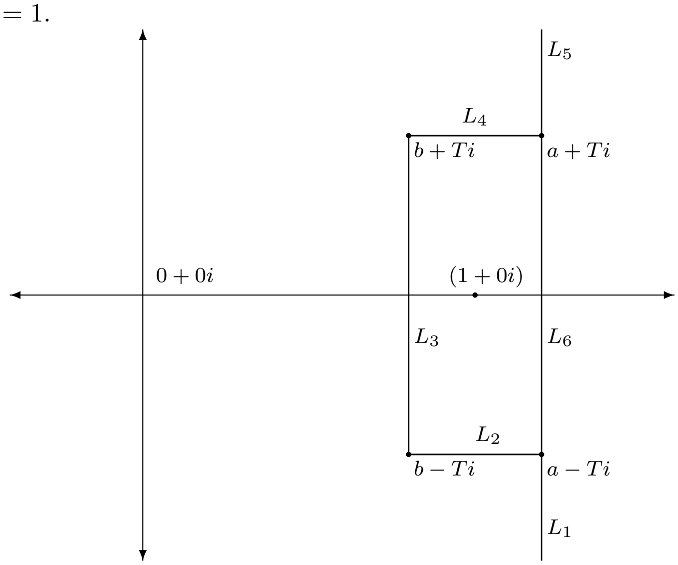
3.2. **The Existence and Residue of the Simple Pole at $s = 1$.** The following Theorem indicates that as long as there are no zeros in the interior of the path L_6, L_4, L_3, L_2 defined above, the integral along said path is $x^2/2$. This result comes from the Cauchy Integration Formula. A detailed discussion of the intricacies of complex analysis is beyond the scope of this paper, and we assume these theorems without proof. For a more detailed discussion of residues and complex integration, consult Chapter 3 of [3].

Theorem 3.1. *The function*

$$g_x(s) = -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}$$

has a simple pole at $s = 1$ with residue $x^2/2$.

FIGURE 1. The paths of integration. Note that the path L_6, L_4, L_3, L_2 forms a closed, counterclockwise loop around the pole at $s = 1$.



Proof. Using the continuations established in Lemma A.8 and Lemma A.13, we know

$$g_x(s) = -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)} = -\frac{\frac{-1}{(s-1)^2} - f'(s)}{\frac{s}{s-1} - f(s)}\frac{x^{s+1}}{s(s+1)},$$

where $f(s) = s\int_1^\infty \{x\}x^{-s-1}dx$. Note that in this proof we assume that $f'(s)$ exists near $s = 1$, a fact that is easily verified given that $f(s)$ is analytic near $s = 1$. From there, some manipulation yields:

$$g(s) = -\frac{\frac{-1}{(s-1)^2} - f'(s)}{\frac{s}{s-1} - f(s)}\frac{x^{s+1}}{s(s+1)} = -\frac{-1-(s-1)^2 f'(s)}{s(s-1)-(s-1)^2 f(s)}\frac{x^{s+1}}{s(s+1)}.$$

If the function has a simple pole at s=1, the following limit must exist and, if the second statement in the theorem is true, the limit must be equal to $x^2/2$. Luckily,

$$\lim_{s\to 1}(s-1)g(s) = \lim_{s\to 1} -\frac{-1-(s-1)^2 f'(s)}{s-(s-1)f(s)}\frac{x^{s+1}}{s(s+1)} = \frac{x^2}{2}.$$

This completes the proof. □

The previous Theorem proved that

$$\begin{aligned}\psi_1(x) &= \int_{a-i\infty}^{a+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds \\ &= \int_{L_1} g_x(s)ds + \int_{L_5} g_x(s)ds + \frac{x^2}{2} - \int_{L_2} g_x(s)ds - \int_{L_3} g_x(s)ds - \int_{L_4} g_x(s)ds,\end{aligned}$$

where we choose $a > 1$, $b < 1$, and T (a function of x) such that there are no zeros in the interior of the path L_6, L_4, L_3, L_2. Thus it remains to be shown that

$$\int_{L_1} g_x(s)ds + \int_{L_5} g_x(s)ds - \int_{L_2} g_x(s)ds - \int_{L_3} g_x(s)ds - \int_{L_4} g_x(s)ds = O(f(x)),$$

where $f(x)$ is such that $\lim_{x\to\infty} f(x)/x^2 = 0$.

3.3. **Estimating the Remainder.** Recall that we assumed $x \geq e$, chose $a = 1+1/\log x$, set $1/2 < 1-c/(\log T)^9 = b < 1$, and restricted T such that $e \leq T \leq x$. In this section we will use two hypothetical bounds on $|\zeta'(s)/\zeta(s)|$ to estimate the remainder of $\psi_1(x)$. These bounds will be stated as givens in this section, and proven later.

