THE EFFECT OF GRAVITATIONAL RADIATION ON THE OBSERVATIONAL PERIOD OF PULSARS

ALEXANDER TOLISH

ABSTRACT. According to Einstein's theory of general relativity, gravity can be explained as the curvature of a spacetime manifold. The presence of mass, energy, and momentum curves spacetime in accordance with the Einstein field equations. One solution to the Einstein equations in vacuum is a wave of gravitational radiation, in which a self-propagating ripple of curviture travels through space. This curviture deforms the metric and thus changes the course of geodesics through spacetime. Rays of electromagnetic radiation follow null geodesics; if gravitational waves pass between an astronomical body which periodically emits electromagnetic signals (such as a pulsar) and the Earth, an observer on the Earth will see the signals as deflected by the wave and thus observe them at a different frequency than the one at which they were emitted.

Contents

1. A Brief Introduction to General Relativity

NB: The reader is assumed to know some basic topics of differential geometry, including differentiable manifolds, curves and tangent vectors, scalar, vector, and tensor fields, metric geometry, the covariant derivative, and the Riemann curvature tensor. For the sake of brevity, we will not review these topics here; when in doubt, the reader can refer to standard textbooks, such as [1] and [2]. Also, in this paper we will use the index notation most popular in relativity literature: a Latin index refers only to spatial components, while a Greek index can refer to any component, and if a single character appears in both upper and lower positions in one term, we sum over all possible components.

Definition 1.1. In general relativity, *spacetime*—the arena in which the physical world exists—is a four-dimensional differentiable manifold endowed with a Lorentzian (signature (1,3)) metric $g_{\mu\nu}$. A single point in spacetime is called an *event*. The curve of events traced out by a physical entity is called a *worldline*.

A massive particle traveling along the worldline $x^\mu(\lambda)$ always has a negative squared tangent norm: $\dot{x}^\mu \dot{x}_\mu < 0$. We can try to introduce a generalized sense of "distance" along a worldline in spacetime by integrating the square root of the squared tangent norm along the curve. In order to keep this quantity real, though, we must take the square root of the negative squared tangent norm:

$$\tau = \int_\gamma \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\mu}{d\lambda}} d\lambda.$$

An observer riding the worldline does not see himself moving through space, only through time. Therefore he will recognize this generalized distance as the change in coordinate time within his own frame of reference; for this reason we call this quantity *proper time*.

One of the axioms of the theory of relativity is the principle of least action: a body which is not suffering any non-gravitational forces moves along a worldline of stationary proper time. We can use the Euler-Lagrange equation from the calculus of variations to find a differential equation describing the allowed worldlines. We wind up finding that they are geodesics. Nonmassive bodies, like photons (and thus electromagnetic signals), do not have a proper time *per se*, but a similar curve-minimizing axiom requires that they also follow geodesics.

The theory of general relativity hypothesizes that the curvature of spacetime—and thus deviations from the simple "straight" geodesics encountered in special relativity—arises due to the presence of matter, energy, and momentum. Einstein gave a tensor equation, known today as the Einstein field equation, relating curvature to the presence of physical "stuff":

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.2}$$

Here, $R_{\mu\nu}$ is the Ricci tensor, the contraction of the Riemann tensor; R is the scalar curvature, the trace of the Ricci tensor; $T_{\mu\nu}$ is the stress-energy tensor field, a coordinate-independent measure of the matter, energy, and momentum present at a point in spacetime; G is the gravitational constant of classical physics; and c is the speed of light. We often shorten the curvature side of the equation, introducing the new tensor field

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

known as the Einstein tensor.

2. The Linear Approximation, Gauge Freedom, and Gravitational Waves

Note: This material in this section can be found in almost any general relativity textbook that discusses gravitational waves. See, for example, [3] and [4].

A quick glance at the constant of proportionality in the Einstein field equations (1.2) gives one a rough feeling of much stess-energy is needed to curve space. In SI units, the gravitational constant G is about $6.67 \cdot 10^{-11}$ m^3·kg^{-1}·s^{-2} while the speed of light c is approximately $3.00 \cdot 10^8$ m·s^{-1}; the field equations read

$$G_{\mu\nu} = 2.08 \cdot 10^{-43} \text{ s}^2 \cdot \text{kg}^{-1} \cdot \text{m}^{-1} \, T_{\mu\nu}.$$

The sun has an average mass-energy density (the dominant component of the stress-energy tensor) of $T^{00} = 1.27 \cdot 10^{20}$ kg · m^{-1} · s^{-2}. The corresponding component of the Einstein tensor within the sun is $G_{00} = 2.64 \cdot 10^{-23}$ m^{-2}. By comparison, the Einstein tensor of the flat Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is identically 0. We need to look at hyperenergetic phenomena, like a collapsing star, to find an Einstein tensor component appreciably greater than this. Even though the spacetime metric $g_{\mu\nu}$ is not generally flat, throughout most of the universe (including interstellar space and average solar systems such as our own) it is "flat enough" to be considered a small perturbation of a flat background metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and each component of the perturbation tensor $h_{\mu\nu}$ has absolute value much less than 1, as does all of its derivatives $\partial_\xi h_{\mu\nu}$. We can recalculate geometric quantities like the Christoffel symbols, Riemann tensor, and Einstein tensor in terms of the perturbation tensor; because its components are so small, we can ignore terms containing products of components (in particular, we can raise and lower indices with the Minkowski metric instead of the full metric) and derivatives. Because these quantities are linear in the perturbation tensor, the use of these simplifications is called the *linear approximation* to general relativity or *linearized gravity.*

Lemma 2.2. *In the linear approximation, the Christoffel symbols are*

$$\Gamma^{\alpha}{}_{\beta\gamma} = \eta^{\alpha\mu}\Gamma_{\mu\beta\gamma} = \eta^{\alpha\alpha}\frac{1}{2}\left(\partial_\gamma h_{\alpha\beta} + \partial_\beta h_{\alpha\gamma} - \partial_\alpha h_{\beta\gamma}\right).$$

Lemma 2.3. *In the linear approximation, the Riemann tensor is*

$$R^{\alpha}{}_{\beta\gamma\delta} = \eta^{\alpha\mu}R_{\mu\beta\gamma\delta} = \eta^{\alpha\alpha}\frac{1}{2}\left(\partial_\beta\partial_\gamma h_{\alpha\delta} + \partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\alpha\partial_\gamma h_{\beta\delta} - \partial_\beta\partial_\delta h_{\alpha\gamma}\right).$$

We can contract the linearized Riemann tensor to find the linearized Ricci tensor and scalar curvature fields. Of more importance is the Einstein tensor.

Lemma 2.4. *In the linear approximation, the Einstein tensor is*

$$G_{\mu\nu} = \frac{1}{2}\left(\Box\overline{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\overline{h}_{\alpha\beta} - \partial^\alpha\partial_\nu\overline{h}_{\mu\alpha} - \partial^\alpha\partial_\mu\overline{h}_{\nu\alpha}\right),$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ *with* $h = h^{\mu}{}_{\mu}$ *and* $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \partial^\alpha\partial_\alpha$ *is the d'Alembertian wave operator.*

Remark 2.5. Now the Einstein field equations can be written as

$$\Box\overline{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\overline{h}_{\alpha\beta} - \partial^\alpha\partial_\nu\overline{h}_{\mu\alpha} - \partial^\alpha\partial_\mu\overline{h}_{\nu\alpha} = -\frac{16\pi G}{c^4}T_{\mu\nu}.$$

While longer than the full field equations (1.2), the linearized equations are actually much simpler. The Einstein tensor is a complicated quantity constructed solely for the purpose of satisfying the field equations and has limited physical meaning itself. The linearized equations, on the other hand, equate the stress-energy tensor with a linear combination of second derivatives of the metric itself.

Since general relativity is essentially concerned with geometric, coordinate-independent quantities, we have a great deal of freedom in choosing the coordinate systems with which we describe the dynamics of spacetime, matter, and fields.

Definition 2.6. The choice of coordinates $\{x^\mu\}$ with which we describe spacetime is called a *gauge*. If a quantity is unchanged under gauge transformation, it is said to be *gauge invariant*.

Lemma 2.7. *Within the linearized approximation, let $\{x^\mu\}$ be a gauge on spacetime and $\{x'^\mu = x^\mu + \xi^\mu(x)\}$ be a new gauge, where the coordiante change functions have derivatives $\partial_\nu \xi_\mu(x)$ on the order of smallness as the perturbation components. Then in the linearized approximation, the perturbation tensor is*

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) - (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu).$$

Theorem 2.8 (The Lorenz Gauge)**.** *We can choose a gauge so that*

$$\partial^\nu \overline{h}_{\mu\nu} = 0,$$

where $\overline{h}_{\mu\nu}$ is the modified perturbation defined in lemma 4.4. This is called the Lorenz gauge.

