REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

ABHINAV SHRESTHA

Abstract. This paper studies the representations of semisimple Lie algebras, with care given to the case of $\mathfrak{sl}_n(\mathbb{C})$. We develop and utilize various tools, including the adjoint representation, the Killing form, root space decomposition, and the Weyl group to classify the irreducible representations of semisimple Lie algebras.

Contents

1. Introduction

Lie algebras arise as vector spaces of linear transformations. In most examples we will study, they arise as a subspace of the vector space of all linear transformations, endowed with the commutator bracket.

Definition 1.1. A *Lie Algebra* is a vector space L over a field k (for our purposes, $k = \mathbb{C}$) with an operation $[\cdot, \cdot] : L \times L \to L$, called the bracket or commutator satisfying the following axioms:

(i) The bracket operation is bilinear.
(ii) $[x, x] = 0$ for all $x \in L$.
(iii) (Jacobi Identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in L$.

Axioms (i) and (ii) together imply that the bracket is anti-commutative, that is, $[x, y] = -[y, x]$, for $x, y \in L$.

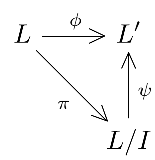
We now define suitable equivalents of concepts from linear algebra for Lie algebras. A subspace K of L is called a *subalgebra* if $[x, y] \in K$, for any $x, y \in K$. The subalgebra K is called an *ideal* if $[x, y] \in K$ for any $x \in K, y \in L$.

Given two Lie algebras L and L', a linear transformation $\phi : L \to L'$ is a *homomorphism* of Lie algebras if $\phi([x, y]) = [\phi(x), \phi(y)]$. If ϕ is an isomorphism

Date: DEADLINE AUGUST 26, 2011.

of vector spaces, ϕ is an *isomorphism* of Lie algebras, and L and L' are called *isomorphic*. We can define the kernel and image of a linear transformation ϕ in the natural way. We note that the kernel of ϕ is an ideal of L, and the image is a subalgebra of L'. The following standard homomorphism theorems apply to homomorphisms and ideals of Lie algebras:

Proposition 1.2. *(a) If $\phi : L \to L'$ is a homomorphism of Lie algebras, then $L/\operatorname{Ker}\phi \cong \operatorname{Im}\phi$. Furthermore, if I is any ideal of L included in $\operatorname{Ker}\phi$, there exists a unique homomorphism $\psi : L/I \to L'$ such that the following diagram commutes (π is the canonical projection map):*



(b) If I and J are ideals of L, such that $I \subset J$, then J/I is an ideal of L/I and $(L/I)/(J/I)$ is naturally isomorphic to L/J.
(c) If I, J are ideals of L, there is a natural isomorphism between $(I + J)/J$ and $I/(I \cap J)$.

We can use the commutator to define a few algebraic structures closely related to their counterparts in group and ring theory. The *center* of a Lie algebra L is defined by $Z(L) = \{z \in L | [x, z] = 0 \ for \ all \ x \in L\}$. We call L *abelian* if $L = Z(L)$, or equivalently, if its *derived algebra*, $[L, L]$, is trivial. Finally, we define the normalizer of a subalgebra K of L by $N_K(L) = \{x \in L | [x, K] \subset K\}$ and the *centralizer* of a subset X of L by $C_X(L) = \{x \in L | [x, X] = 0\}$. By the Jacobi identity, both the normalizers and centralizers are subalgebras of L.

We now turn our attention to our first few examples of Lie algebras. Given a vector space V, consider $\operatorname{End}(V)$, the set of linear transformations from V to itself, which is itself a vector space over k of dimension n^2 (where n=dim V). We can construct a Lie algebra from $\operatorname{End}(V)$ with the same underlying vector space by defining a new operation $[x, y] = x \cdot y - y \cdot x$, where $\cdot$ is multiplication in $\operatorname{End}(V)$. We call this the *general linear algebra*, and denote it $\mathfrak{gl}(V)$, or $\mathfrak{gl}_n(\mathbb{C})$. Our primary interest will be in a particular infinite family of subalgebras of $\mathfrak{gl}_n(\mathbb{C})$, given in the following example.

Example 1.3. Denote by $\mathfrak{sl}(V)$ or $\mathfrak{sl}_n(\mathbb{C})$ the set of linear transformations of $\operatorname{End}(V)$ with trace zero. This is called the *special linear algbera*. Because the trace of a linear transformation is independent of the choice of basis, we can verify if a linear transformation ϕ is in $\mathfrak{sl}(V)$ by summing the diagonal elements of the matrix of ϕ with respect to any basis. Furthermore, since we know $\operatorname{Tr}(xy) = \operatorname{Tr}(yx)$ and $\operatorname{Tr}(x + y) = \operatorname{Tr}(x) + \operatorname{Tr}(y)$, $\mathfrak{sl}_n(\mathbb{C})$ is a subalgebra of $\mathfrak{gl}(V)$.

We now define a powerful tool in the analysis of Lie algberas that will follow. For any $x \in L$, we can define a map $\operatorname{ad}(x)$ by $y \mapsto [x, y]$. The map $x \mapsto \operatorname{ad}(x)$ is called the *adjoint representation* of L. On occasion, we may want to view x as an element of both L and a subalgebra K. In such cases, we use the notation $\operatorname{ad}_L(x)$ and $\operatorname{ad}_K(x)$ to avoid ambiguity. The adjoint representation is a special case of the more general notion of a representation. In the latter sections of this paper, we will write $x(y)$ to denote $\operatorname{ad}(x)(y)$. Parentheses may be omitted.

Definition 1.4. A *representation* of a Lie algebra L is a homomorphism $\phi : L \to \mathfrak{gl}(V)$.

It is clear that $\mathrm{ad} : L \to \mathfrak{gl}(V)$ is a linear transformation, so we need to check that it preserves the Lie bracket.

$$\begin{aligned}
[\mathrm{ad}(x), \mathrm{ad}(y)](z) &= \mathrm{ad}(x)\,\mathrm{ad}(y)(z) - \mathrm{ad}(y)\,\mathrm{ad}(x)(z) \\
&= \mathrm{ad}(x)([y, z]) - \mathrm{ad}(y)([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [[x, z], y] \\
&= [[x, y], z] \\
&= \mathrm{ad}([x, y])(z)
\end{aligned}$$

We will restrict our analysis to a special subset of Lie algebras for which the representations can be clearly understood, and, in particular, their finite dimensional representations. In order to precisely describe these Lie algebras, we provide the following definitions:

Definition 1.5. Let L be a Lie algebra.