Lemma 3.2. *Suppose the following bound is valid for $\sigma \geq 1-c$, $|t| \leq 2$, $s \neq 1$:*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\max(1, 1/|\sigma - 1|)).$$

Then,

$$\int_{L_1} g_x(s)ds = \int_{L_1} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds = O\left(\frac{x^2\log x}{T}\right).$$

Proof. We begin by bounding $|\zeta'(s)/\zeta(s)|$ on L_1

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^a} = -\frac{\zeta'(a)}{\zeta(a)} = O\left(\frac{1}{|1-a|}\right) = O\left(\log x\right). \tag{2}$$

Note the second to last step comes from our hypothetical bound on $|\zeta'(s)/\zeta(s)|$, which we can use because we fixed $x \geq e$. Then we bound $|x^{s+1}/s(s+1)|$,

$$\left|\frac{x^{s+1}}{s(s+1)}\right| \leq \frac{x^{a+1}}{|s||s+1|} \leq \frac{x^{a+1}}{t^2} = \frac{ex^2}{t^2}. \tag{3}$$

Recall that $s = \sigma + ti$. From (2) and (3) we conclude that

$$\int_{L_1} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds < \int_T^{\infty}(\log x)\frac{x^2}{t^2}dt = O\left(\log x\frac{x^2}{T}\right).$$

□

Note that because our assumed bound also holds over L_5 we can bound $\int_{L_5} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds$ in the same manner. We will now bound the integrals over L_2 and L_4.

Lemma 3.3. *Suppose we have for all s in the range $\sigma \geq b$, $|t| \geq 2$, $s \neq 1$*

$$\left|\frac{\zeta'(s)}{\zeta s}\right| = O((\log|t|)^9).$$

Then we have

$$\int_{L_2} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds = O\left(\frac{x^2\log|T|^9}{T^2}\right).$$

Proof. We bound $|\zeta'(s)/\zeta(s)|$ by our hypothesis and $|x^{s+1}/s(s+1)|$ using the following inequality

$$\left|\frac{x^{s+1}}{s(s+1)}\right| \leq \frac{x^{a+1}}{|s||s+1|} \leq \frac{x^{a+1}}{t^2} = O\left(\frac{x^2}{T^2}\right).$$

We can see that by these estimates

$$\int_{L_2} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds < \int_a^b(\log T)^9\frac{x^2}{T^2}d\sigma = O\left((\log T)^9\frac{x^2}{T^2}\right).$$

Because L_4 is also within the range of our hypothetical bound, we can use the same argument to bound the integral on L_4. □

We will now estimate

$$\int_{L_3} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds = \int_{b-Ti}^{b+Ti} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds.$$

Lemma 3.4. *Assuming the same hypothetical bounds for $|\zeta'(s)/\zeta(s)|$:*

$$\left|\frac{\zeta'(s)}{\zeta s}\right| = O((\log|t|)^9) \quad (\sigma \geq b, |t| \geq 2, s \neq 1),$$

and

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\max(1, 1/|\sigma - 1|)) \quad (\sigma \geq 1 - c, |t| \leq 2, s \neq 1),$$

we have

$$\int_{L_3} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds = O(x^{b+1}(\log T)^9).$$

Proof. Between our two hypothetical estimates of $|\zeta'(s)/\zeta(s)|$, we can bound the function on the path in question

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\max(1, (\log T)^9/c, (\log|t|)^9))) = O((\log T)^9).$$

We then bound $|x^{s+1}/s(s+1)|$

$$\left|\frac{x^{s+1}}{s(s+1)}\right| = \frac{x^{b+1}}{|s||s+1|} = O(x^{b+1}\min(1, t^{-2})).$$

We combine the two bounds to conclude

$$\int_{L_3} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1}}{s(s+1)}ds = O\left(\int_{-T}^{T} x^{b+1}\min(1, t^{-2})(\log T)^9 dt\right) = O(x^{b+1}(\log T)^9).$$

□

Theorem 3.5. *If there exists a c such that the zeta function is zero-free for all s such that $\sigma \geq 1 - c/(\log T)^9$, and if*

$$\left|\frac{\zeta'(s)}{\zeta s}\right| = O((\log|t|)^9) \quad (\sigma \geq b, |t| \geq 2, s \neq 1),$$

and

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\max(1, 1/|\sigma - 1|)) \quad (\sigma \geq 1 - c, |t| \leq 2, s \neq 1).$$

are valid bounds, then we have the Prime Number Theorem.

