LEBESGUE MEASURE AND L2 SPACE.

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Abstract. This paper begins with an introduction to measure spaces and the Lebesgue theory of measure and integration. Several important theorems regarding the Lebesgue integral are then developed. Finally, we prove the completeness of the $\mathcal{L}^2(\mu)$ space and show that it is a metric space, and a Hilbert space.

Contents

1. Measure Spaces

Definition 1.1. Suppose X is a set. Then X is said to be a **measure space** if there exists a σ-ring $\mathfrak{M}$ (that is, $\mathfrak{M}$ is a nonempty family of subsets of X closed under countable unions and under complements)of subsets of X and a non-negative countably additive set function μ (called a measure) defined on $\mathfrak{M}$. If $X \in \mathfrak{M}$, then X is said to be a **measurable space**.

For example, let $X = R^p$, $\mathfrak{M}$ the collection of Lebesgue-measurable subsets of R^p, and μ the Lebesgue measure. Another measure space can be found by taking X to be the set of all positive integers, $\mathfrak{M}$ the collection of all subsets of X, and $\mu(E)$ the number of elements of E.

We will be interested only in a special case of the measure, the *Lebesgue measure.* The Lebesgue measure allows us to extend the notions of length and volume to more complicated sets.

Definition 1.2. Let R^p be a p-dimensional Euclidean space . We denote an interval of R^p by the set of points $x = (x_1, ..., x_p)$ such that

$$a_i \leq x_i \leq b_i \ \ (i = 1, \ldots, p) \tag{1.3}$$

Definition 1.4. Let I be an interval in R^p and define

$$m(I) = \prod_{i=1}^{p}(b_i - a_i) \tag{1.5}$$

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If A is a union of finite intervals $A = I_1 \cup \ldots \cup I_n$ and these intervals are pairwise disjoint, we set

$$m(A) = m(I_1) + \ldots + m(I_n) \tag{1.6}$$

Definition 1.7. Let f be a function defined on a measurable space X with values in the extended real number system. The function f is said to be **measurable** if the set

$$\{x | f(x) > a\} \tag{1.8}$$

is measurable for all real a.

Measurability is also preserved with respect to addition, multiplication, and limit processes of measurable functions.

2. Lebesgue Integration

Definition 2.1. A real-valued function s defined on X is called a *simple function* if the range of s is finite.

Let $E \in X$ and define

$$K_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases} \tag{2.2}$$

K_E is called the *characteristic function* or *indicator function* of E.

Any simple function can be written as a finite linear combination of characteristic functions. Suppose s is a simple function which takes on values $c_1, \ldots, c_n$ and let

$$E_i = \{x | s(x) = c_i\} \;\; i = 1, \ldots, n \tag{2.3}$$

Then

$$s = \sum_{i=1}^{n} c_i K_{E_i} \tag{2.4}$$

Theorem 2.5. *Let f be a real function on X. For every $x \in X$, there exists a sequence $\{s_n\}$ such that $s_n(x) \to f(x)$ as $n \to \infty$. If f is measurable $\{s_n\}$ may be chosen as a sequence of measurable functions. If f is nonnegative, $\{s_n\}$ may be chosen to be a monotonically increasing sequence.*

This theorem shows that any measurable function can be approximated by simple functions, and therefore also by linear combinations of simple functions. We will use this to define the Lebesgue integral.

Definition 2.6. Let

$$s(x) = \sum_{i=1}^{n} c_i K_{E_i}(x) \;\; (x \in X, c_i > 0) \tag{2.7}$$

as in (2.1). Suppose s is measurable, and suppose $E \in \mathfrak{M}$. Define

$$I_E(s) = \sum_{i=1}^{n} c_i \mu(E \cap E_i) \tag{2.8}$$

If f is measurable and nonnegative, we define

$$\int_E f \, \mathrm{d}\mu = \sup I_E(s) \tag{2.9}$$

where the sup is taken over all measurable simple functions s such that $0 \leq s \leq f$ The left side of (2.9) is called the **Lebesgue integral** of f (with respect to the measure μ over the set E).

To extend the integral to functions that are not nonnegative is an easy addition.

Definition 2.10. Let f be defined on a measure space X with values in the extended real numbers. We may write

$$f = f^+ - f^- \tag{2.11}$$

where

$$f^+(x) = \begin{cases} f(x) & \text{if f(x) ¿ 0} \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if f(x) ¡ 0} \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

Then f^+ and f^- are nonnegative measurable functions. We define

$$\int_E f = \int_E f^+ + \int_E f^- \tag{2.14}$$

Integration can also be extended to complex functions.