(i) The *derived series* of L is a sequence of ideals, $L^{(0)} = L, L^{(1)} = [L, L], L^{(2)} = [L^{(1)}, L^{(1)}], \ldots, L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. We say that L is *solvable* if $L^{(n)} = 0$ for some n.
(ii) The *radical* of L, denoted $\mathrm{Rad}L$, is the unique maximal solvable ideal of L. We say that L is *semisimple* if $\mathrm{Rad}L = 0$.
(iii) The *descending central series* of L is a sequence of ideals, $L^0 = L, L^1 = [L, L], L^2 = [L, L^1], \ldots, L^i = [L, L^{i-1}]$. We say that L is *nilpotent* if $L^n = 0$ for some n. Clearly, nilpotent algebras are solvable.

In addition, we also utilize the definitions for nilpotent and semisimple linear transformations from linear algebra. This provides us the following decomposition of endomorphisms:

Proposition 1.6 (Jordan-Chevalley decomposition)**.** *Let V be a finite dimensional vector space over $\mathbb{C}$, and $x \in \mathrm{End}(V)$. Then there exists $x_s, x_n \in \mathrm{End}(V)$, such that $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, and x_s and x_n commute.*

We refer to x_s as the semisimple part of x, and x_n as the nilpotent part. Finally, we will state the following theorems, that provide criteria for the nilpotency and solvability of a Lie algebra. We omit the proofs.

Theorem 1.7. *(Engel) If for all $x \in L$, x is* ad*-nilpotent, then L is nilpotent.*

Theorem 1.8. *(Lie) Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, with V finite dimensional. Then V contains a common eigenvector for all endomorphisms in L.*

Theorem 1.9. *(Cartan Criterion) Let L be a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x \in [L, L], y \in L$. Then L is solvable.*

2. The Killing Form

We want to now find a criterion for the semisimplicity of a Lie algebra.

Definition 2.1. Given a Lie algebra L, and any $x, y \in L$, the *Killing form* is defined by $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$.

The Killing form is a symmetric bilinear form on L. The bilinearity of the Killing form follows from the bilinearity of the bracket operation, and symmetry from the fact that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$. In addition, the Killing form is *associative*, in the sense that $\kappa([x, y], z) = \kappa(x, [y, z])$. To see this, first, note that $[x, y]z = xyz - yxz$, and $x[y, z] = xyz - xzy$. Since $\mathrm{Tr}(y(xz)) = \mathrm{Tr}((xz)y)$, $\mathrm{Tr}([x, y]z) = \mathrm{Tr}(x[y, z])$. Since ad is a Lie algebra homomorphism, it follows that the Killing form is associative.

The Killing form gives us a necessary and sufficient condition for the semisimplicity of a Lie algebra. This is related to the nondegeneracy of the Killing form. In general, a symmetric bilinear form, β, is nondegenerate if its *radical*, defined by $\{x \in L | \beta(x, y) = 0 \text{ for all } y \in L\}$, is 0. Because the Killing form is associative, the radical of κ is an ideal of L. For arbitrary bilinear forms β and Lie algebras L, the radical of β should not be confused with Rad L, the maximal solvable ideal of L. However, if L is semisimple, the radical of the Killing form coincides with the radical of L. This is shown in the following theorem:

Theorem 2.2. *Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate.*

Proof. Suppose that L is semisimple, that is, Rad $L = 0$. Let S be the radical of κ. Then, by definition, for all $x \in S$, $y \in L$, we have that $\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$. In particular, this holds for $y \in [S, S]$, so $\mathrm{ad}_L(S)$ is solvable by Cartan's Criterion, and hence, so is S. However, as we just noted, S is an ideal of L, so $S \subset \text{Rad } L = 0$, and κ is nondegenerate.

Conversely, suppose that κ is nondegenerate. By definition, $S = 0$. To show that L is semisimple, we first notice that L is semisimple if and only if it contains no nonzero abelian ideals, because Rad L contains an abelian term in its derived series (which is nonzero, if Rad L is nonzero). and conversely, any abelian ideal must be contained in Rad L. Thus, it suffices to show that every abelian ideal I of L is contained in S. Let $x \in I, y \in L$. Then $\mathrm{ad}(x)\,\mathrm{ad}(y)$ maps L into I, and I into $[I, I] = 0$. Thus, $\mathrm{ad}(x)\,\mathrm{ad}(y)$ is nilpotent, and $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$, so $I \subset S = 0$. □

Corollary 2.3. *Let L be a semisimple Lie algebra. Then there exist ideals $L_1, \dots L_t$ of L, which are simple (as Lie algberas), such that $L = L_1 \oplus \cdots \oplus L_t$.*

The corollary follows from taking an arbitrary ideal I, and showing that its orthogonal complement (with respect to the Killing form), $I^\perp = \{x \in L | \kappa(x, y) = 0 \text{ for all } y \in I\}$, is an ideal, such that $L = I \oplus I^\perp$. We then proceed inductively. We omit a rigorous proof of the corollary.

3. Reducibility of Representations

To discuss the decomposition and reducibility of Lie algebra representations, we would like to use the language of modules. As it turns out, the definition of modules over a Lie algebra is a reformulation of the definition of a representation.

Definition 3.1. Given a Lie algebra L, an L-module is a vector space V with a bilinear operation $L \times V \to V$ satisfying the following condition:

$$[x, y]v = xyv - yxv$$

where $x, y \in L$, $v, w \in V$, $a, b \in \mathbb{C}$.

Any representation $\phi : L \to \mathfrak{gl}_n(\mathbb{C})$ can be viewed as an L-module by the action $xv = \phi(x)v$. Conversely, given an L-module, we can construct a representation ϕ from this definition. Hence, these are equivalent definitions. The definition of modules gives rise to the the following definitions, and a fundamental lemma in representation theory, which will be critical in our analysis.

Definition 3.2. (i) A *homomorphism* of L-modules is a linear map $\psi : V \to W$ such that $\psi(xv) = x\psi(v)$, for $x \in L, v \in V$. We call ψ an isomorphism of L-modules if it is an isomorphism of vector spaces.
(ii) An L-module V is called *irreducible* if its only L-submodules are itself and 0. We do not consider the zero dimensional vector space irreducible, but we do allow one dimensional vector spaces.
(iii) An L-module B is called *completely reducible* if V is the direct sum of irreducible L-submodules.