Proof. According to lemma 2.7, in the linearized approximation the gauge transformation of the metric perturbation tensor is

$$h'_{\mu\nu} = h_{\mu\nu} - (\partial_\nu \xi_\mu - \partial_\mu \xi_\nu).$$

The trace of this transformed perturbation tensor is

$$\begin{aligned} h' = \eta^{\mu\nu} h'_{\mu\nu} =& \eta^{\mu\nu} \left(h_{\mu\nu} - (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) \right) \\ =& \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu} (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) \\ =& h - 2\partial_\alpha \xi^\alpha. \end{aligned}$$

The modified perturbation tensor in this gauge, $\bar{h}'_{\mu\nu}$, can be written as

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' \\ &= h_{\mu\nu} - (\partial_\nu \xi_\mu - \partial_\mu \xi_\nu) - \frac{1}{2}\eta_{\mu\nu}(h - 2\partial_\alpha \xi^\alpha) \\ &= \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \right) - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta \partial_\alpha \xi^\alpha \\ &= \overline{h}_{\mu\nu} - (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha). \end{aligned}$$

Therefore

$$\begin{aligned} \partial^\nu \bar{h}'_{\mu\nu} =& \partial^\nu \overline{h}_{\mu\nu} - \partial^\nu (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha) \\ =& \partial^\nu \overline{h}_{\mu\nu} - \partial^\nu \partial_\nu \xi_\mu - \partial_\mu \partial^\nu \xi_\nu + \eta_{\mu\nu} \partial^\nu \partial_\alpha \xi^\alpha \\ =& \partial^\nu \overline{h}_{\mu\nu} - \Box \xi_\mu - \partial_\mu (\partial^\nu \xi_\nu - \partial_\alpha \xi^\alpha). \end{aligned}$$

We see $\partial^\nu \xi_\nu = \eta^{\alpha\nu} \partial_\alpha \xi_\nu = \partial_\alpha \xi^\alpha$, so the quantity within the parenthases vanishes; we are left finding

$$\partial^\nu \bar{h}'_{\mu\nu} = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu. \tag{2.9}$$

If we begin working in a gauge in which $\partial^\nu \overline{h}_{\mu\nu}(x) = f_\mu(x)$, we can transform into the Lorenz gauge by using coordinate change functions $\{\xi^\mu(x)\}$ which satisfy the equation $\Box \xi_\mu(x) = f_\mu(x)$. □

An arbitrary type $(0,2)$ tensor on a 4-dimensional manifold has 16 components and thus 16 degrees of freedom. Because the metric must be symmetric, the perturbation tensor has at most 10 degrees of freedom. Making a gauge choice requires that we specify 4 component change functions, so the perturbation tensor in the Lorenz gauge has 6 degrees of freedom.

Remark 2.10. In the Lorenz gauge, the linearized Einstein equation, presented in Remark 2.5, becomes

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

Definition 2.11. In the absence of matter and physical fields, $T_{\mu\nu} = 0$ and the linearized Einstein equations (written in the Lorenz gauge) become

$$\Box \bar{h}_{\mu\nu} = 0. \tag{2.12}$$

Allowed solutions of the metric perturbation are tensor fields whose components evolve like independent scalar fields according to the wave equation. Such solutions are called *gravitational waves.*

Remark 2.13. Written explicitly, Equation 2.12 becomes

$$-\frac{\partial^2 h_{\mu\nu}}{\partial {x^0}^2} + \frac{\partial^2 h_{\mu\nu}}{\partial {x^1}^2} + \frac{\partial^2 h_{\mu\nu}}{\partial {x^2}^2} + \frac{\partial^2 h_{\mu\nu}}{\partial {x^3}^2} = 0.$$

The coordinate x_0 equals ct and $\{x^1, x^2, x^3\} = \{x, y, z\}$, so the above equation can be again re-written as

$$\frac{\partial^2 h_{\mu\nu}}{\partial t^2} = c^2 \left(\frac{\partial^2 h_{\mu\nu}}{\partial x^2} + \frac{\partial^2 h_{\mu\nu}}{\partial y^2} + \frac{\partial^2 h_{\mu\nu}}{\partial z^2} \right).$$

Physically, gravitational waves propagate through space at the speed of light.

Gravitational waves have 6 degrees of freedom. We can reduce this to 2 degrees and greatly simplify the appearance of the perturbation by specifying another gauge transformation.

Theorem 2.14 (The Transverse-Traceless Gauge)**.** *Within the Lorenz gauge, we can make further coordinate specifications requiring that the perturbation tensor ($h_{\mu\nu}$ in the Lorenz gauge and $h'_{\mu\nu}$ now) satisfies the conditions*

$$h'^{0\mu} = 0, \qquad {h'}^{i}_{\ i} = 0, \qquad \partial^j h'_{ij} = 0.$$

This is called the transverse-traceless gauge; the perturbation tensor in the transverse-traceless gauge is denoted $h^{TT}_{\mu\nu}$.

Proof. Let $\{\zeta^\mu\}$ be a new set of infinitessimal coordinate change functions. If we require $\Box \zeta_\mu = 0$, then we will also find

$$\Box \zeta_{\mu\nu} \equiv \Box \left(\partial_\nu \zeta_\mu + \partial_\mu \zeta_\nu - \eta_{\mu\nu} \partial_\alpha \zeta^\alpha \right) = 0,$$

because the wave operator commutes with partial derivatives. Under such a gauge transformation, the perturbation tensor becomes (by Lemma 2.7)

$$h'_{\mu\nu} = h_{\mu\nu} - (\partial_\nu \zeta_\mu + \partial_\mu \zeta_\mu),$$

and the derivative of the modified perturbation becomes (by Equation 2.9)

$$\partial^\nu \bar{h}'_{\mu\nu} = \partial^\nu \bar{h}_{\mu\nu} - \Box \zeta_\mu = \partial^\nu \bar{h}_{\mu\nu} = 0.$$

The perturbation still satisfies the condition of the Lorenz gauge; a solution which had the form of a gravitational wave before the new gauge transformation still has the form of a gravitational wave, but we can further control the perturbation's components. We begin by choosing ζ^0 so that the transformed modified trace $\bar{h}' = \bar{h}'^{\mu}{}_{\mu}$ vanishes and follow up finding ζ^i so that $\bar{h}'_{0i} = 0$. We know

$$\bar{h}' = \eta^{\mu\nu}\bar{h}'_{\mu\nu} = \eta^{\mu\nu}\left(h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h'\right) = h' - 2h' = -h',$$

so $h' = \bar{h}' = 0$ and $\bar{h}'_{\mu\nu} = h'_{\mu\nu}$. Now the requirement for the Lorenz gauge reads

$$\partial^{\nu}h'_{\mu\nu} = 0.$$

For $\mu = 0$, this equation can be rewritten as

$$\partial^0 h'_{00} + \partial^i h'_{0i} = 0.$$

We already know $h'_{0i} = 0$ for each i, so we can conclude that $\partial^0 h'_{00} = 0$; the 00 component of the metric perturbation is constant in time. In relativity, we are primarily interested in the derivatives of the metric, so h'_{00} may as well be 0. Now $h'_{0\mu} = 0$ for all μ; we can easily raise the indices to find $h'^{0\mu} = 0$ as well. The traceless condition $h'^{\mu}{}_{\mu} = 0$ can be strengthened to $h'^{i}{}_{i} = 0$, and the Lorenz condition can be strengthened to $\partial^j h'_{ij} = 0$. □

The simplest form of gravitation wave is a plane wave,

$$h_{\mu\nu} = \Re[\mathcal{A}_{\mu\nu}e^{ik_{\alpha}x^{\alpha}}],$$

where $\Re[\,\cdot\,]$ takes the real part of a complex argument and $\mathcal{A}_{\mu\nu}$ is a complex amplitude. The 4-dimensional wavevector is $k^{\mu} = \eta^{\mu\nu}k_{\nu}$. Within a coordinate system, the spatial components of the wavevector form a 3-vector $\vec{k}$ which points in the direction of the wave's propagation. Let's say for a moment that it is propagating in the x^3 direction: now the wavevector can be written as

$$k^{\mu} = \begin{pmatrix} k^0 \\ 0 \\ 0 \\ k^3 \end{pmatrix}.$$

Because a gravitational wave travels at the speed of light, it follows a null geodesic; the wavevector has a squared magnitude

$$\begin{aligned} |k^{\mu}|^2 &= 0 \\ \eta_{\mu\nu}k^{\mu}k^{\nu} &= 0 \\ -k_0^2 + k_3^2 &= 0. \end{aligned}$$

The components of the wavevector in the direction of time and spatial propagation have equal magnetude. The other components vanish.