Proof. We can now combine our estimates for the integrals established in Lemmas 3.2, 3.3, and 3.4 to find an estimate for $\psi_1(x)$. By the previous lemmas we can show that

$$\begin{aligned}\psi_1(x) &= \frac{x^2}{2} + 2O\left(\frac{x^2\log x}{T}\right) + 2O\left(\frac{x^2\log T^9}{T^2}\right) + O((x^{b+1})(\log T)^9)) \\ &= \frac{x^2}{2} + O\left(x^2\left(\frac{\log x}{T} + \frac{(\log T)^9}{T^2} + x^{b-1}(\log T)^9\right)\right).\end{aligned}$$

We note that because $x \geq e$ and $x \geq T \geq e$, $(\log T)^9 T^{-2}$ is of smaller order than the term $(\log x)T^{-1}$, and therefore it can be dropped. Now we have

$$O\left(\frac{\log x}{T} + x^{b-1}(\log T)^9\right) = O\left(\frac{\log x}{T} + \exp\left(-c\frac{\log x}{(\log T)^9}\right)(\log T)^9\right).$$

We also choose $T = \exp((\log x)^{1/10})$. Because $x \geq e$ this choice fits within our restriction $x \geq T \geq e$. After this choice we obtain

$$O\left(\frac{\log x}{T} + \exp\left(-c\frac{\log x}{(\log T)^9}\right)(\log T)^9\right) = O\left(\frac{\log x}{\exp((\log x)^{1/10})} + \frac{((\log x)^{9/10})}{\exp\left(c(\log x)^{1/10}\right)}\right)$$
$$= O(\exp\left(-c'(\log x)^{1/10}\right)),$$

where $c' < c$. Therefore

$$\psi_1(x) = \frac{x^2}{2} + O\left(x^2 \exp\left(-c'(\log x)^{1/10}\right)\right).$$

By Theorem 3.1 and Theorem 2.11, all that remains to be shown to get from the above estimate to the Prime Number Theorem, is that

$$\lim_{x\to\infty} \frac{1}{e^{c'(\log x)^{1/10}}} = 0.$$

This fact is obvious because $(\log x)^{1/10}$ is unbounded. □

4. Bounds on the Zeta Function and a Zero-Free Region

In the previous sections we have been able to show that the Prime Number Theorem boils down to estimates on the function $\zeta'(s)/\zeta(s)$ and the establishment of a zero-free region. To be more precise, we are finished if we establish the following three facts:

(4) $\quad$ There exists a c such that $\zeta(s) \neq 0$ for all $\sigma \geq 1 - c/(\log T)^9$

(5) $\quad \left|\frac{\zeta'(s)}{\zeta(s)}\right| = O((\log|t|)^9) \quad (\sigma \geq 1 - c/(\log T)^9, |t| \geq 2, s \neq 1)$

(6) $\quad \left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\max(1, 1/|\sigma - 1|)) \quad (\sigma \geq 1 - c, |t| \leq 2, s \neq 1).$

4.1. Upper Bounds on the Zeta Function and its Derivative.

Before we can prove these statements we need to get preliminary bounds on $\zeta(s)$ and $\zeta'(s)$.

Theorem 4.1. *For $|t| \geq 2$, $1/2 \leq \sigma_0 < 1$, $\sigma \geq \sigma_0$ the following holds,*

$$|\zeta(s)| \leq 4\frac{|t|^{1-\sigma_0}}{1-\sigma_0}.$$

Furthermore there is a positive, real number A_1 such that

$$|\zeta(s)| \leq A_1 \log|t| \quad (|t| \geq 2, \sigma \geq 1 - \frac{1}{4\log|t|}).$$

We will need two lemmas to prove this result.

Lemma 4.2.

$$|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{N^{1-\sigma}}{|t|} + \frac{|s|}{\sigma}N^{-\sigma} \quad (N \in \mathbb{N}, \sigma > 0, t \neq 0).$$

Proof. Left as an exercise. Hint: use Corollary A.9. □

Lemma 4.3.

$$|\zeta(s)| \leq \frac{N^{1-\sigma_0}}{1-\sigma_0} + \frac{N^{1-\sigma_0}}{|t|} + \left(1 + \frac{|t|}{\sigma_0}\right)N^{-\sigma_0} \quad (N \in \mathbb{N}, 1/2 < \sigma_0 < 1, \sigma \geq \sigma_0, t \neq 0).$$

Proof. Left as an exercise. Hint: use Lemma 4.2. □

Proof of Theorem 4.1. To prove the first statement we apply Lemma 4.3 with $N = [|t|]$, where $[x]$ is the greatest integer less than or equal to x. By the hypothesis $0 < \sigma_0 < 1$, we have $N^{1-\sigma_0} \leq |t|^{1-\sigma_0}$. So by Lemma 4.3

$$|\zeta(s)| \leq \frac{|t|^{1-\sigma_0}}{1-\sigma_)}\left(1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]}\right).$$

Now it simply remains to be shown that

$$1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]} \leq 4.$$

Using the hypotheses $|t| \geq 2$ and $0 < \sigma_0 < 1$,

$$\frac{(1-\sigma_0)}{|t|} \leq \frac{(1-\sigma_0)}{[|t|]} \leq \frac{1}{2}. \tag{7}$$

The givens $[|t|] \geq |t|/2$ and $1/2 \leq \sigma_0 < 1$ can be used to bound

$$\frac{(1-\sigma_0)|t|}{(\sigma_0[|t|])} \leq \frac{(1-\sigma_0)}{(\sigma_0)} \leq 2. \tag{8}$$

Thus

$$1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]} \leq 1 + 1/2 + 1/2 + 2 = 4.$$

This proves the first statement.