Lemma 3.3 (Schur)**.** *If V and W are irreducible L-modules, and ψ is a L-module homomorphism, then either ψ is an isomorphism or $\psi = 0$. Furthermore, if $V = W$, then $\psi = \lambda I$ for some $\lambda \in \mathbb{C}$, and I the identity.*

Proof. For the first claim, $\operatorname{Ker}\psi$ and $\operatorname{Im}\psi$ are both L-submodules, and the result follows from applying the module isomorphism theorems. Since V is irreducible, $\operatorname{Ker}\psi$ is a submodule, and hence either 0 or V. If $\operatorname{Ker}\psi = 0$, the map is injective, and maps onto a nonzero submodule of W, which must be all of W.

For the second claim, notice that ψ is a linear map over $\mathbb{C}$, and so must have an eigenvalue λ. Thus, $\psi - \lambda I$ is also a linear map with nontrivial kernel. But as we have just shown, this implies $\psi - \lambda I = 0$, and $\psi = \lambda I$. □

Before we continue analyzing the decomposition of L-modules, we will mention standard ways of producing new L-modules from given ones. Let V be an L-module. Then the dual vector space V^* is also an L-module if we define the action of L on V by $(xf)(v) = -f(xv)$, for $f \in V^*, v \in V, x \in L$. Similarly, if V and W are L-modules, then $V \otimes W$, the tensor product of the underlying vector spaces, is also an L-module. The module structure is given by $x(v \otimes w) = xv \otimes w + v \otimes xw$. We can generalize this to tensor powers, denoted $V^{\otimes n}$. Because $\operatorname{Hom}(V, W) \cong V^* \otimes W$, this provides a module structure on sets of linear transformations.

In addition to tensor powers, we can use the related notion of symmetric and alternating powers to construct L-modules. In particular, if V and W are L-modules, then $\operatorname{Sym}^n V$ is an L-module, where $\operatorname{Sym}^n V$ is the module formed by $V^{\otimes n}$ modulo the submodule generated by elements $(v_1 \otimes v_2 \otimes \cdots \otimes v_n) - (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)})$, with σ in the symmetric group on n elements, S_n. Likewise, the alternating, or exterior powers $\bigwedge^n V$ (defined as $V^{\otimes n}$ modulo the submodule generated by the elements $(v_1 \otimes v_2 \otimes \cdots \otimes v_n) + (-1)^\sigma (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}))$ are representations.

Finally, we state a major theorem concerning decomposition of the representations of semisimple Lie algebras. We omit the proof.

Theorem 3.4 (Weyl)**.** *Let $\phi : L \to \mathfrak{gl}(V)$ be a (finite dimensional) representation of a semisimple Lie algebra L. The ϕ is completely reducible.*

4. Representations of $\mathfrak{sl}_2(\mathbb{C})$

We begin our analysis of $\mathfrak{sl}_2(\mathbb{C})$ by taking the standard basis:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These elements satisfy the following relations:

$$[h, x] = 2x, [h, y] = -2y, [x, y] = h.$$

Multiplication in $\mathfrak{sl}_2(\mathbb{C})$ is completely determined by the identities described above. We notice that in particular, x and y are eigenvectors of $\mathrm{ad}(h)$, with eigenvalues 2 and -2 respectively. Suppose $I \neq 0$ is an ideal of $\mathfrak{sl}_2(\mathbb{C})$, and $ax + by + ch$ is an arbitrary nonzero element of I. Applying $\mathrm{ad}(x)$ twice, we get $[x, [x, ax+by+ch]] = [x, a[x, x]+b[x, y]+c[x, h]] = [x, bh-2cx] = b[x, h] = -2bx \in I$. Similarly, applying $\mathrm{ad}(y)$ twice, we get that $-2ay \in I$. Thus, if either a or b is nonzero, then I contains y or x, respectively, and it follows that $I = L$. On the other hand, if $a = b = 0$, then $ch \in I$, and is nonzero, and by applying $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$, we get $I = L$ again. Thus, $\mathfrak{sl}_2(\mathbb{C})$ is simple. Furthermore, by the arguments we applied to I, any Lie algebra with elements satisfying those multiplication identities generate a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

We now introduce the following definitions to understand the representations of $\mathfrak{sl}_2(\mathbb{C})$.

Definition 4.1. Let V be an arbitrary $\mathfrak{sl}_2(\mathbb{C})$-module, and $V_\lambda = \{v \in V | h(v) = \lambda v\}$, for $\lambda \in \mathbb{C}$. If V_λ is nontrivial, we call λ a *weight* of h, and V_λ a *weight space*.

These definitions will be modified in the future to study the representations of other Lie algebras. Presently, the definitions of weight and weight space are exactly the definitions of eigenvalue and eigenspace for vector spaces.

Lemma 4.2. *If $v \in V_\lambda$, then $xv \in V_{\lambda+2}$ and $yv \in V_{\lambda-2}$.*

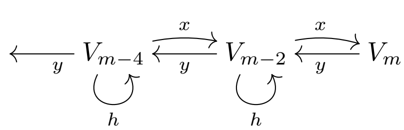
Proof. This follows from the following computation, using the multiplication on $\mathfrak{sl}_2(\mathbb{C})$ described earlier in this section.

$$h(xv) = [h, x]v + (xh)v = (2x)v + (\lambda x)v = (\lambda + 2)xv.$$

An analogous computation holds for y. □

Remark 4.3. The lemma implies that x and y are nilpotent endomorphisms.

Pictorially, the lemma gives us the following description of the action of $\mathfrak{sl}_2(\mathbb{C})$ on the weight spaces.



Because we are working with a finite dimensional L-module, there must exist some $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. We call any nonzero vector in V_λ a *maximal vector* of weight λ. Our next step will be to use a maximal vector to help us classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. To do this, we require the following lemma.

Lemma 4.4. *Let V be an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. Choose a maximal vector $v_0 \in V_\lambda$; set $v_{-1} = 0$, $v_i = \dfrac{y^i}{i!}v_0$, $(i \geq 0)$. The following identities hold for $i \geq 0$.*

(a) $hv_i = (\lambda - 2i)v_i$,
(b) $yv_i = (i+1)v_{i+1}$,
(c) $xv_i = (\lambda - i + 1)v_{i-1}$

Proof. Part (a) follows by induction on i. The case for $i = 0$ is obvious. Now suppose $h(v_i) = (\lambda - 2i)(v_i)$. We know that $h(v_{i+1}) = h(\frac{1}{i+1}yv_i)$. By inductive hypothesis, $v_i \in V_{\lambda-2i}$, and $yv_i \in V_{\lambda-2i-2} = V_{\lambda-2(i+1)}$, by Lemma 4.2. Thus, $h(v_{i+1}) = (\lambda - 2(i+1))v_{i+1}$. Part (b) follows from our definition of v_i. Part (c) requires an induction argument similar to that of part (a). The base case, $i = 0$ is clear, since we defined $v_{-1} = 0$. Now, we make the following calculation:

$$\begin{aligned} ix(v_i) &= x(yv_{i-1}) \\ &= [x, y]v_{i-1} + yx(v_{i-1}) \\ &= hv_{i-1} + yx(v_{i-1}) \\ &= (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)yv_{i-2} \\ &= (\lambda - 2i + 2)v_{i-1} + (i-1)(\lambda - i + 2)v_{i-1} \\ &= i(\lambda - i + 1)v_i. \end{aligned}$$

Dividing both sides by i gives us our result. □

We are now ready to describe all irreducible representations via their weights and weight spaces.