This is all true in any coordinate system. However, we can say more in the transverse-traceless gauge. If $h^{TT}_{0\mu}$ is to vanish at every point in spacetime, then we must require that the amplitudes $\mathcal{A}_{0\mu}$ must also vanish in this gauge. To satisfy the Lorenz condition, we require

$$\begin{aligned}
0 =&\partial^i h^{TT}_{ji} \\
=&\partial^1 h^{TT}_{j1} + \partial^2 h^{TT}_{j2} + \partial^3 h^{TT}_{j3} \\
=&\partial_1 h^{TT}_{j1} + \partial_2 h^{TT}_{j2} + \partial_3 h^{TT}_{j3} \\
=&\partial_1 \Re[\mathcal{A}_{j1} e^{ik_\alpha x^\alpha}] + \partial_2 \Re[\mathcal{A}_{j2} e^{ik_\alpha x^\alpha}] + \partial_3 \Re[\mathcal{A}_{j3} e^{ik_\alpha x^\alpha}] \\
=&\Re[i\left(\mathcal{A}_{j1}k_1 + \mathcal{A}_{j2}k_2 + \mathcal{A}_{j3}k_3\right) e^{ik_\alpha x^\alpha}].
\end{aligned}$$

Since the wave is traveling in the x^3 direction, we know the components k_1 and k_2 vanish while k_3 does not. The Lorenz condition becomes

$$0 = \Re[ik_3 \mathcal{A}_{j3} e^{i(-k_0x^0 + k_3x^3)}].$$

If this is to hold true everywhere, we need the amplitude itself to equal 0 for every j. We already know $\mathcal{A}_{30} = 0$, so we find $\mathcal{A}_{3\mu} = 0$ for every μ, whether spatial or temporal. We are left with only four potentially nonzero components of the perturbation: h^{TT}_{11}, h^{TT}_{12}, h^{TT}_{21}, and h^{TT}_{22}. By the symmetry of the metric, we know $h^{TT}_{12} = h^{TT}_{21} \equiv h_\times$; because the perturbation is traceless in the transverse-traceless gauge, we can also conclude $h^{TT}_{22} = -h^{TT}_{11} \equiv -h_+$. We are left with two distinct components of the perturbation, corresponding to the two degrees of freedom left over after we adopted the transverse-traceless gauge. The perturbation itself, in matrix form, is

$$h^{TT}_{\mu\nu} = \Re\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(k_0x^0 + k_3x^3)}\right].$$

Of course, there aren't many pure plane waves radiating through space. But we can use plane waves as Fourier components to build up more general waveforms.

Definition 2.15. A *planar pulse wave* moving in the $\vec{k}$ direction is a gravitational wave all of whose perturbation components can be written as a function of one variable—$h^{TT}_{\mu\nu} = f_{\mu\nu}(x^0 - k_ix^i)$—and satisfy the limit

$$\lim_{x^0 - k_ix^i \to \pm\infty} h^{TT}_{\mu\nu}(x^0 - k_ix^i) = 0.$$

We can evaluate the partial derivatives with the chain rule:

$$\begin{aligned}
\partial_0 h^{TT}_{\mu\nu} &= f'_{\mu\nu}(k_0x^0 - k_ix^i) \\
\partial_j h^{TT}_{\mu\nu} &= -k_j f'_{\mu\nu}(k_0x^0 - k_ix^i).
\end{aligned}$$

If $\vec{n}$ is a 3-vector normal (in space) to $\vec{k}$, then $n^j \partial_j h^{TT}_{\mu\nu} = 0$. Therefore $h^{TT}_{\mu\nu}$ is constant throughout the spatial plane to which $\vec{k}$ is normal. Such a plane is called a plane of equal phase.

Although planar pulses extend throughout spacetime, they are mathematically functions of one variable and can thus be resolved into the integral of a Fourier spectrum of plane waves with parallel wavevectors. Assume, without loss of generality, that $\vec{k}$ lies in the x^3 direction. Now $k_3 = k_0$ for the whole wave and each component. For the whole wave, we can assume the wavenumber into the wave function;

we will leave them as the variable of Fourier decomposition in the components:

$$h_{\mu\nu}^{TT}(x^0 - x^3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re[\hat{h}_{\mu\nu}(k_0)e^{ik_0(x^0-x^3)}]dk_0,$$

where $\hat{h}_{\mu\nu}(k_0)$ is the Fourier transform of the wave perturbation $h_{\mu\nu}^{TT}(x^0 - x^3)$ itself. Since the entire perturbation is a gravitational wave solution in the transverse-traceless gauge and the wave operator is linear, each Fourier component of the perturbation must also be a gravitational wave solution in the transverse-traceless gauge. The components are plane waves, and as we saw above, the $h_{\mu 3}^{TT}$ components of a plane wave traveling in the x^3 direction vanish. If this is to be true thoughout all spacetime, we must require that $\hat{h}_{\mu 3}(k_0) = 0$ for all frequencies k_0. This implies that the $(\mu 3)$ components of the full perturbation also vanish. To satisfy the other requirements of the transverse-traceless gauge, we must write the full perturbation (in matrix form) as

$$h_{\mu\nu}^{TT}(x^0 - x^3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(x^0 - x^3) & h_\times(x^0 - x^3) & 0 \\ 0 & h_\times(x^0 - x^3) & -h_+(x^0 - x^3) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This tensor recalls the perturbation of a plane wave, but while all the nonzero components of the plane wave perturbation have the same functional form (namely, that of a sinusoid) with component-dependent amplitude, the perturbation of planar pulse has components of different functional forms.

3. The Change in Observational Period of Pulsar Signals

Definition 3.1. A *pulsar* is collapsed star consisting almost entirely of neutrons which emits a collimated beam of electromagnetic radiation and quickly rotates about some fixed axis. If an observer at rest with respect to the pulsar lies in the beam of radiation at some time, he will lie in it again after the pulsar rotates through a period; these are the only times that the observer can measure the radiation, so he sees them as periodic pulses, explaning the pulsar's name. This leads us to a simplified but often satisfactory definition: for an observer far removed from a pulsar, the pulsar may be regarded as an object that gives off bursts of electromagnetic radiation periodically according to the pulsar's proper time.

A radiation signal released by the pulsar follows a null geodesic through spacetime. The geometry of spacetime between the pulsar and an observer determines what curves qualify as geodesics and thus what the observer sees. We wish to find what effect a simple gravitational wave would have on an observer's measurements. To do this, we will find the Christoffel symbols for the gravitational wave and from them the allowed geodesics. But before we begin that process, we must demonstrate that the gravitational wave will not affect the pulsar and observer themselves.

Theorem 3.2. *In the transverse-traceless gauge, any body initially at rest* ($\frac{dx^i}{d\tau} = 0$, $\frac{dx^0}{d\tau} = 1$) *will remain at rest, with the progression of time undisturbed, under the influence of a planar-pulse gravitational wave on a flat background spacetime (assuming no non-gravitational forces are at work).*

Proof. Because no non-gravitational forces are acting on the body in question, it will follow a geodesic through spacetime. The equation for the geodesic, $x^\mu(\tau)$, satisfies the geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}.$$

Because the body is initially at rest, the spatial components of the geodesic's derivative $\frac{dx^i}{d\tau}$ are zero at $\tau = 0$. This means that any term in the geodesic equation with a spatial lower index in the Christoffel symbol will be zero:

$$\Gamma^\alpha{}_{i\mu}\frac{dx^i}{d\tau}\frac{dx^\mu}{d\tau} = \Gamma^\alpha{}_{\mu i}\frac{dx^\mu}{d\tau}\frac{dx^i}{d\tau} = 0.$$

The geodesic equation depends only on the term with two temporal lower indices in the Christoffel symbol. We find

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{00}\left(\frac{dx^0}{d\tau}\right)^2.$$

Now, $\frac{dx^0}{d\tau}$ certainly doesn't equal zero at $\tau = 0$. However, when we look at this component of the Christoffel symbol for the gravitational wave,

$$\Gamma^\alpha{}_{00} = \frac{1}{2}\eta^{\alpha\alpha}\left(\partial_0 h^{TT}_{\alpha 0} + \partial_0 h^{TT}_{0\alpha} - \partial_\alpha h^{TT}_{00}\right),$$

we find that it does equal zero. Every term in the component's definition involves a (0μ) component of $h^{TT}_{\mu\nu}$, and as we saw in the previous section, all these components vanish. Therefore $\Gamma^\alpha{}_{00} = 0$, and all the second derivatives disappear:

$$\frac{d^2x^\alpha}{d\tau^2} = 0.$$

If the second derivates are all zero, then the spatial derivates remain zero and the temporal derivative remains 1. The body remains at rest, undisturbed by the wave. □

Now we can begin the actual problem. Consider a planar-pulse gravitational wave in the transverse-traceless gauge, moving in the x^3 direction. It exhibits a metric perturbation

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(x^0 - x^3) & h_\times(x^0 - x^3) & 0 \\ 0 & h_\times(x^0 - x^3) & -h_+(x^0 - x^3) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Because this wave propagates on a flat, uniform spacetime, we can calculate the Christoffel symbols directly from the perturbation:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\alpha}(\partial_\mu h^{TT}_{\alpha\nu} + \partial_\nu h^{TT}_{\alpha\mu} - \partial_\alpha h^{TT}_{\mu\nu}).$$

It is immediately clear that many symbols will be zero. The perturbation is a function of only two components, x^0 and x^3; any derivatives in the x^1 or x^2 direction will be zero. Therefore any symbols with only indices of 1 and 2 will equal zero. At the same time, any perturbation component with an index of 0 or 3 is zero, so a Christoffel symbol will vanish if it does not have at least two indices of 1 or 2.