To prove the second statement, set $\sigma_0 = 1 - 1/(4\log|t|)$ and apply the first statement

$$|\zeta(s)| \leq 4\frac{|t|^{1/(4\log|t|)}}{1/(4\log|t|)} = \frac{4e^{1/4}}{1/\log|t|} = 16e^{1/4}\log|t|.$$

This proves the second statement, with $A_1 = 16e^{1/4}$. □

Theorem 4.4. *For $|t| \geq 2$, and $\sigma \geq 1 - \frac{1}{12\log|t|}$ the following bound holds*

$$|\zeta'(s)| \leq A_2 \log^2|t|. \tag{9}$$

Proof. For $\sigma \geq 2$ note that the Dirichlet series representation of $\zeta'(s)$ implies that $|\zeta'(s)| \leq \sum_{n=1}^{\infty}(\log n)n^{-2}$, so the asserted bound holds trivially, because $(\log n)/n^2$ converges. The analyticity of $\zeta(s)$ in the region $\text{Re}(s) > 0, s \neq 1$ implies that $|\zeta'(s)|$ is uniformly bounded in any compact rectangle contained in this region. Therefore the asserted bound holds in the range $2 \leq |t| \leq 3, 2 \geq \sigma \geq 1/2$. Therefore it only remains to show that such an A_2 exists when $|t| \geq 3$.

Now let s be in the range $\sigma \geq 1 - 1/(12\log|t|), |t| \geq 3$ and let $\delta = 1/(12\log|t|)$. We apply the Cauchy Residue Theorem

$$|\zeta'(s)| = \left|\frac{1}{2\pi i}\oint_{|s-s'|=\delta}\frac{\zeta(s)}{(s'-s)^2}ds\right| \leq \frac{1}{\delta}\max_{|s-s'|=\delta}|\zeta(s')|.$$

We will now show that for $|s - s'| = \delta, \zeta(s')$ is bounded by a constant multiple of $\log|t|$. In order to do this we will show that all s' in that range fall into the range of validity of the second statement of Theorem 4.1. Let $s' = \sigma' + it'$ and $|s' - s| \leq \delta$ is given. By the hypotheses $|t| \geq 3$ and $\sigma \geq 1 - \delta$ we know

$$|t'| \geq |t| - \delta \geq |t| - 1/12 > 2$$

and

$$|t'| \geq |t| + \delta \leq |t| + 1/12 \leq \frac{13}{12}|t| \leq |t|^{3/2}.$$

Therefore

$$\sigma' \geq \sigma - \delta \geq 1 - \frac{1}{6\log|t|} \geq 1 - \frac{1}{6\log|t'|^2/3} = 1 - \frac{1}{4\log|t'|}.$$

Accordingly all s' fall into the range of validity of Theorem 4.1. Thus

$$|\zeta(s')| \leq A_1 \log|t'| \leq (3/2)A_1\log|t| \quad (|s'-s| = \delta).$$

Substituting this back into our first estimate of $\zeta'(s)$ we get

$$|\zeta'(s)| \leq 18A_1\log^2|t|,$$

which is the desired estimate. □

4.2. **Zero-Free Region and Lower Bounds.** We will begin by proving the first statement, that we can choose a c, $0 < c < 1/2$ such that for $\sigma \geq 1 - c/(\log T)^9$ the zeta function is zero-free. Recall that by Corollary A.11 we know that there are no zeros in the half plane $\sigma > 1$. We shall prove that the zeta function is zero-free on a strip to the left of the line $\sigma = 1$ by establishing a lower bound for $\zeta(s)$ on that strip. We will provide one bound for $|t| \leq 2$ and another for $|t| \geq 2$.

Theorem 4.5. *The zeta function has no zeros on the half-plane $\sigma \geq 1$. Additionally there exist constants $c_1 > 0$ and $A_2 > 0$ such that*

$$\left|\frac{1}{\zeta(s)}\right| \leq A_2 \quad (\sigma > 1 - c_1, |t| \leq 2)$$

and constants $c_2 > 0$ and $A_3 > 0$ such that

$$\left|\frac{1}{\zeta(s)}\right| \leq A_3(\log|t|)^7 \quad (\sigma \geq 1 - c_2/(\log|t|)^9, |t| \geq 2). \tag{10}$$

In order to prove this Theorem we will need a few Lemmas.