Theorem 4.5. *Let V be an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, with vectors v_i as defined above in Lemma 4.4, and let $m + 1 = \dim(V)$.*

(a) Relative to h, V is the direct sum of weight spaces V_μ, $\mu = m, m-2, \ldots, -(m-2), -m$, and $\dim(V_\mu) = 1$ for each μ.
(b) The module V has (up to nonzero scalar multiples) a unique maximal vector, whose weight is m.
(c) The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is given explicitly by the above formulas. In particular, there exists at most one irreducible $\mathfrak{sl}_2(\mathbb{C})$-module (up to isomorphism) of each possible dimension $m + 1$, $m \geq 1$.

Proof. By part (a) of Lemma 4.4, all the nonzero v_i are linearly independent. Thus, the number of nonzero v_i is bounded above by $\dim V + 1$, and in particular, they are finite in number. Thus, we can let m be the smallest integer such that $v_m \neq 0$, but $v_{m+1} = 0$. For all $i > 0$, $v_{m+i} = 0$. Thus, the weights corresponding to the v_i form a finite arithmetic progression of length $m + 1$.

The subspace spanned by $\{v_0, \ldots, v_m\}$ is an L-submodule, due to Lemma 4.4. Since this subspace is nonzero and V is irreducible, this subspace is all of V. This proves V is the direct sum of the weight spaces associated with the v_i.

In addition, if we apply the formula in part (c) of Lemma 4.4, letting $i = m+1$, we see that $x(v_{m+1}) = (\lambda - m)v_m$. But since $v_{m+1} = 0$, λ must be equal to m. Thus, the weight of a maximal vector is a nonnegative integer, equal to one less than $\dim V$. Furthermore, each weight space contains exactly one v_i, and $\dim V_\mu = 1$ for each weight μ. We say that each weight has multiplicity one. In particular, the maximal vector is unique up to scalar multiples.

□

Remark 4.6. We call m the highest weight of V.

Corollary 4.7. *Let V be any finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Then the eigenvalues of h on V are all integers, and each occurs along with its negative (an equal number of times). Moreover, there are exactly $\dim(V_0)+\dim(V_1)$ summands in any decomposition of V into a direct sum of irreducible submodules.*

Proof. The case where $V = 0$ is trivial, so we can assume $\dim V \geq 1$. Apply Weyl's Theorem to write V as a direct sum of irreducible L-submodules. The first assertion follows precisely from Theorem 4.5. For the second assertion, we observe that each irreducible L-module contains either a weight space of weight 1 or 0 with multiplicity one, and does not contain both. □

This gives us a complete picture of all the representations of $\mathfrak{sl}_2(\mathbb{C})$. It is worth noting that this combinatorial description of the irreducible representations is easily converted into an algebraic description by taking symmetric powers of the standard representation, $V = \mathbb{C}^2$. Let v_1, v_2 be the standard basis for $\mathbb{C}^2$. Then a basis for $W = \operatorname{Sym}^2 V$ is given by $\{x^2, xy, y^2\}$. We calculate the action of h on each of these basis vectors.

$$h(x^2) = xh(x) + h(x)x = 2x^2$$
$$h(xy) = xh(y) + h(x)y = 0$$
$$h(y^2) = yh(y) + h(y)y = -2y^2$$

So, the representation W is the irreducible representation of highest weight 2. Similarly, the n-th symmetric power, $\operatorname{Sym}^n V$ has basis $\{x^n, x^{n-1}y, \ldots, y^n\}$, and $h(x^{n-k}y^k) = (n-k)h(x)x^{n-k-1}y^k + kh(y)x^{n-k}y^{k-1} = (n-2k)x^{n-k}y^k$. Thus, the weights of $\operatorname{Sym}^n V$ are $n, n-2, \ldots, -n$, each occurring with multiplicity one, so $\operatorname{Sym}^n V$ is the irreducible representation of highest weight n.

5. Root Space Decomposition

Our analysis of $\mathfrak{sl}_2(\mathbb{C})$ will guide us in examining the irreducible representations of semisimple Lie algebras in general. Our first step will be to find an analogue of h from $\mathfrak{sl}_2(\mathbb{C})$. In a general semisimple Lie algebra, however, no single element will serve the function of h. Instead, we have an abelian subalgebra which acts diagonally. Such a subalgebra is called a *Cartan subalgebra*.

As it turns out, in the case of semisimple Lie algebras over $\mathbb{C}$, we have a precise characterization of the Cartan subalgebra. Let L be a semisimple Lie algebra. We can always find some $x \in L$ for which $\operatorname{ad} x$ is not nilpotent. Otherwise, L is nilpotent by Engel's theorem, which is absurd, since a nilpotent Lie algebra cannot be semisimple. Thus, there always exists a nonzero subalgebra consisting of elements that act diagonally on L. We call such an algebra a toral subalgebra.

Lemma 5.1. *A toral subalgebra of L is abelian.*

Proof. Let T be a toral subalgebra. We want to show that $\operatorname{ad}_T(x) = 0$ for all $x \in T$. This is equivalent to showing that $\operatorname{ad}_T(x)$ has no nonzero eigenvalues. Suppose for contradiction that $\operatorname{ad}_T(x)(y) = [x, y] = ay, a \neq 0$, for some nonzero $y \in T$. Then, $\operatorname{ad}_T(y)(x)$ is an eigenvector of eigenvalue 0 of $\operatorname{ad}_T(y)$. Since $\operatorname{ad}_T(y)$ is diagonalizable, we can decompose T into eigenspaces of $\operatorname{ad}_T(y)$, and write x as a linear combination of eigenvectors of $\operatorname{ad}_T(y)$. Now apply $\operatorname{ad}_T(y)$ to x. The result must be nonzero, since $-ay \neq 0$. Thus, it is the sum of eigenvectors of nonzero eigenvalue. Thus, $\operatorname{ad}_T(y)$ cannot vanish on $\operatorname{ad}_T(y)(x) = -ay$, a contradiction. □

Let H be a fixed maximal toral subalgebra of L. We know from linear algebra that L can be decomposed into the direct sum of subspaces $L_\alpha = \{x \in L | [h,x] = \alpha(h)x \text{ for all } h \in H\}$, where $\alpha \in H^*$. If $L_\alpha \neq 0$, we say that α is a *root* of L. We denote by Φ, the set of all roots of L. This gives us a root space decomposition of L, given by $L = C_L(H) \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$. We have already seen one example of a root space decomposition, that of $\mathfrak{sl}_2(\mathbb{C})$ into $L_{-2}, L_0 = H$, and L_2, spanned by y, h, and x respectively.