We can conclude that the only nonzero symbols have one index of 0 or 3 and two indices of 1 and/or 2. We find

$$\begin{array}{ll}
\Gamma^0{}_{11} = \frac{1}{2}h'_+(x^0-x^3) & \Gamma^0{}_{12} = \frac{1}{2}h'_\times(x^0-x^3) \\
\Gamma^0{}_{21} = \frac{1}{2}h'_\times(x^0-x^3) & \Gamma^0{}_{22} = -\frac{1}{2}h'_+(x^0-x^3) \\
\\
\Gamma^1{}_{01} = \frac{1}{2}h'_+(x^0-x^3) & \Gamma^1{}_{02} = \frac{1}{2}h'_\times(x^0-x^3) \\
\Gamma^1{}_{10} = \frac{1}{2}h'_+(x^0-x^3) & \Gamma^1{}_{13} = -\frac{1}{2}h'_+(x^0-x^3) \\
\Gamma^1{}_{20} = \frac{1}{2}h'_\times(x^0-x^3) & \Gamma^1{}_{23} = -\frac{1}{2}h'_\times(x^0-x^3) \\
\Gamma^1{}_{31} = -\frac{1}{2}h'_+(x^0-x^3) & \Gamma^1{}_{32} = -\frac{1}{2}h'_\times(x^0-x^3) \\
\\
\Gamma^2{}_{01} = \frac{1}{2}h'_\times(x^0-x^3) & \Gamma^2{}_{02} = -\frac{1}{2}h'_+(x^0-x^3) \\
\Gamma^2{}_{10} = \frac{1}{2}h'_\times(x^0-x^3) & \Gamma^2{}_{13} = -\frac{1}{2}h'_\times(x^0-x^3) \\
\Gamma^2{}_{20} = -\frac{1}{2}h'_+(x^0-x^3) & \Gamma^2{}_{23} = \frac{1}{2}h'_+(x^0-x^3) \\
\Gamma^2{}_{31} = -\frac{1}{2}h'_\times(x^0-x^3) & \Gamma^2{}_{32} = \frac{1}{2}h'_+(x^0-x^3) \\
\\
\Gamma^3{}_{11} = \frac{1}{2}h'_+(x^0-x^3) & \Gamma^3{}_{12} = \frac{1}{2}h'_\times(x^0-x^3) \\
\Gamma^3{}_{21} = \frac{1}{2}h'_\times(x^0-x^3) & \Gamma^3{}_{22} = -\frac{1}{2}h'_+(x^0-x^3).
\end{array}$$

Of course, the unwritten symbols are all zero. We can use the complete set of Christoffel symbols to find the differential equations describing geodesics in the presence of the gravitational wave:

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda};$$

$$\begin{aligned}
\frac{d^2x^0}{d\lambda^2} =& -\Gamma^0{}_{11}\left(\frac{dx^1}{d\lambda}\right)^2 + 2\Gamma^0{}_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} + \Gamma^0{}_{22}\left(\frac{dx^2}{d\lambda}\right)^2 \\
=& \frac{1}{2}h'_+(x^0-x^3)\left(\left(\frac{dx^2}{d\lambda}\right)^2 - \left(\frac{dx^1}{d\lambda}\right)^2\right) - h'_\times(x^0-x^3)\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda};
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x^1}{d\lambda^2} =& -2\Gamma^1{}_{01}\frac{dx^0}{d\lambda}\frac{dx^1}{d\lambda} - 2\Gamma^1{}_{02}\frac{dx^0}{d\lambda}\frac{dx^2}{d\lambda} - 2\Gamma^1{}_{13}\frac{dx^1}{d\lambda}\frac{dx^3}{d\lambda} - 2\Gamma^1{}_{23}\frac{dx^2}{d\lambda}\frac{dx^3}{d\lambda} \\
=& h'_+(x^0-x^3)\left(\frac{dx^3}{x\lambda} - \frac{dx^0}{d\lambda}\right)\frac{dx^1}{d\lambda} + h'_\times(x^0-x^3)\left(\frac{dx^3}{d\lambda} - \frac{dx^0}{d\lambda}\right)\frac{dx^2}{d\lambda};
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x^2}{d\lambda^2} =& -2\Gamma^2{}_{01}\frac{dx^0}{d\lambda}\frac{dx^1}{d\lambda} - 2\Gamma^2{}_{02}\frac{dx^0}{d\lambda}\frac{dx^2}{d\lambda} - 2\Gamma^2{}_{13}\frac{dx^1}{d\lambda}\frac{dx^3}{d\lambda} - 2\Gamma^2{}_{23}\frac{dx^2}{d\lambda}\frac{dx^3}{d\lambda} \\
=& h'_\times(x^0-x^3)\left(\frac{dx^3}{x\lambda} - \frac{dx^0}{d\lambda}\right)\frac{dx^1}{d\lambda} + h'_+(x^0-x^3)\left(\frac{dx^0}{d\lambda} - \frac{dx^3}{d\lambda}\right)\frac{dx^2}{d\lambda};
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x^3}{d\lambda^2} =& -\Gamma^3{}_{11}\left(\frac{dx^1}{d\lambda}\right)^2 + 2\Gamma^3{}_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} + \Gamma^3{}_{22}\left(\frac{dx^2}{d\lambda}\right)^2 \\
=& \frac{1}{2}h'_+(x^0-x^3)\left(\left(\frac{dx^2}{d\lambda}\right)^2 - \left(\frac{dx^1}{d\lambda}\right)^2\right) - h'_\times(x^0-x^3)\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda}.
\end{aligned}$$

We can divide the problem into two major cases. One case takes the displacement from the pulsar to the Earth to be in the direction the wave is traveling (i.e., the x^3 direction), so that the signal's worldline will be a slight perturbation from the background curve of

$$x^\mu(\lambda) = \begin{pmatrix} B^0 + D\lambda \\ 0 \\ 0 \\ D\lambda \end{pmatrix},$$

where D is the spatial distance from the pulsar to the Earth and we use the unit parameterization, $\lambda \in [0,1]$; the other case considers the displacement to be perpendicualr to the direction of the wave—withouth loss of generality, say the displacement is in the x^1 direction and take unit parameterization, so the signal worlidline is a perturbation from

$$x^\mu(\lambda) = \begin{pmatrix} B^0 + D\lambda \\ D\lambda \\ 0 \\ 0 \end{pmatrix}.$$

A general planar pulse can be written as a sum of planar pulses traveling parallel and perpendicular to the pulsar-Earth displacement, so it is sufficient to solve the basis cases individually.

4. A Wave Traveling Parallel to the Pulsar-Earth Displacement

Let a pulsar and the Earth lie at rest in the transverse traceless gauge, tracing out the worldlines

$$x^\mu_{\text{pulsar}}(\tau) = \begin{pmatrix} \tau \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x^\mu_{\text{Earth}}(\tau) = \begin{pmatrix} \tau \\ 0 \\ 0 \\ D \end{pmatrix}.$$

A photon emited by the pulsar, following the worldline $x^\mu(\lambda)$, will be observed on the earth if there is some value λ^* at which the worldline passes through the event

$$x^\mu(\lambda^*) = \begin{pmatrix} T \\ 0 \\ 0 \\ D \end{pmatrix}.$$

The first component, T, is the coordinate time at which an observer on Earth sees the photon: as long as the Earth is at rest, we consider a photon observed no matter what T is. Since the pulsar emits photon ebursts periodically, an Earthbound astronomer will observe them periodically, but we have no reason to assume that the time interval between observations will be the same as the interval between emissions, or even constant. In general, we expect the allowed null geodesics—and thus the observation times—to depend on the geometry of spacetime (and in particualr, the form of any gravitational wave) between the pulsar and the earth.

Above, we have the differential equations for geodesics in the presence of a planar pulse wave moving in the x^3 direction. It's a second-order equation, and we already know where the photons' worldlines originate (as we saw above, if the pulsar begins at rest, it remains at rest), so we need to give an initial wavevector $k^\mu = \dot{x}^\mu =$

$\frac{dx^\mu}{d\lambda}$ to specify a single worldline. We're interested in the wavevectors that yield photon worldlines that intersect the Earth's worldline. It's tempting to consider the simplest possible wavevector, the one associated with the background curve:

$$\dot{x}^\mu(0) = \begin{pmatrix} D \\ 0 \\ 0 \\ D \end{pmatrix},$$

with unit parameterization. Every term in the geodesic equation we found above contains either $\dot{x}^1$ or $\dot{x}^2$, which are both zero at $\lambda = 0$. Therefore the second derivative of this worldline the instant it is emitted from the pulsar is

$$\ddot{x}^\mu(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

if we differentiate the geodesic equations to find the higher-order derivatives, we will find that they are are also 0 for the same reason. If all of the curve's higher-order derivatives are zero, then the first derivative must remain constant: $\dot{x}^\mu(\lambda) = \dot{x}^\mu(0)$ for all λ. If a photon is released from the pulsar at the origin at time t_0, the coordinate expression of the geodesic takes the form

$$x^\mu(\lambda) = \begin{pmatrix} ct_0 + D\lambda \\ 0 \\ 0 \\ D\lambda \end{pmatrix}.$$

It happens that this is exactly the kind of geodesic we were looking for: at $\lambda = 1$, $x^1 = x^2 = 0$ while $x^3 = D$—the light's worldline intersects the Earth's worldline. As long as the separation from the pulsar to the Earth is along the same direction as the gravitational wave's propagation, the signals follows the background worldline, regardless of the functional form of the wave.