Lemma 4.6 (3-4-1 Lemma)**.** *For any real number θ we have*

$$3 + 4\cos\theta + \cos(2\theta) \geq 0.$$

Proof. This follows immediately from the fact that

$$\begin{aligned} 0 \leq (1+\cos\theta)^2 &= 1 + 2\cos\theta + \cos^2\theta \\ &= 1 + 2\cos\theta + (1/2)(1+\cos(2\theta)) \\ &= (1/2)(3 + 4\cos\theta + \cos(2\theta)). \end{aligned}$$

□

Lemma 4.7. *We have*

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4|\zeta(\sigma+2it)| \geq 1 \quad (\sigma > 1, t \in \mathbb{R}).$$

Proof. We begin by noting that for $\sigma > 1$ we have

$$\begin{aligned} \log|\zeta(s)| &= \log\left|\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\right| = -\mathrm{Re}\sum_p \log\left(1 - \frac{1}{p^s}\right) \\ &= \mathrm{Re}\sum_p\sum_{m\geq 1}\frac{1}{mp^{ms}} = \sum_p\sum_{m\geq 1}\frac{\cos(t\log p^m)}{mp^{m\sigma}}. \end{aligned}$$

We apply this relation to $\sigma, \sigma + it$, and $\sigma + 2it$:

$$\begin{aligned}\log|\zeta(\sigma)^3\zeta(\sigma+it)^4|\zeta(\sigma+2it)| \\ &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it) \\ &= \sum_p \sum_{m\geq 1} \frac{3 + 4\cos(t\log p^m) + \cos(2t\log p^m)}{mp^{m\sigma}} \\ &\geq 0.\end{aligned}$$

The last step follows from Lemma 4.6. Then the result follows from the fact that if $\log x \geq 0$ then $x \geq 1$. □

Proof of Theorem 4.5. We will first prove that the zeta function has no zeros on the line $\sigma = 1$. This will complete the proof that the ζ function is zero-free for $\sigma \geq 1, s \neq 1$ because we have proved earlier that there are no zeros when $\sigma > 0$. Suppose for contradiction that $\zeta(1+it_0) = 0$. We have shown that the zeta function has a simple pole at $s = 1$, so necessarily $t_0 \neq 0$

We hope to apply Lemma 4.7, therefore we consider the behavior of $\zeta(\sigma)$, $\zeta(\sigma + it_0)$, and $\zeta(\sigma + 2it_0)$. Because $\zeta(s)$ has a pole at $s = 1$, $\zeta(\sigma)(\sigma - 1)$ is bounded as $\sigma \to 1+$. Additionally, by the analyticity of $\zeta(s)$,

$$\frac{\zeta(\sigma+it_0)}{\sigma-1} = \frac{\zeta(\sigma+it_0) - \zeta(1+it_0)}{(\sigma+it_0) - (1+it_0)}$$

stays bounded as $\sigma \to 1+$. Finally because $\zeta(s)$ is analytic at $1+2it_0$, $\zeta(s)$ converges to $\zeta(1 + 2it_0)$ as $\sigma \to 1+$, and is bounded. The function

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4|\zeta(\sigma+2it)| = (\sigma-1)|\zeta(\sigma)(\sigma-1)|^3 \left|\frac{\zeta(\sigma+it_0)}{\sigma-1}\right|^4 |\zeta(\sigma+2it_0)|$$

is of order $O(\sigma-1)$ and tends to zero as $\sigma \to 1+$. We have arrived at a contradiction by Lemma 4.7. Therefore, the zeta function cannot have a zero on the line $\sigma = 1$.

The proof of the second statement ($\left|\frac{1}{\zeta(s)}\right| \leq A_2$ for $\sigma > 1 - c_1, |t| \leq 2$) is trivial for $\sigma \geq 2$:

$$|\zeta(s)| \geq 1 - \sum_{n\geq 2} \frac{1}{n^2} = 1 - \left(\frac{\pi^2}{6} - 1\right) > 0$$

and therefore

$$\frac{1}{|\zeta(s)|} \leq (2 - \pi^2/6)^{-1} \quad (\sigma \geq 2).$$

It only remains to show that $1/\zeta(s)$ is uniformly bounded on the compact rectangle $1 - c_1 \leq \sigma \leq 2, |t| \leq 2$ for a sufficiently small c_1. We proved in the first statement that $1/\zeta(s)$ is bounded on the half plane $\sigma \geq 1$, and therefore on any compact region contained in this half-plane. We can choose the rectangle $1 \leq \sigma \leq 2, |t| \leq 2$. By compactness, $1/\zeta(s)$ must be bounded on any sufficiently small neighborhood of this rectangle. This implies we can choose a c_1 small enough such that the rectangle $1 - c_1 \leq \sigma \leq 2, |t| \leq 2$ is contained in that neighborhood.