Proposition 5.2. *Given a a Lie algebra L,*
(a) For all $\alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.
(b) If $x \in L_\alpha$, $\alpha \neq 0$, then $\operatorname{ad}(x)$ is nilpotent.
(c) If $\alpha + \beta \neq 0$, then L_α is orthogonal to L_β with respect to the Killing form.

Proof. For part (a), we make a computation using the Jacobi identity for $x \in L_\alpha$, $y \in L_\beta$:

$$[h,[x,y]] = [[h,x],y] + [x,[h,y]] = [\alpha(h)x, y] + [x, \beta(h)y] = (\alpha+\beta)(h)[x,y].$$

For part (b), observe that there are finitely many roots of L, and by (a), $\operatorname{ad}^n(x) : L_\beta \to L_{n\alpha+\beta}$. For sufficiently large n, $L_{n\alpha+\beta} = 0$, so $\operatorname{ad}^n x = 0$. Finally, for part (c), choose $h \in H$ such that $(\alpha+\beta)(h) \neq 0$. Then, $\kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y])$. Thus, $\alpha(h)\kappa(x,y) = -\beta(h)\kappa(x,y)$, and $\kappa(x,y) = 0$. □

Corollary 5.3. *The restriction of the Killing form to $C_L(H)$ is nondegenerate.*

Proof. From the proposition, we know that $C_L(H) = L_0$ is orthogonal to L_α, for all $\alpha \in \Phi$. Suppose $z \in L_0$ is orthogonal to L_0. Then, $\kappa(z, L) = \kappa(z, L_0 \oplus \bigoplus_{\alpha\in\Phi} L_\alpha) = 0$. However, since κ is nondegenerate on L, this is only possible if $z = 0$. □

We state the following proposition.

Proposition 5.4. *Let H be a maximal toral subalgebra of L. Then, $H = C_L(H)$.*

We will sketch the proof. The full proof can be found in Chapter 2 of [1]. First, we briefly introduce abstract Jordan decomposition. If L is a semisimple Lie algebra, and $x \in L$, then $\operatorname{ad}(x)$ can be decomposed via the usual Jordan decomposition. It is a result that there exist x_s and x_n such that $x = x_s + x_n$, $\operatorname{ad}(x_s)$ is semisimple, and $\operatorname{ad}(x_n)$ is nilpotent. This is called the abstract Jordan decomposition of x. We call x_s and x_n the semisimple and nilpotent parts of x respectively. Note that this definition overlaps with that of Jordan-Chevalley decomposition as we previously defined it.

Proof Sketch. We can show that for any element $x \in C_L(H)$, both the nilpotent and semisimple parts of x are contained in $C_L(H)$. In particular, all the semisimple elements of $C_L(H)$ lie in H. Thus, if $C_L(H) \neq H$, then $C_L(H)$ contains a nonzero nilpotent element. We can also show that $C_L(H)$ is abelian. In addition, it is a result from linear algebra that if M and N are commuting endomorphisms of a finite dimensional vector space, then $\operatorname{Tr}(MN) = 0$. Thus, for all $y \in C_L(H)$, $\kappa(x,y) = \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$, contradicting Corollary 5.3.

□

It follows immediately from this proposition and Corollary 5.3, that the restriction of κ to H is nondegenerate. This fact allows us to relate H and H^*

in the following manner. For each $\phi \in H^*$, define an element $t_\phi \in H$ satisfying $\phi(h) = \kappa(t_\phi, h)$ for all $h \in H$. This allows us to extend the Killing form to H^*: if $\alpha, \beta \in H^*$, $\kappa(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$. Additionally, this notation will be helpful as we try to understand how representations of general semisimple Lie algebras relate to the representations of $\mathfrak{sl}_2(\mathbb{C})$, as we will see soon. Before we observe this relationship, however, let us apply our definitions to the case of $\mathfrak{sl}_n(\mathbb{C})$.

Example 5.5. In the case of $\mathfrak{sl}_n(\mathbb{C})$, the Cartan subalgebra is the subalgebra of diagonal matrices, $H = \{a_1h_1 + a_2h_2 + \cdots + a_nh_n \mid a_1 + a_2 + \cdots + a_n = 0\}$, where $h_i = E_{i,i}$ and $E_{i,j}$ is the map sending e_j to e_i and sending all other e_k to 0. The dual can be written as $H^* = \mathbb{C}\{\ell_1, \ell_2, \ldots, \ell_n\}/(\ell_1 + \ell_2 + \cdots + \ell_n)$, where the ℓ_i form a dual basis via $\ell_i(h_j) = \delta_{i,j}$ (where $\delta_{i,j}$ is the Kronecker delta function). To find the roots of $\mathfrak{sl}_n(\mathbb{C})$, we perform the following calculation:

$$\begin{aligned}\mathrm{ad}(a_1h_1 + a_2h_2 + \cdots + a_nh_n)(E_{i,j}) &= a_i[h_i, E_{i,j}] + a_j[h_j, E_{i,j}] \\ &= a_i(h_i(E_{i,j})) + a_j(E_{i,j}(-h_j)) = (a_i - a_j)(E_{i,j}).\end{aligned}$$

Thus, the $E_{i,j}$ are eigenvectors of eigenvalue $\ell_i - \ell_j$, and the roots of $\mathfrak{sl}_n(\mathbb{C})$ are precisely these pairwise differences of ℓ_i.

Proposition 5.6. *Let L be a semisimple Lie algbera, H a Cartan subalgebra of L, and Φ, its set of roots.*

(a) The roots of L, Φ, span H^*.
(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.
(c) Let $\alpha \in \Phi$, $x \in L_\alpha$, and $y \in L_{-\alpha}$. Then $[x, y] = \kappa(x, y)t_\alpha$.
(d) If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is one-dimensional, with basis t_α.
(e) For $\alpha \in \Phi$, $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.
(f) If $\alpha \in \Phi$ and x_α is any nonzero element of L_α, then there exists $y_\alpha \in L_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a three-dimensional simple subalgebra of L isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, via

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(g) The element $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$, and $h_\alpha = -h_{-\alpha}$.