The pulsar will emit bursts at intervals of $\Delta\tau$ along its own proper time. Since the pulsar remains at rest in our coordinate system, its proper time equals the coordinate time; with a convenient choice for $x^0 = 0$, we find that the pulsar emits bursts withe worldlines begining at the events

$$B_n^\mu = \begin{pmatrix} n\Delta\tau \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for integer n; the worldline $x_n^\mu(\lambda)$ beginning at B_n^μ is observed at the event

$$O_n^\mu \equiv x_n^\mu(1) = \begin{pmatrix} n\Delta\tau + D \\ 0 \\ 0 \\ D \end{pmatrix}.$$

The Earth, too, is at rest, so the coordinate time also equals the Earth's proper time. The time between the observations of subsequent bursts is

$$O_{n+1}^0 - O_n^0 = [((n+1)\Delta\tau + D) - (n\Delta\tau + D)] = \Delta\tau.$$

The time interval between the observation of subsequent bursts equals the time between the bursts' emissions. If the Earth lies at $x^3 = -D$, so that the photon must pass through the wave rather than ride along with it, we can choose the worldlines

$$x^\mu(\lambda) = \begin{pmatrix} n\Delta\tau + D\lambda \\ 0 \\ 0 \\ -D\lambda \end{pmatrix}.$$

These curves also satisfy the geodesic equations and the boundary conditions; they yeild observation periods equal to the emission periods. In both cases, the observation period is unaffected by the gravitational wave, which makes sense: the wave does not affect distances in the x^3 direction, so there is no reason for the photon's worldline to deviate from the simplest, most efficient path–straight along the x^3 axis.

5. A Wave Traveling Perpendicular to the Pulsar-Earth Displacement

Such an argument won't hold true if the displacement of the Earth from the pulsar is perpendicular to the direction of the wave's propagation. Let the pulsar follow the same worldline it did above, but now let the Earth follow the worldline

$$x^\mu_{\text{Earth}}(\tau) = \begin{pmatrix} \tau \\ D \\ 0 \\ 0 \end{pmatrix}.$$

Again, let a planar pulse traveling in the x^3 direction encounter the system. In order to travel a distance D in the x^1 direction, the dominant spatial component of the photon worldline's first derivative will have to be $\dot{x}^1$, so the second derivatives of the worldline will not, in general, vanish; we'll have to take the functional form of the wave (h_+ and $h_\times$ for the two modes of polarizations) and explicitly solve the geodesic equations.

We can, however, make a significant simplification. Since we are working in the linear approximation, the magnitudes of the metric perturbations are far less than one. Geodiscs passing though the wave should only deviate slightly from the unperturbed background geodesics, and the deviation should be about on the same order of smallness as the perturbation magnitude. The geodesic equations involve a product of the derivative of the perturbation (arising from the Christoffel symbols) and two derivatives of the geodesic; accounting for the geodesic's deviations in this equation will give us terms on the order of the perturbation squared or cubed, which can safely be ignored. Therefore, when calculating the real geodesic sharing the endpoints of a background geodesic using the above equations, we may as well substitute in the first derivatives of the background geodesic. The background geodesic (with unit parameterization) between the pulsar and the Earth in this configuration is

$$x^\mu(\lambda) = \begin{pmatrix} D\lambda \\ D\lambda \\ 0 \\ 0 \end{pmatrix}.$$

The linearized geodesic equations, adjusted about this background geodesic, are

$$\begin{aligned}
\frac{d^2x^0}{d\lambda^2} &= -\frac{1}{2}D^2 h'_+(x^0 - x^3), \\
\frac{d^2x^1}{d\lambda^2} &= -D^2 h'_+(x^0 - x^3), \\
\frac{d^2x^2}{d\lambda^2} &= -D^2 h'_\times(x^0 - x^3), \\
\frac{d^2x^3}{d\lambda^2} &= -\frac{1}{2}D^2 h'_+(x^0 - x^3).
\end{aligned}$$

At this point, we need to provide the gravitational wave's waveforms and a photon's initial wavevector to solve the equations and find the future trajectory of the light's worldline through the wave. We will begin by finding the family of geodesics emanating from the pulsar in the presence of a specific gravitational wave. Then we will find which null geodesics intersect the Earth's worldline.

Perhaps the simplest nontrivial planar pulse wave is a triangle pulse of height a and width l:

$$h_+(x^0 - x^3) = \begin{cases} 0 & x^0 - x^3 < -\frac{l_+}{2} \\ \frac{2a_+}{l_+}(x^0 - x^3 + \frac{l_+}{2}) & -\frac{l_+}{2} < x^0 - x^3 < 0 \\ \frac{2a_+}{l_+}(\frac{l_+}{2} - (x^0 - x^3)) & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3, \end{cases}$$

$$h_\times(x^0 - x^3) = \begin{cases} 0 & x^0 - x^3 < -\frac{l_\times}{2} \\ \frac{2a_\times}{l_\times}((x^0 - x^3) + \frac{l_\times}{2}) & -\frac{l_\times}{2} < x^0 - x^3 < 0 \\ \frac{2a_\times}{l_\times}(\frac{l_\times}{2} - (x^0 - x^3)) & 0 < x^0 - x^3 < +\frac{l_\times}{2} \\ 0 & \frac{l_\times}{2} < x^0 - x^3, \end{cases}$$

where we have chosen our x^0 coordinates so that the gravitational wave passes through the $x^3 = 0$ plane at $x^0 = 0$. Of course, a perturbation can't really take the form of a triangular pulse: we require a metric to be smooth, and the triangular pulse has three points where its first derivative fails to exist. Still, a triangular pulse is similar enough in form to a more realistic waveform—say, for example, a Gaussian pulse—that we should be able to use it to draw some qualitative conculsions.

The linearized geodesic equations in the presence of a triangular pulse are

$$\frac{d^2x^0}{d\lambda^2} = \begin{cases} 0 & x^0 - x^3 < -\frac{l_+}{2} \\ -\frac{D^2ka_+}{l_+} & -\frac{l_+}{2} < x^0 - x^3 < 0 \\ \frac{D^2ka_+}{l_+} & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3, \end{cases}$$

$$\frac{d^2x^1}{d\lambda^2} = \begin{cases} 0 & x^0 - x^3 < -\frac{l_+}{2} \\ -\frac{2D^2ka_+}{l_+} & -\frac{l_+}{2} < x^0 - x^3 < 0 \\ \frac{2D^2ka_+}{l_+} & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3, \end{cases}$$

$$\frac{d^2x^2}{d\lambda^2} = \begin{cases} 0 & x^0 - x^3 < -\frac{l_+}{2} \\ -\frac{2D^2ka_\times}{l_\times} & -\frac{l_+}{2} < x^0 - x^3 < 0 \\ \frac{2D^2ka_\times}{l_\times} & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3, \end{cases}$$

$$\frac{d^2x^3}{d\lambda^2} = \begin{cases} 0 & x^0 - x^3 < -\frac{l_+}{2} \\ -\frac{D^2ka_+}{l_+} & -\frac{l_+}{2} < x^0 - x^3 < 0 \\ \frac{D^2ka_+}{l_+} & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3. \end{cases}$$

We can divide spacetime into four distinct regions: $x^0 - x^3 < -\frac{l_+}{2}$, the flat region "before" the wave; $-\frac{l_+}{2} < x^0 - x^3 < 0$, the region of the wave where the perturbation is increasing; $0 < x^0 - x^3 < \frac{l_+}{2}$, the region of the wave where the perturbation is decreasing; and $\frac{l_+}{2} < x^0 - x^3$, the flat region "after" the wave. Within each of these regions, the geodesic's second derivatives are constant; given the initial position and derivative of a worldline (i.e., the coordinates of its emission event and its initial wavevector), we can calculate its trajectory through the region in which it is emitted. Then we can take the worldline's four-position and wavevector at the boundary between this and the next region and repeat the process, calculating the worldline through the second region. We can repeat this process until the worldline has encountered the Earth or passed into the fourth region—here spacetime is flat, so geodesics will continue on with constant wavevector.

If we assume that the distance between the pulsar and the Earth is much greater than the width of the wave pulse (i.e., $D >> l_+, l_\times$), then we can consider five relevent cases:

(1) The signal is emitted in the third spacetime region.
(2) The signal is emitted in the second spacetime region.
(3) The signal passes all the way through the wave in deep space.
(4) The signal is observed in the third spacetime region.
(5) The signal is observed in the second spacetime region.

Of course, if the signal is observed in the first region or emitted in the fourth, it does not pass through the wave and we can take the simple, unperturbed solution.