As for the third statement ($\left|\frac{1}{\zeta(s)}\right| \leq A_3(\log|t|)^7$ for $\sigma \geq 1 - c_2/(\log|t|)^9, |t| \geq 2$), we use the same argument as above for the case $\sigma \geq 2$. Therefore all that remains is the case $\sigma \leq 2$. We fix a constant A to be chosen later and let t $|t| \geq 2$. First consider the range $1 + A(\log|t|)^{-9} \leq \sigma \leq 2$. By Lemma 4.7 we have

$$|\zeta(\sigma + it)| \geq \frac{1}{\zeta(\sigma)^{3/4}} \frac{1}{|\zeta(\sigma + 2it)|^{1/4}}.$$

But because $\zeta(s)$ has a simple pole at $s = 1$, there exists an absolute constant c_3 such that

$$\zeta(\sigma) \leq c + 3(\sigma - 1)^{-1}$$

for $1 < \sigma \leq 2$. Also, by Theorem 4.1, we have

$$|\zeta(\sigma + 2it)| \leq A_1 \log|2t| \leq 2A_1 \log|t|.$$

We insert these bounds into the result from Lemma 4.7, and restrict the range to $1 + A(\log|t|)^{-9} \leq \sigma \leq 2$. After this process we have

$$|\zeta(\sigma + it)| \geq c_3^{-3/4}(2A_1)^{-1/4}(\sigma - 1)^{3/4}(\log|t|)^{-1/4} \geq c_4 A^{3/4}(\log|t|)^{-7},$$

where $c_4 = c_3^{-3/4}(2A_1)^{-1/4}$ is an absolute constant. To complete the proof we show that if A is chosen sufficiently small then a bound of the same type holds in the range $1 - A(\log|t|)^{-9} \leq \sigma \leq 1 + A\log|t|^{-9}$.

Let

$$\sigma_1 = 1 - A(\log|t|)^{-9}$$
$$\sigma_2 = 1 + A(\log|t|)^{-9}.$$

Note that for $\sigma_1 \leq \sigma \leq \sigma_2$ we have

$$\begin{aligned} |\zeta(\sigma + it)| &= \left| \zeta(\sigma_2 + it) - \int_\sigma^{\sigma_2} \zeta'(u + it)du \right| \\ &\geq |\zeta(\sigma_2 + it)| - (\sigma_2 - \sigma_1) \max_{\sigma_1 \leq u \leq \sigma_2} |\zeta'(u + it)|. \end{aligned}$$

The first term on the right can be approximated by $c_4 A^{3/4}(\log|t|)^{-7}$. We also have bounded $\zeta'(s)$ by $A_2(\log|t|)^2$, when σ satisfies $\sigma \geq 1 - (12\log|t|)^{-1}$. We can choose A small enough that the bound for $\zeta'(s)$ holds on our new range. Therefore,

$$|\zeta(\sigma + it)| \geq c_4 A^{3/4}(\log|t|)^{-7} - 2A(\log|t|)^{-9} A_2(\log|t|)^2.$$

Choosing A small enough the coefficient $c_4 - 2A^{1/4}A_2$ becomes positive and we obtain $|\zeta(\sigma + it)| \geq c_5(\log|t|)^{-7}$. □

Theorem 4.8 (The Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\log(x)}.$$

Proof. Note that (5) follows for $c =$ some c_1 from combining the estimates (9) and (10) from Theorems 4.1 and 4.5 respectively. Theorem 4.5 also proves (4), for $c =$ some sufficiently small c_2. (6) follows from the analytic properties of $\zeta(s)$. Because $1/\zeta(s)$ is analytic in $\sigma \geq 1$ and $\zeta(s)$ is analytic in $\sigma \geq 0$, except for a simple pole at $s = 1$, the logarithmic derivative $\zeta'(s)/\zeta(s)$ is analytic in $\sigma \geq 1$ except for a simple pole at $s = 1$. Therefore $(s-1)\zeta'(s)/\zeta(s)$ is analytic in $\sigma \geq 1$. By compactness the analyticity extends to a neighborhood of any compact set contained in that interval, such as $\sigma \geq 1 - c, |t| \leq 2$, provided $c = c_3$ is sufficiently small. It follows that the function is bounded in the region $1-c \leq \sigma \leq 2, |t| \leq 2$ so $\zeta'(s)/\zeta(s) = O(1/|\sigma-1)|$ in this region. Because for $\sigma \geq 2$ the function is trivially bounded by $\sum_{n\leq 1} \Lambda(n)n^{-1}$, we obtain the third and final claim. We make sure all three claims are true for the same c by choosing $c = \min(c_1, c_2, c_3)$. The proofs of these claims prove the Prime Number Theorem by Theorem 3.5. □

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Appendix A. Arithmetic Functions

This appendix contains some background results about arithmetic functions used in the proof of the Prime Number Theorem.

A.1. Dirichlet Convolutions.