Proof. (a) Suppose for contradiction that Φ does not span H^*. Then by duality, there exists some nonzero $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. Then, $[h, L_\alpha] = 0$ for all $\alpha \in \Phi$. However, $[h, H] = 0$, because H is abelian. Since $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, this means $h \in Z(L)$. This is a contradiction, since $Z(L)$ is a solvable ideal, and hence 0.

(b) Let $\alpha \in \Phi$, and suppose for contradiction that $-\alpha \notin \Phi$. Then $L_{-\alpha} = 0$, so $\kappa(L_\alpha, L_\beta) = 0$ for all $\beta \in H^*$ by Proposition 5.2. Thus, $\kappa(L_\alpha, L) = 0$, which contradicts the nondegeneracy of κ on L.

(c) Let $h \in H$ be arbitrary. Because κ is associative, we have the following computation:

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(h, \kappa(x, y)t_\alpha).$$

Thus, $\kappa(h, [x, y] - \kappa(x, y)t_\alpha) = 0$, and H is orthogonal to $[x, y] - \kappa(x, y)t_\alpha$, forcing $[x, y] = \kappa(x, y)t_\alpha$, since κ is nondegenerate.

(d) Part (c) shows that t_α spans $[L_\alpha, L_{-\alpha}]$ if $[L_\alpha, L_{-\alpha}]$ is nonzero. Let $x \in L_\alpha$ be nonzero. By the argument in part (b), $\kappa(x, L_{-\alpha}) \neq 0$, or κ would be degenerate. Thus, there is some $y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$, and so, $[x, y] \neq 0$.

(e) Suppose for contradiction that $\alpha(t_\alpha) = 0$. Since $t_\alpha \in H$, $[t_\alpha, x] = \alpha(t_\alpha)x = 0 = [t_\alpha, y]$, for all $x \in L_\alpha$, $y \in L_{-\alpha}$. As in (d), we can find x, y that satisfy $\kappa(x, y) \neq 0$, and in fact $\kappa(x, y) = 1$, by multiplying by scalars. Then $[x, y] = t_\alpha$ by (c). The subspace S spanned by x, y, t_α is a three-dimensional solvable algebra (using (d) to show solvability). In fact, $S \cong \mathrm{ad}_L(S) \subset \mathfrak{gl}(L)$. In particular, $\mathrm{ad}_L(s)$ is nilpotent for $s \in [S, S]$ by Lie's Theorem. Thus, $\mathrm{ad}_L(t_\alpha)$ is both nilpotent and semisimple, hence zero. But then, $t_\alpha \in Z(L) = 0$, contradicting our choice of t_α.

(f) Given nonzero $x_\alpha \in L_\alpha$, we want to find $y_\alpha \in L_{-\alpha}$ such that $\kappa(x_\alpha, y_{-\alpha}) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$. This is possible because of (e) and the fact that $\kappa(x_\alpha, L_{-\alpha}) \neq 0$. Let $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. Then $[x_\alpha, y_\alpha] = h_\alpha$ by (c). Furthermore, $[h_\alpha, x_\alpha] = \frac{2}{\alpha(t_\alpha)}[t_\alpha, x_\alpha] = \frac{2}{\alpha(t_\alpha)}(\alpha(t_\alpha)x_\alpha) = 2x_\alpha$. Similarly, we have that $[h, y_\alpha] = -2y_\alpha$. Thus, $x_\alpha, y_\alpha, h_\alpha$ span a three-dimensional subalgebra of L isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

(g) Since $\kappa(t_\alpha, h) = \alpha(h)$ defines t_α, $\kappa(t_{-\alpha}, h) = -\alpha(h)$, and $t_\alpha = -t_{-\alpha}$. It follows that $h_\alpha = h_{-\alpha}$ by the definition of h_α.

□

We will use the notation $\mathfrak{s}_\alpha$ for the subalgebra of L constructed in part (f). We have that $\mathfrak{s}_\alpha = L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$, and that $\dim L_\alpha = 1$, for any $\alpha \in \Phi$.

Proposition 5.7. *If α is a root, the only scalar multiples of α that are roots are α and $-\alpha$.*

Proof. Consider the subspace M of L spanned by H along with all root spaces of the form $L_{c\alpha}$, for a scalar $c \in \mathbb{C}$. Clearly, M is an $\mathfrak{s}_\alpha$-submodule of L. The only weights of h_α on M are the integers 0 and $2c$, with nonzero c for which, $L_{c\alpha} \neq 0$. Thus, c must be a multiple of $\frac{1}{2}$. Furthermore, the action of $\mathfrak{s}_\alpha$ on $\mathrm{Ker}\,\alpha$ is trivial. Since $\mathrm{Ker}\,\alpha$ is a subspace of codimension one in H, complementary to $\mathbb{C}h_\alpha$, and $\mathfrak{s}_\alpha$ is an irreducible $\mathfrak{s}_\alpha$-submodule of M, $\mathrm{Ker}\,\alpha$ and $\mathfrak{s}_\alpha$ account for all occurrences of the weight 0 for h_α. Thus, the only even weights are 0 and ± 2. However, $\frac{1}{2}\alpha$ cannot be a root either, so 1 is not a weight of h_α. Thus, by Corollary 4.7, we have that the only scalar multiples of α that are roots are α and $-\alpha$. □

It turns out that not only do the roots of L have integral values on the h_α, but in fact, the weights of any finite-dimensional representation attain integral values on the h_α. This allows us to define the weight lattice, $\Lambda_W = \{\beta \in H^* | \beta(\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. Since Φ is a subset of Λ_W, we can consider the lattice generated by Φ. This forms a sublattice of finite index in Λ_W, denoted Λ_Φ, the root lattice. All roots of representations of L will lie in Λ_W.

Before we proceed, we return to our example of $\mathfrak{sl}_n(\mathbb{C})$. As we said, the roots are of the form $\ell_i - \ell_j$. Thus, the root lattice of $\mathfrak{sl}_n(\mathbb{C})$ can be stated via generators and relations as follows:

$$\Lambda_\Phi = \{\sum_{i=1}^n a_i \ell_i | a_i \in \mathbb{Z}, \sum_{i=1}^n a_i = 0\} / (\sum_{i=1}^n \ell_i = 0).$$

We can also find the subalgebras, $\mathfrak{s}_\alpha$, isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. The root space $L_{\ell_i - \ell_j}$ is generated by $E_{i,j}$, so the corresponding subalgebra $\mathfrak{s}_\alpha$ is generated by $E_{i,j}$, $E_{j,i}$, and $[E_{i,j}, E_{j,i}] = h_i - h_j$. The action of $h_i - h_j$ on $E_{i,j}$ has eigenvalue $(\ell_i - \ell_j) = 2$, by the calculation performed previously (5.5), so this is indeed isomorphic to $\mathfrak{sl}_n(\mathbb{C})$.