CASE 1: THE SIGNAL IS EMITTED IN THE THIRD REGION

Here we consider photons emitted while they are in the third region, so the photons' worldlines must pass through parts of regions three and four. The curves' second derivatives in the relevant vicinitees are

$$\frac{d^2x^0}{d\lambda^2} = \begin{cases} D^2a_+/l_+ & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3, \end{cases}$$

$$\frac{d^2x^1}{d\lambda^2} = \begin{cases} 2D^2a_+/l_+ & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3, \end{cases}$$

$$\frac{d^2x^2}{d\lambda^2} = \begin{cases} 2D^2a_\times/l_\times & 0 < x^0 - x^3 < \frac{l_\times}{2} \\ 0 & \frac{l_\times}{2} < x^0 - x^3, \end{cases}$$

$$\frac{d^2x^3}{d\lambda^2} = \begin{cases} D^2a_+/l_+ & 0 < x^0 - x^3 < \frac{l_+}{2} \\ 0 & \frac{l_+}{2} < x^0 - x^3. \end{cases}$$

If a geodesic has the initial conditions

$$x^\mu(0) = \begin{pmatrix} B^0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \dot{x}^\mu(0) = \begin{pmatrix} \dot{x}^0(0) \\ \dot{x}^1(0) \\ \dot{x}^2(0) \\ \dot{x}^3(0) \end{pmatrix}$$

(where $0 < B^0 < l_+/2$), we can reconstruct the geodesic within region 3 by finding its components as MacLaurin series in λ. Since the second derivatives are constant, the component functions will be quadratic:

$$\begin{aligned} x^\mu(\lambda) &= x^\mu(0) + \dot{x}^\mu(0)\lambda + \frac{1}{2}\ddot{x}^\mu(0)\lambda^2 \\ &= \begin{pmatrix} B^0 + \dot{x}^0(0)\lambda + \frac{D^2a_+}{2l_+}\lambda^2 \\ \dot{x}^1(0)\lambda + \frac{D^2a_+}{l_+}\lambda^2 \\ \dot{x}^2(0)\lambda + \frac{D^2a_\times}{l_\times}\lambda^2 \\ \dot{x}^3(0)\lambda + \frac{D^2a_+}{2l_+}\lambda^2 \end{pmatrix}. \end{aligned}$$

The worldline leaves the thrid region when $x^0 - x^3 = l_+/2$. By inserting the reconstructed components, we find that this occurs at the point of parameter

$$\lambda_1 = \frac{\frac{l_+}{2} - B^0}{\dot{x}^0(0) - \dot{x}^3(0)};$$

the worldline's position and first derivative at this point are

$$x^\mu(\lambda_1) = \begin{pmatrix} B^0 + \dot{x}^0(0)\lambda_1 \frac{D^2a_+}{2l_+}\lambda_1^2 \\ \dot{x}^1(0)\lambda_1 + \frac{D^2a_+}{l_+}\lambda_1^2 \\ \dot{x}^2(0)\lambda_1 + \frac{D^2a_\times}{l_\times}\lambda_1^2 \\ \dot{x}^3(0)\lambda_1 + \frac{D^2a_+}{2l_+}\lambda_1^2 \end{pmatrix}, \quad \dot{x}^\mu(\lambda_1) = \begin{pmatrix} \dot{x}^0(0) + \frac{D^2a_+}{l_+}\lambda_1 \\ \dot{x}^1(0) + \frac{2D^2a_+}{l_+}\lambda_1 \\ \dot{x}^2(0) + \frac{2D^2a_\times}{l_\times}\lambda_1 \\ \dot{x}^3(0) + \frac{D^2a_+}{l_+}\lambda_1 \end{pmatrix}.$$

If we require the worldline to have a continuous first derivative—the strongest condition we can impose given the nonsmoothness of the perturbation—then we can extend it into the fourth region by taking its components' Taylor series in λ about λ_1. The second derivatives vanish in this region, so the curve must be linear:

$$x^\mu(\lambda) = \begin{pmatrix} \left[B^0 - \frac{D^2 a_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{l_+}{4} - B^0 + \frac{B^{0\,2}}{l_+}\right)\right] \\ \quad + \left[\dot{x}^0(0) + \frac{D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\left(\frac{1}{2} - \frac{B^0}{l_+}\right)\right]\lambda \\ \\ \left[-\frac{D^2 a_+}{(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{l_+}{4} - B^0 + \frac{B^{0\,2}}{l_+}\right)\right] \\ \quad + \left[\dot{x}^1(0) + \frac{2D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\left(\frac{1}{2} - \frac{B^0}{l_+}\right)\right]\lambda \\ \\ \left[-\frac{D^2 a_\times}{(\dot{x}^0(0)-\dot{x}^3(0))^2}\frac{l_+}{l_\times}\left(\frac{l_+}{4} - B^0 + \frac{B^{0\,2}}{l_+}\right)\right] \\ \quad + \left[\dot{x}^2(0) + \frac{2D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\frac{l_+}{l_\times}\left(\frac{1}{2} - \frac{B^0}{l_+}\right)\right]\lambda \\ \\ \left[-\frac{D^2 a_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{l_+}{4} - B^0 + \frac{B^{0\,2}}{l_+}\right)\right] \\ \quad + \left[\dot{x}^0(0) + \frac{D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\left(\frac{1}{2} - \frac{B^0}{l_+}\right)\right]\lambda \end{pmatrix}.$$

We need to find the components of the initial first derivative such that the geodesic encounters the earth. If we again choose the unit parameterization, we have four unknowns:

$$\dot{x}^0(0),\ \ \dot{x}^1(0),\ \ \dot{x}^2(0),\ \ \dot{x}^3(0),$$

and four equations (taken from the conditions $x^1(1) = D,\ x^2(0) = 0,\ x^3(0) = 0,\ \dot{x}^\mu \dot{x}_\mu = 0$):

$$\begin{aligned}
&-D^2 a_+\left(\frac{l_+}{4} - B^0 + \frac{B^{0^2}}{l_+}\right) \\
&\qquad = \left(D - \dot{x}^1(0)\right)\left(\dot{x}^0(0) - \dot{x}^3(0)\right)^2 - D^2 a_+\left(1 - \frac{2B^0}{l_+}\right)\left(\dot{x}^0(0) - \dot{x}^3(0)\right), \\
&D^2 a_\times \frac{l_+}{l_\times}\left(\frac{l_+}{4} - B^0 + \frac{B^{0^2}}{l_+}\right) \\
&\qquad = \dot{x}^2(0)\left(\dot{x}^0(0) - \dot{x}^3(0)\right)^2 + D^2 a_\times \frac{l_+}{l_\times}\left(1 - \frac{2B^0}{l_+}\right)\left(\dot{x}^0(0) - \dot{x}^3(0)\right), \\
&D^2 a_+\left(\frac{l_+}{8} - \frac{B^0}{2} + \frac{B^{0^2}}{2l_+}\right) \\
&\qquad = \dot{x}^3(0)\left(\dot{x}^0(0) - \dot{x}^3(0)\right)^2 + D^2 a_+\left(\frac{1}{2} - \frac{B^0}{2l_+}\right)\left(\dot{x}^0(0) - \dot{x}^3(0)\right), \\
&\left(\dot{x}^0(0)\right)^2 = \left(\dot{x}^1(0)\right)^2 + \left(\dot{x}^2(0)\right)^2 + \left(\dot{x}^3(0)\right)^2.
\end{aligned}$$

This system of equations can't be solved analytically, but if we provide physical numbers for the parameters $D, a_+, a_\times, l_+, l_\times$ and B^0, we can find an approximate solution by numerical methods. One set of realistic astrophysical numbers is

$$\begin{aligned} D =& 10^3 \text{ ly}, \\ a_+ =& 10^{-15}, \\ a_\times =& 10^{-15}, \\ l_+ =& 10^{-6} \text{ ly}, \\ l_\times =& 10^{-6} \text{ ly}. \end{aligned}$$

Here, we aren't interested in the worldline of any single burst; we want to compare the intersection of subsequent worldlines with the Earth. Therefore, rather than assign one number to B^0, we want a set of numbers. A pulsar period $\Delta\tau$ of 10^{-7} ly will give us nine or ten pulsar bursts throughout the gravitational wave; therefore we take

$$B^0_n = n\Delta\tau = n \cdot 10^{-7} \text{ ly}.$$

For these parameters, we find the solutions

n	$\dot{x}^0(0)$	$\dot{x}^1(0)$	$\dot{x}^2(0)$	$\dot{x}^3(0)$
0	1000	1000	−0.0050	−0.0025
1	1000	1000	−0.000000040	−0.000000020
2	1000	1000	−0.000000030	−0.000000015
3	1000	1000	−0.000000020	−0.000000010
4	1000	1000	−0.000000010	−0.000000005
5	1000	1000	0	0.

These results make qualitative sense. The burst at $n = 0$ needs to pass through an entire half of the wave, so it will be bent the most; the burst at $n = 5$ doesn't pass through the wave at all, so we can use the flat-spacetime solution (note that we chose $D = 1000$).