Definition A.1. *Let $f(n)$ be an arithmetic function. Then the series*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is called the Dirichlet series *associated with f.*

Definition A.2. *Let $f(n)$ and $g(n)$ be arithmetic functions. The Dirichlet convolution of f and g, written $f * g(n)$, is defined as*

$$\sum_{d|n} f(n)g(n/d).$$

Theorem A.3. *Let f, g, and h be arithmetic functions, and let F, G, and H be the corresponding Dirichlet series. Then suppose $h = f * g$, where $f * g$ is the Dirichlet convolution of f and g. If f and g converge absolutely at some point s, so does H, and $F(s)G(s) = H(s)$.*

Proof. By definition,

$$\begin{aligned} F(s)G(s) &= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{f(k)g(m)}{k^s m^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{km=n} f(k)g(m) = \sum_{n=1}^{\infty} \frac{f * g(n)}{n^s}. \end{aligned}$$

The absolute convergence of H comes from the following inequality:

$$\sum_{n=1}^{\infty} \left| \frac{h(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \sum_{km=n} |f(k)| \cdot |g(m)| = \sum_{k=1}^{\infty} \left| \frac{f(k)}{k^s} \right| \sum_{m=1}^{\infty} \left| \frac{g(m)}{m^s} \right|.$$

□

A.2. **Perron Formulas and Mellin Transform Representation.** These identities will be stated here without proof. For more information on Perron Formulas, see chapter four of [1]. For more information on the Mellin Transform, see chapter four of [1], or for the Mellin Transform from a complex analytic point of view, see chapter four of [3].

Theorem A.4 (Perron Formulas). *Given an arithmetic function $f(n)$, where $M(f,x) = \sum_{n\leq x} f(n)$, the Perron Formulas are:*

$$M_1(f,x) := \int_1^\infty M(f,y)dy = \sum_{n\leq x} f(n)(n-x)$$

and

$$M_1(f,x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}ds.$$

Theorem A.5 (Mellin Transform Representations). *Let $M(f,x)$ be as above and $L(f,x) = \sum_{n\leq x} f(n)/n$. Then for a given arithmetic function $f(n)$, with Dirichlet series $F(s)$ convergent on a half plane $\sigma > \sigma_c$, the Mellin Transform Representations are:*

$$F(s) = s\int_1^\infty M(f,x)x^{-s-1}dx \quad (\sigma > \max(0,\sigma_c))$$

and

$$F(s) = (s-1)\int_1^\infty L(f,x)x^{-s}dx \quad (\sigma > \max(1,\sigma_c)).$$

A.3. **The Zeta Function.**

Definition A.6. *The Dirichlet series associated with the constant arithmetic function $f(n) = 1$*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

is called the zeta function.

Lemma A.7. *The zeta function, as defined above, converges absolutely on the half plane $\sigma > 1$.*

Proof. Left as an exercise. □

Lemma A.8. *The zeta function $\zeta(s)$ has an analytic continuation to a function defined on the half plane $\sigma > 0$ and is analytic in this half plane with the exception of a simple pole at $s = 1$. This continuation is also denoted by $\zeta(s)$ and has the integral representation*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}s^{-s-1}dx \quad (\sigma > 0). \tag{11}$$

Proof. Let $M(f,x) = \sum_{n\leq x} f(n)$, and $L(f,x) = \sum_{n\leq x} f(n)/n$. We can then take the Mellin Transform of the zeta function, to relate the function to its sum:

$$\zeta(s) = s\int_1^\infty M(1,x)x^{-s-1}dx \quad (\sigma > \max(0,1)).$$

It is clear that for $f(x) = 1$, $M(1, x) = x - \{x\}$. Therefore,

$$\begin{aligned}\zeta(s) &= s\int_1^\infty (x - \{x\})x^{-s-1}dx \quad (\sigma > \max(0,1)) \\ &= s\left(\int_1^\infty x^{-s}dx - \int_1^\infty \{x\}x^{-s-1}dx\right) \quad (\sigma > 1) \\ &= \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}dx \quad (\sigma > 1).\end{aligned}$$

This gives the desired equality on the half plane $\sigma > 1$. Given any $\epsilon > 0$ the integral in the above formula can be bounded for $\sigma > \epsilon$ through the following inequality:

$$\left|\int_1^\infty \{x\}x^{-s-1}dx\right| \leq \int_1^\infty \{x\}x^{-\sigma-1}dx \leq \int_1^\infty \{x\}x^{-\epsilon-1}dx \leq \frac{1}{\epsilon}.$$

Because we chose ϵ to be arbitrarily small, the integral represents an analytic function on the half plane $\sigma \geq \epsilon$. Thus it follows that this integral is analytic on the half plane $\sigma > 0$. Now it only remains to show that $\frac{s}{s-1}$ has a simple pole at $s = 1$, and is analytic elsewhere (it is obvious that if this part of the function has a pole at $s = 1$ the complete function will as well). If $\frac{s}{s-1}$ has a simple pole at $s = 1$ then the limit

$$\lim_{s\to 1}(s-1)\frac{s}{s-1}$$

will exist. The limit does exist and is equal to one.

$$\lim_{s\to 1}(s-1)\frac{s}{s-1} = \lim_{s\to 1} s = 1.$$

□

Corollary A.9.