Finally, we introduce orderings on the roots. This can be done by picking an arbitrary real linear functional f on the lattice Λ_Φ, irrational with respect to the lattice. This allows us to decompose our set of weights $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+ = \{\alpha | f(\alpha) > 0\}$. We call the roots of Φ^+ the positive roots, and those in Φ^- negative roots. This ordering allows us to now define the notion of a highest weight vector.

Definition 5.8. Let V be a representation of L. A *highest weight vector* is a nonzero vector $v \in V$ that is both a weight vector for the action of H, and in the kernel of the action of L_α, for all $\alpha \in \Phi^+$. We say that the weight associated with v is a *highest weight* of the representation.

In our analysis of $\mathfrak{sl}_2(\mathbb{C})$, the ordering amounts to a choice of α or $-\alpha$, and this corresponds to whether the vectors in L_α or $L_{-\alpha}$ are the highest weight vectors. While the uniqueness of v is trivial in the case of $\mathfrak{sl}_2(\mathbb{C})$, we will see that this is actually quite general. However, note that in general, a representation does not have a unique highest weight vector, or highest weight.

Proposition 5.9. *Let L be a semisimple Lie algebra. Every finite dimensional representation V of L possesses a highest weight vector. If, in addition, V is irreducible, the highest weight vector is unique up to scalars. Furthermore, the subspace W of V generated by the images of a highest weight vector under root spaces L_β for $\beta \in \Phi^-$ is an irreducible subrepresentation.*

Proof. The existence follows from finite-dimensionality. Take α to be the weight in V for which $f(\alpha)$ is maximal. Any vector nonzero vector $v \in V_\alpha$ will be a highest weight vector, because $V_{\alpha+\beta} = 0$ for all $\beta \in \Phi^+$, and hence is in the kernel of the action of all L_α.

The uniqueness is an immediate result of the final part of the proposition, so we will prove this first. This requires a calculation analogous to the one we performed for Lemma 4.2 for $\mathfrak{sl}_2(\mathbb{C})$. Let W_n be the subspace spanned by all $w_n(v)$, where w_n is a word of length at most n of elements of L_β, for $\beta \in \Phi^-$. We will induct on the length, n, with the case for $n = 0$ being easy. We claim that for any $x \in \Phi^+$, $x(W_n) \subset W_n$. To show this, consider a generator of W_n of the form yw, $w \in W_{n-1}$. Now apply the commutation relation $xyw = yxw + [x, y]w$ and the fact that $[x, y] \in H$ to verify the claim. Now, let $W = \bigcup_{n=1}^{\infty} W_n$. This is a subrepresentation of V. We claim W is irreducible. To see this, we write $W = W' \oplus W''$. Either W' or W'' contains the weight space W_α, since it is one-dimensional, and so must be all of W. □

6. Weyl Group and Weyl Chamber

The symmetry that we see from the integrality of the weights is not the full picture. To complete the picture, we introduce the *Weyl group*, denoted $\mathfrak{W}$. The Weyl group is generated by reflections along hyperplanes perpendicular to a root α with respect to the Killing form. An explicit formula for such reflections is given by $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \beta(h_\alpha)\alpha$.

At this point, we could axiomatize root systems. We will instead pursue a somewhat more concrete path, using the Weyl group to better understand the relationship between roots in representations. First, we will look at the case of $\mathfrak{sl}_n(\mathbb{C})$. The Weyl group is generated by reflections across the hyperplane perpendicular to the $\ell_i - \ell_j$. In particular, $\sigma_{\ell_i - \ell_j}$ exchanges ℓ_i and ℓ_j, while leaving all other ℓ_k alone. The Weyl group then acts as the symmetric group on the generators ℓ_i.

Proposition 6.1. *Suppose V is a finite-dimensional representation of L. Then the set of weights of V is invariant under the Weyl group. In particular, the set of weights congruent to any β modulo α will be invariant under σ_α.*

Proof. Suppose that V is a representation of L, with weight space decomposition $V = \bigoplus V_\beta$. The weights β in this decomposition can be broken into equivalence classes modulo α. Each of these equivalence classes can then be broken up into weight spaces as follows:

$$V_{[\beta]} = \bigoplus_{n \in \mathbb{Z}} V_{\beta + n\alpha}$$

This $V_{[\beta]}$ will be a subrepresentation of V for $\mathfrak{s}_\alpha$

We now introduce the notion of strings of weights. Since there are finitely many weights, we can take β to be such that $V_{\beta - n\alpha} = 0$, for any positive n. By our analysis of $\mathfrak{sl}_2(\mathbb{C})$, we know that the weights that correspond to nonzero summands in our decomposition of $V_{[\beta]}$ are $\beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + m\alpha$, with $m = -\beta(h_\alpha)$, an uninterrupted string. If we fix β and $m \geq 0$ so that it corresponds to the decomposition of $V_{[\beta]}$, we have that

$$\beta(h_\alpha), (\beta + \alpha)(h_\alpha) = \beta(h_\alpha) + 2, \ldots, (\beta + m\alpha)(h_\alpha) = \beta(h_\alpha) + 2m$$

is symmetric about zero, and $\beta(h_\alpha) = -m$. In particular,

$$\sigma_\alpha(\beta + k\alpha) = \beta + (-\beta(h_a) - k)\alpha = \beta + (m - k)\alpha.$$

□

We say that a root in Φ^- is a *simple* negative root if it cannot be the expressed as the sum of two negative roots. We have an analogous notion of positive simple roots. Since we have only finitely many roots, any negative root can be written as a sum of negative simple roots, and likewise for positive roots.

We can use simple roots to simplify our characterization of the Weyl group. It is a theorem that the Weyl group can be generated by just the reflections σ_α where α is a positive simple root. The proof of this fact is omitted in this paper, but details can be found in Chapter 3 of [1].