We wish to find the period between observation of subsequent pulsar bursts. We can do this by calculating $O^0_{n-1} - O^0_n(1)$. The term

$$\frac{D^2 a_+}{(\dot{x}^0(0) - \dot{x}^3(0))^2}\left(\frac{l_+}{8} - \frac{B^0}{2} + \frac{{B^0}^2}{2l_+}\right)$$

is of the order 10^{-22}; meanwhile, the term

$$\frac{D^2 a_+}{\dot{x}^0(0) - \dot{x}^3(0)}\left(-\frac{B^0}{l_+}\right)$$

is of the order 10^{-13}, so it is more significant in our analysis. Including this term but not the former, we find

$$\begin{aligned} O^0_5 - O^0_4 &= 10^{-7} - 1.00 \cdot 10^{-13} \text{ ly}, \\ O^0_4 - O^0_3 &= 10^{-7} - 1.00 \cdot 10^{-13} \text{ ly}, \\ O^0_3 - O^0_2 &= 10^{-7} - 1.00 \cdot 10^{-13} \text{ ly}, \\ O^0_2 - O^O_1 &= 10^{-7} - 1.00 \cdot 10^{-13} \text{ ly}, \\ O^0_1 - O^0_0 &= 10^{-7} - 1.00 \cdot 10^{-13} \text{ ly}. \end{aligned}$$

The major term—10^{-7}—is simply the period between emissions; we expect to see it no matter the curvature of spacetime. However, each observation period is shortened by $1.00 \cdot 10^{-13}$ ly. There is a real difference between the observation period and emission period due to the gravitational wave. The emission period, in more familiar units, is about 3.16 s; the difference is $3.16 \cdot 10^{-6}$ s. The difference is not large, but it is within the scope of measurement.

CASE 2: THE SIGNAL IS EMITTED IN THE SECOND REGION

We now turn our attention to bursts released in the second region of spacetime ($B^0 \in [-\frac{l_+}{2}, 0]$). The geodesics of these bursts must pass through a portion of the second region (that of increasing metric perturbation), all of the third region (decreasing perturbation), and a large stretch of the fourth region (flat spacetime). We will therefore need to piece together three separate solutions rather than two, as we did in case 1. Watching the process is not particularly enlightening, so we can jump forward to the results. Within region 4, the signal's worldline takes the form

$$x^\mu(\lambda) = \begin{pmatrix} \left[B^0 + \frac{D^2 a_+}{2(\dot{x}^0 - \dot{x}^3)^2}\left(\frac{B^{02}}{l_+} + B^0 - \frac{l_+}{4}\right)\right] + \left[\dot{x}^0(0) + \frac{D^2 a_+}{2(\dot{x}^0 - \dot{x}^3)}\left(\frac{2B^0}{l_+} + 1\right)\right]\lambda \\ \left[\frac{D^2 a_+}{(\dot{x}^0 - \dot{x}^3)^2}\left(\frac{B^{02}}{l_+} + B^0 - \frac{l_+}{4}\right)\right] + \left[\dot{x}^1(0) + \frac{D^2 a_+}{(\dot{x}^0 - \dot{x}^3)}\left(\frac{2B^0}{l_+} + 1\right)\right]\lambda \\ \left[\frac{D^2 a_\times}{(\dot{x}^0 - \dot{x}^3)^2}\frac{l_+}{l_\times}\left(\frac{B^{02}}{l_+} + B^0 - \frac{l_+}{4}\right)\right] + \left[\dot{x}^2(0) + \frac{D^2 a_+}{(\dot{x}^0 - \dot{x}^3)}\frac{l_+}{l_\times}\left(\frac{2B^0}{l_+} + 1\right)\right]\lambda \\ \left[\frac{D^2 a_+}{2(\dot{x}^0 - \dot{x}^3)^2}\left(\frac{B^{02}}{l_+} + B^0 - \frac{l_+}{4}\right)\right] + \left[\dot{x}^3(0) + \frac{D^2 a_+}{2(\dot{x}^0 - \dot{x}^3)}\left(\frac{2B^0}{l_+} + 1\right)\right]\lambda \end{pmatrix}.$$

This is the section of the geodesic which will encounter the Earth, so by choosing the unit parameterization and adding the conditions

$$x^\mu(1) = \begin{pmatrix} T \\ D \\ 0 \\ 0 \end{pmatrix} \text{ and } \dot{x}^\mu \dot{x}_\mu = 0,$$

we can again solve for the signals' inital wavevectors numerically. With these conditions and the astrophysical values we used in case 1, we find the final results

$$\begin{aligned} O^0_{\ 0} - O^0_{\ -1} &= 10^{-7} + 1.00 \cdot 10^{-13}, \\ O^0_{\ -1} - O^0_{\ -2} &= 10^{-7} + 1.00 \cdot 10^{-13}, \\ O^0_{\ -2} - O^0_{\ -3} &= 10^{-7} + 1.00 \cdot 10^{-13}, \\ O^0_{\ -3} - O^0_{\ -4} &= 10^{-7} + 1.00 \cdot 10^{-13}, \\ O^0_{\ -4} - O^0_{\ -5} &= 10^{-7} + 1.00 \cdot 10^{-13}. \end{aligned}$$

As in case 1, the gravitational wave introduces a correction one tenth the durration of the emission period. However, now the correction is an addition: The observed period is now increased by about $3.16 \cdot 10^{-6}$ seconds, while the emission period is still just 3.16 seconds.

CASE 3: THE SIGNAL CROSSES THE WAVE IN DEEP SPACE

If a signal is emitted before $B^0 = -l_+/2$, then it begins its worldline by passing through the flat first region and then encountering the wave. However, if the burst begins its journey too soon, it will arrive at Earth durring or even before passing through any of the wave. Therefore we must set a lower bound for the burst times for this case, one which will depend on the nature of the geodesics. We will begin by finding the form of the allowed geodesics radiating from the pulsar, and then determining the time range of interest.

We can construct geodesics the same way we did in cases 1 and 2, by solving the geodesic equation in each region and fitting each solution together with continuity and differentiability. In the region 4—after passing through the wave—the geodesic takes the form

$$x^\mu(\lambda) = \begin{pmatrix} B^0 - \frac{D^2 a_+ l_+}{4(\dot{x}^0(0)-\dot{x}^3(0))^2} + \dot{x}^0(0)\lambda \\ -\frac{D^2 a_+ l_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2} + \dot{x}^1(0)\lambda \\ -\frac{D^2 a_\times l_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\frac{l_+}{l_\times} + \dot{x}^2(0)\lambda \\ -\frac{D^2 a_+ l_+}{4(\dot{x}^0(0)-\dot{x}^3(0))^2} + \dot{x}^3(0)\lambda \end{pmatrix}.$$

It is interesting to note that the four equations used to solve for the initial wavevector, $x^1(1) = D$, $x^2(1) = x^3(1) = 0$, and $\dot{x}^\mu(0)\dot{x}_\mu(0) = 0$, are all independent of the burst time B^0. Therefore the initial wavevectors of signals considered in case 3 are time-independent (as are the observation periods). For the astrophysical conditions we've used above, the initial wavevector is

$$x^\mu(0) = \begin{pmatrix} 1000 \\ 1000 \\ 5.0 \cdot 10^{-17} \\ 2.5 \cdot 10^{-17} \end{pmatrix}.$$

This is negligibly different from the initial wavevector of the background geodesics, and we will find similar results for any realistic physical setup. For the sake of calculations, we may as well say that the real initial wavevector is equal to the background wavevector:

$$x^\mu(0) = \begin{pmatrix} D \\ D \\ 0 \\ 0 \end{pmatrix}.$$

Above, we required that the signal pass all the way through the wave before being observed on Earth. We know that the latest possible signal to fall into this case is released at $B^0 = -\frac{l_+}{2}$; any signal released after this is emitted while the pulsar lies in the wave. The earliest possible signal included in this case is observed just as the wave encounters the Earth, so

$$\begin{aligned} x^0(1) - x^3(1) &= \frac{l_+}{2} \\ B^0 + \dot{x}^0(0) - \dot{x}^3(0) &= \frac{l_+}{2} \\ B^0 &= \frac{l_+}{2} - (\dot{x}^0(0) - \dot{x}^3(0)). \end{aligned}$$

In almost any realistic system, this will effectively reduce to

$$B^0 = -D + \frac{l_+}{2},$$

so case three considers signals emitted between $x^0 = -D + \frac{l_+}{2}$ and $x^0 = -\frac{l_+}{2}$. The observational period of any subsequent signals released in this time range is

$$\begin{aligned} x^0_{n+1}(1) - x^0_n(1) &= \left[B^0_{n+1} - \frac{D^2 a_+ l_+}{4(\dot{x}^0(0) - \dot{x}^3(0))^2} + \dot{x}^0(0)\right] \\ &\quad - \left[B^0_n - \frac{D^2 a_+ l_+}{4(\dot{x}^0(0) - \dot{x}^3(0))^2} + \dot{x}^0(0)\right] \\ &= [(n+1)\Delta\tau - (n)\Delta\tau] \\ &= \Delta\tau. \end{aligned}$$

We didn't even need to find a numeric solution for this case because the initial wavevector—the very reason we needed to use numeric solutions earlier—is the same for each signal and can be factored out, leaving the difference between one quantity and itself. Regardless of the parameters of the gravitational wave, two subsequent signals (provided they both pass all the way through the wave) will be observed with the same period as which they are emitted.