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \{u\}u^{-s-1}du \quad (N \in \mathbb{N}, \sigma > 0).$$

Proof. As in Lemma A.8 we begin by taking a Mellin Transform. However this time we use a different series:

$$F(s) = \zeta(s) - \sum_{n=1}^{N}\frac{1}{n^s}.$$

After the Mellin Transform we have for $\sigma > 1$

$$\begin{aligned}F(s) &= s\int_1^\infty M(f,x)x^{-s-1}dx = s\int_N^\infty (x - \{x\} - N)x^{-s-1}dx \\ &= s\int_N^\infty x^{-s}dx - sN\int_N^\infty x^{-s-1}dx - s\int_N^\infty \{x\}x^{-s-1}dx \\ &= -\frac{sN^{1-s}}{1-s} - N^{1-s} - s\int_N^\infty \{x\}x^{-s-1}dx \\ &= -\frac{N^{1-s}}{1-s} - s\int_N^\infty \{x\}x^{-s-1}dx.\end{aligned}$$

By definition, $\zeta(s) = F(s) + \sum_{n=1}^{N}\frac{1}{n^s}$. So this yields the desired result for $\sigma > 1$. The zeta function is analytic in the larger half plane $\sigma > 0$, by Lemma A.8 (with the exception of the pole at $s = 1$). It is clear that $-\frac{N^{1-s}}{1-s}$ is also analytic on the half plane $\sigma > 0$, with the exception of a pole at $s = 1$ (the argument is almost identical to the argument in Lemma A.8). The integral $\int_N^\infty \{x\}x^{-s-1}dx$ is uniformly convergent, and therefore analytic, in any half plane $\sigma \geq \epsilon, \epsilon > 0$. Together, these statements imply that the relation remains valid on the larger half plane $\sigma > 0$. □

Lemma A.10. *For complex numbers s such that $\sigma > 1$,*

$$\prod_{i=1}^{\infty}(1 - p_i^{-s})^{-1} = \sum_{k=1}^{\infty}\frac{1}{k^s} = \zeta(s),$$

where k spans the integers, and p_i is the ith prime number.

Proof.

$$\prod_{i=1}^{\infty}(1 - p_i^{-s})^{-1} = \prod_{i=1}^{\infty}(1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \dots). \tag{12}$$

When the second half of this equation is multiplied out, it will produce the sum of all possible combinations of the prime numbers to the power of s. From an application of the fundamental theorem of arithmetic, this is the same as the sum of the integers to the s power. Thus,

$$\prod_{i=1}^{\infty}(1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \dots) = \sum_{k=1}^{\infty}\frac{1}{k^s}. \tag{13}$$

It is clear that (12) and (13) above complete the proof. □

Corollary A.11. *The zeta function has no zeros on the half plane $\sigma > 1$.*

Proof. Follows from Lemma A.10 □

A.4. **The von Mangold Function.** Recall that the von Mangold function is defined as $\Lambda(n) = \log(p)$, if $n = p^m$, and as $\Lambda(n) = 0$ for other values of n. Also recall the identity $(\Lambda * 1)(n) = \log(n)$.

Lemma A.12. *The Dirichlet series associated with the function $\log x$ is $-\zeta'(s)$, and it converges absolutely in the half plane $\sigma > 1$.*

Proof. Left as an exercise. □

Lemma A.13. *There is continuation of $\zeta'(s)$ on the half plane $\sigma > 0$, and it is analytic on that half plane, with the exception of a pole at $s = 1$.*

Proof. We begin with the continuation of $\zeta(s)$ on the plane $\sigma > 0$ as defined in Lemma A.8:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}dx.$$

We differentiate this with respect to s:

$$\zeta'(s) = \frac{s-1-s}{(s-1)^2} - f'(s) = \frac{-1}{(s-1)^2} - f'(s),$$

where $f(s) = s\int_1^\infty \{x\}x^{-s-1}dx$. It was shown in Lemma A.8 that $f(s)$ is analytic on the half plane $\sigma > 0$. It then follows that $f'(s)$ is analytic on the same half plane. With the exception of a pole of order 2 at $s = 1$, the function $\frac{-1}{(s-1)^2}$ is also analytic on the half plane $\sigma > 0$. Thus the continuation of $\zeta'(s)$ is analytic on $\sigma > 0$, with the exception of a pole at $s = 1$. □

Lemma A.14. *The Dirichlet series of the von Mangold function equals* $\zeta'(s)/\zeta(s)$*.*

Proof. We use Theorem A.3 to relate the Dirichlet series of the von Mangold function to the Dirichlet series for the logarithm function and to the Dirichlet series of the constant function 1, or the zeta function.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\log n}{n^s}.$$

We can use the definition of the zeta function and Lemma A.13 to rewrite the above equation:

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\zeta'(s).$$

From this equation it is a simple matter of dividing both sides by $\zeta(s)$ to get the desired result. Recall that Corollary A.11 asserts that on the half plane $\sigma > 1$ $\zeta(s) \neq 0$, so this quotient is defined. This equation also makes it clear that the Dirichlet series for the von Mangold function converges on the half plane $\sigma > 1$. □

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