Our next goal is to construct the Weyl chamber, denoted $\mathfrak{C}$, and explore its relationship to the Weyl group and orderings on roots. First, the roots of a representation span an euclidean space, $\mathbb{E}$, with the Killing form as the inner product. For each root $\alpha \in \Phi$, we denote by P_α, the perpendicular space of α in $\mathbb{E}$. The P_α partition $\mathbb{E}$ into finitely many connected components. These connected components are called the Weyl chambers of $\mathbb{E}$.

We say that a weight α is *dominant* if $\kappa(t_\alpha, t_\gamma) = \alpha(t_\gamma) \geq 0$ for all $\gamma \in \Phi^+$. In particular, the highest weight vector of a finite dimensional representation is dominant. The Weyl chamber containing the dominant weights is called the fundamental Weyl chamber. Clearly, the fundamental Weyl chamber depends on how one separates Φ into Φ^+ and Φ^-, that is, the ordering of the roots.

The action of the Weyl group permutes the Weyl chambers. In particular, it acts simply transitively on the set of Weyl chambers, and likewise on the set of orderings on the roots. The proof of this fact and the precise formulation of the Weyl chamber using root systems and bases is provided in chapter 3 of Introduction to Lie Algebras and Representation Theory [1].

For $\mathfrak{sl}_n(\mathbb{C})$, let $c_1 > c_2 > \cdots > c_n$ with $\sum_{i=1}^{n} c_i = 0$. We define a linear functional f, by the following:

$$f(\sum_{i=1}^{n} a_i \ell_i) = \sum_{i=1}^{n} c_i a_i$$

The corresponding ordering on the roots will result in $\Phi^- = \{\ell_i - \ell_j | j < i\}$. The simple negative roots will then be the $\ell_{i+1} - \ell_i$. The fundamental Weyl chamber associated with this ordering is $\mathfrak{C} = \{\sum_{i=1}^{n} a_i \ell_i | a_1 \geq a_2 \geq \cdots \geq a_n\}$.

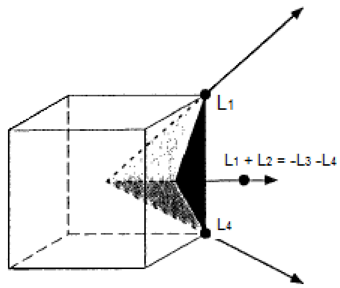
7. Classification of Irreducible Representations

Theorem 7.1. *For any $\alpha \in \mathfrak{C} \cap \Lambda_W$, there exists a unique irreducible, finite-dimensional Γ_α of L with highest weight α.*

Proof of Uniqueness. Suppose V and W are two irreducible, finite-dimensional representations of L with highest weight vectors v and w respectively, each with weight α. Then, the vector $(v, w) \in V \oplus W$ will be a highest weight vector of weight α in $V \oplus W$. Now consider $U \subset V \oplus W$, the irreducible subrepresentation generated by (v, w). The projections $\pi_1 : U \to V$ and $\pi_2 : U \to W$ are nonzero, and hence must be isomorphisms by Schur's Lemma. $\square$

Note that this theorem provides a bijection between $\mathfrak{C} \cap \Lambda_W$ and the set of irreducible representations of L. We will not prove the existence of such an irreducible representation, but we will demonstrate it for the case of $\mathfrak{sl}_n(\mathbb{C})$. A general construction using Verma modules is given in [1]. In order to prove the $\mathfrak{sl}_n(\mathbb{C})$ case, we will define *fundamental weights*. These are the elements $\omega_i \in H^*$ such that $\omega_i(h_{\alpha_j}) = \delta_{i,j}$, with $\alpha_1, \ldots \alpha_n$ simple roots relative to some fixed ordering on the roots. All dominant weights can be expressed uniquely as a nonnegative integral linear combination of fundamental weights, and we write $\Gamma_{a_1,\ldots,a_n}$ for the irreducible representation with weight $a_1\omega_1 + \cdots + a_n\omega_n$.

In the case of $\mathfrak{sl}_n(\mathbb{C})$, the fundamental weights (relative to the ordering induced by f, seen previously) are $\omega_i = \ell_1 + \ell_2 + \cdots + \ell_i$. We claim that the intersection of $\mathfrak{C}$ with Λ_Φ is in fact the free semigroup generated by the fundamental weights. First, we consider $\mathfrak{sl}_4(\mathbb{C})$, since we can visualize its weight diagram. The fundamental weights form a 2-simplex. Each fundamental weight lies on an edge of $\mathfrak{C}$, and so $\mathfrak{C}$ is the cone over the 2-simplex with vertex at 0. Furthermore, the faces of the Weyl chamber are the orthogonal compliments of the negative simple roots. This is expressed in the following diagram from [2]



We note that the fundamental weights are each a dominant weight of an exterior power of the standard representation: V, $\bigwedge^2 V$, and $\bigwedge^3 V$ have highest weight ℓ_1, $\ell_1 + \ell_2$, and $\ell_1 + \ell_2 + \ell_3$ respectively. To see this, we will perform a more general calculation, that will apply to the exterior powers of $\mathfrak{sl}_n(\mathbb{C})$. Let $1 \leq k \leq n$.

We want to find a highest weight in $\bigwedge^k V$ (V, the standard representation of $\mathfrak{sl}_n(\mathbb{C})$). Consider the vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$, and let α be a positive root, with respect to f. Our choice of f forces α to α take e_i to e_j, where $j \leq i$. Thus, the action of L_α must either take some e_i in $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ to 0, or to some e_j already in the term, and so must be zero. Thus, $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a highest weight of $\bigwedge^k V$.

As a result, $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\bigwedge^2 V) \otimes \mathrm{Sym}^c(\bigwedge^3 V)$ contains a highest weight vector with weight $a(\ell_1) + b(\ell_1 + \ell_2) + c(\ell_1 + \ell_2 + \ell_3)$, and consequently contains an irreducible representation with this highest weight.

The case of $\mathfrak{sl}_n(\mathbb{C})$ is analogous. The Weyl chamber is the cone over the $(n-2)$-simplex, edges generated by the fundamental weights, and faces the hyperplanes perpendicular to the negative simple roots, $\ell_{i+1} - \ell_i$. The exterior powers $\bigwedge^k V$ are also irreducible, with highest weight $\sum_{i=1}^{k} \ell_i$. As a result, there exists an irreducible representation of weight $(a_1 + a_2 + \cdots + a_{n-1})\ell_1 + \cdots + a_{n-1}\ell_{n-1}$, which appears inside $\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\bigwedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\bigwedge^{n-1} V)$.

Acknowledgments. I would like to thank my mentor Aaron Marcus for all his help over the course of this project, in unravelling the details of proofs, giving exposition on unclear topics, and filling in the necessary background knowledge.

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