However, that's not to say that the signals are unaffected by the wave. We can compare the bent geodesics to the background geodesic by subtracting the arrival time of a bent worldline (i.e., $x^0_n(1)$) from the arrival time of a background worldlne, D. We find

$$\Delta x^0_n = \frac{D^2 a_+ l_+}{4(\dot{x}^0(0) - \dot{x}^3(0))^2}.$$

The n^{th} signal arrives at a time slightly different than it would if there were no gravitational wave. Passing all the way through the wave does not affect the signals' observational period, but it does affect their phase.

Case 4: The signal is observed in the third region

If a signal is released before the time $x^0 = -D + \frac{l_+}{2}$, it may be observed while the gravitational wave is passing by the Earth. We have already done the work in finding what the worldline of such a signal would look like: when studying the third case, we calculated what a geodesic flowing from the first region, through the second and into the third looks like. It will take this form regardless of whether it continues into the fourth region, so we can use this equation by choosing an emission time such that $x^\mu(1)$ intersects the Earth's worldline within the third region. We find that the final form of such a geodesic looks like

$$x^\mu(\lambda) = \begin{pmatrix} \left[B^0 + \frac{D^2 a_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{B^{0\,2}}{l_+} - B^0 - \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^0(0) + \frac{D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\left(\frac{B^0}{l_+} - \frac{1}{2}\right)\right]\lambda + \left[\frac{D^2 a_+}{2l_+}\right]\lambda^2 \\ \\ \left[\frac{D^2 a_+}{(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{B^{0\,2}}{l_+} - B^0 - \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^1(0) + \frac{2D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\left(\frac{B^0}{l_+} - \frac{1}{2}\right)\right]\lambda + \left[\frac{D^2 a_+}{l_+}\right]\lambda^2 \\ \\ \left[\frac{D^2 a_\times}{(\dot{x}^0(0)-\dot{x}^3(0))^2}\frac{l_+}{l_\times}\left(\frac{B^{0\,2}}{l_+} - B^0 - \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^2(0) + \frac{2D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\frac{l_+}{l_\times}\left(\frac{B^0}{l_+} - \frac{1}{2}\right)\right]\lambda + \left[\frac{D^2 a_\times}{l_\times}\right]\lambda^2 \\ \\ \left[\frac{D^2 a_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{B^{0\,2}}{l_+} - B^0 - \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^3(0) + \frac{D^2 a_+}{\dot{x}^0(0)-\dot{x}^3(0)}\left(\frac{B^0}{l_+} - \frac{1}{2}\right)\right]\lambda + \left[\frac{D^2 a_+}{2l_+}\right]\lambda^2 \end{pmatrix}.$$

Again, numerical evaluations suggest that the difference between the real initial wavevectors and the background wavevectors is negligible, so for our purposes we can assume $(\dot{x}^0(0) - \dot{x}^3(0)) = D$ for all worldlines released around this time. We find the usual phase equation,

$$\begin{aligned} x^0(\lambda) - x^3(\lambda) &= B^0 + (\dot{x}^0(0) - \dot{x}^3(0))\lambda \\ &= B^0 + D\lambda; \end{aligned}$$

Case 4 only considers signals which encounter the Earth in region 3, so $0 < x^0(1) - x^3(1) < \frac{l_+}{2}$; the condition $x^0(1) - x^3(1) < \frac{l_+}{2}$ tells us that the signal must be released at a time $B^0 < -D + \frac{l_+}{2}$, which we already know; the other condition, $0 < x^0(0) - x^3(0)$, tells us that the signal must be released at a time $B^0 > -D$.

Now that we know what time range we are dealing with, we can determine the change in observation period. Since we're assuming $\dot{x}^0(0) - \dot{x}^3(0) = D$ for all worldlines in this case, we don't need to resort to a numerical solution:

$$O^0_{n+1} - O^0_n = \Delta\tau + \frac{a_+}{2}\left(\frac{(2n+1)\Delta\tau^2}{l_+} - \Delta\tau\right) + Da_+\frac{\Delta\tau}{l_+}.$$

For realistic gravitational waves, the first correction term will be orders of magnitude less than the second, so we can simplify to

$$O^0_{n+1} - O^0_n \approx \Delta\tau\left(1 + \frac{Da_+}{l_+}\right).$$

The gravitational wave has again changed the observational period. Using the physical numbers from before, it is extended by $3.16 \cdot 10^{-6}$ s—the same change as in the second case.

CASE 5: THE SIGNAL IS OBSERVED IN THE SECOND REGION

All we have left to consider are signals sent before the time $B^0 = -D$. If the signal is sent before $B^0 = -D - \frac{l_+}{2}$, it can follow the background geodesic and reach the Earth before the wave arives at $x^0 = -\frac{l_+}{2}$, so the only interesting case left is that of signals observed in the second spacetime region, which originate in bursts after $-D - \frac{l_+}{2}$ but before $-D$.

As in case 4, we can take the geodesic calculated in case 3 and require that it terminate in region 2. Doing so, we find the final portion of the curve takes the form

$$x^\mu(\lambda) = \begin{pmatrix} \left[B^0 - \frac{D^2 a_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{B^{0\,2}}{l_+} + B^0 + \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^0(0) - \frac{D^2 a_+}{\dot{x}^0-\dot{x}^3}\left(\frac{B^0}{l_+} + \frac{1}{2}\right)\right]\lambda - \left[\frac{D^2 a_+}{2l_+}\right]\lambda^2 \\ \\ \left[-\frac{D^2 a_+}{(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{B^{0\,2}}{l_+} + B^0 + \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^1(0) - \frac{2D^2 a_+}{\dot{x}^0-\dot{x}^3}\left(\frac{B^0}{l_+} + \frac{1}{2}\right)\right]\lambda - \left[\frac{D^2 a_+}{l_+}\right]\lambda^2 \\ \\ \left[-\frac{D^2 a_\times}{(\dot{x}^0(0)-\dot{x}^3(0))^2}\frac{l_+}{l_\times}\left(\frac{B^{0\,2}}{l_+} + B^0 + \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^2(0) - \frac{2D^2 a_+}{\dot{x}^0-\dot{x}^3}\frac{l_+}{l_\times}\left(\frac{B^0}{l_+} + \frac{1}{2}\right)\right]\lambda - \left[\frac{D^2 a_\times}{l_\times}\right]\lambda^2 \\ \\ \left[-\frac{D^2 a_+}{2(\dot{x}^0(0)-\dot{x}^3(0))^2}\left(\frac{B^{0\,2}}{l_+} + B^0 + \frac{l_+}{4}\right)\right] \\ \quad + \left[\dot{x}^3(0) - \frac{D^2 a_+}{\dot{x}^0-\dot{x}^3}\left(\frac{B^0}{l_+} + \frac{1}{2}\right)\right]\lambda - \left[\frac{D^2 a_+}{2l_+}\right]\lambda^2 \end{pmatrix}.$$

We can try to find the initial wavevectors of the signals observed on Earth, but as in case 4, they differ negligibly from the background wavevector. We may as well again assume $\dot{x}^0(0) - \dot{x}^3(0) = D$ for all the relevant signals. Now the observational period is

$$O^0_{n+1} - O^0_n = \Delta\tau - \frac{a_+}{2}\left(\frac{(2n+1)\Delta\tau^2}{l_+} + \Delta\tau\right) - Da_+\frac{\Delta\tau}{l_+}.$$

Again, we can ignore the first correctional term, leaving an observational period of

$$O^0_{n+1} - O^0_n = \Delta\tau\left(1 - \frac{Da_+}{l_+}\right).$$

For the numerical values we used before, the period will be shortened by about $3.16 \cdot 10^{-6}$ s—the same change as in the first case.

6. Conclusions

We have seen that if a triangular gravitational wave pulse travels down the axis of displacement from a pulsar to the Earth, an Earthbound observer will not see any change in the pulsar's period. And if the wave travels in a direction perpendicular to the separation between the two bodies, we expect the period of the pulsar, as observed on Earth, to quickly shorten and then lengthen back to normal, then to remain constant for a long time, and then to quickly lengthen and then shorten back to normal. The assumptions we made about the gravitational wave weren't very realistic, and in many places we relied on numerical solutions for specific physical parameters, so the quantitative results we have found will probably not be very useful. However, they do provide a fairly good qualitative picture of what is happening. Softening a triangular pulse into a more natural, curved shape would change the geodesic equations, but not radically so; neither would modifying the magnitude or width of the wave (unless we make the perturbation magnitude less than zero, in which case our results suggest that when the period was shortened, it is now lengthened, and vice versa). Furthermore, the geodesic equations presented on page 10 are always valid for linearized gravity, so all that is needed to solve the problem for other (planar-pulse) waves and pulsar-Earth orientations is patience and computing power.

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