Definition 2.15. Let f be a complex-valued function defined on a measure space X and $f = u + iv$, with u and v real. We say that f is measurable if and only if u and v are measurable.

Suppose μ is a measure on X, E is a measurable subset of X, and f is a complex function on X. We say that $f \in \mathcal{L}(\mu)$ on E (and f is *complex square-integrable*) if f is measurable and

$$\int_E |f| \, d\mu < +\infty \tag{2.16}$$

We define the integral of f (with respect to μ and over E) as

$$\int_E f \, d\mu = \int_E u \, d\mu + i \int_E v \, d\mu \tag{2.17}$$

Intuitively, the Lebesgue integral measures the area under a function by making partitions of the range of the function, whereas the Riemann integral partitions the domain. The Lebesgue integral has many desirable properties compared to the Riemann integral. The one we will be most concerned with is the fact that any Lebesgue measurable function is Lebesgue integrable (while a function is Riemann integrable on an interval if and only if it is continuous almost everywhere). This allows us to integrate additional functions on much more diverse sets.

3. $\mathcal{L}^2$ Space

The $\mathcal{L}^2$ space is a special case of an $\mathcal{L}^p$ space, which is also known as the Lebesgue space.

Definition 3.1. Let X be a measure space. Given a complex function f, we say $f \in \mathcal{L}^2$ on X if f is (Lebesgue) measurable and if

$$\int_X |f|^2 \, \mathrm{d}\mu < +\infty \tag{3.2}$$

Then the function f is also said to be *square-integrable*. In other words, $\mathcal{L}^2$ is the set of square-integrable functions.

For $f \in \mathcal{L}^2(\mu)$ define

$$\|f\| = \left(\int_X |f|^2 \, \mathrm{d}\mu \right)^{1/2} \tag{3.3}$$

We call $\|f\|$ the $\mathcal{L}^2(\mu)$ norm of f.

To give a notion of distance in $\mathcal{L}^2(\mu)$, we define the distance between between two functions f and g in $\mathcal{L}^2(\mu)$ as

$$d(f, g) = \|f - g\| \tag{3.4}$$

We define the $\mathcal{L}^p$ space and the $\mathcal{L}^p$ norm similarly (merely switching the "2" above with "p").

Definition 3.5. Let X be a measure space. The measurable function f is said to be in $\mathcal{L}^p$ if it is p-integrable; that is, if

$$\int_X |f|^p \, d\mu < +\infty \tag{3.6}$$

The $\mathcal{L}^p$ norm of f is defined by

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p} \tag{3.7}$$

We also identify functions which differ only on a set of measure zero. This allows $\mathcal{L}^2(\mu)$ to satisfy the properties of a metric space, namely

- $\|f - f\| = 0$
- $\|f - g\| > 0$ if $f \neq g$
- $\|f - g\| = \|g - f\|$
- $\|f - g\| \leq \|f - h\| + \|g - h\|$ for any $h \in \mathcal{L}^2(\mu)$

Theorem 3.8. *(Schwarz inequality) Suppose $f \in \mathcal{L}^2(\mu)$ and $g \in \mathcal{L}^2(\mu)$. Then $fg \in \mathcal{L}^2(\mu)$, and*

$$\int_X |fg| \, \mathrm{d}\mu \leq \|f\| \|g\| \tag{3.9}$$

Proof. We have for all real λ,

$$0 \leq \int_X (|f| + \lambda |g|)^2 \, \mathrm{d}\mu = \|f\|^2 + 2\lambda \int_X |fg| \, \mathrm{d}\mu + \lambda^2 \|g\|^2 \tag{3.10}$$

For $\|g\| \neq 0$, let $\lambda = -\dfrac{\|f\|}{\|g\|}$ to obtain the desired inequality. $\square$

Theorem 3.11. *(Lebesgue's monotone convergence theorem) Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \quad (x \in E) \tag{3.12}$$

Let f be such that

$$f_n(x) \to f(x) \quad (x \in E) \tag{3.13}$$

as $n \to \infty$. Then

$$\int_E f_n \, d\mu \to \int_E f \, d\mu \tag{3.14}$$

Proof. We have

$$\int_E f_n \, d\mu \to \alpha \tag{3.15}$$

as $n \to \infty$. Since $\int f_n \leq \int f$, we have

$$\alpha \leq \int_E f \, d\mu \tag{3.16}$$

Choose c such that $0 < c < 1$ and s a simple measurable function such that $0 \leq s \leq f$. Let

$$E_n = \{x | f_n(x) \geq cs(x)\} \ (n = 1, 2, 3, ...) \tag{3.17}$$

Then $E_1 \subset E_2 \subset E_3 \subset ....$ By (),

$$E = \bigcup_{n=1}^{\infty} E_n \tag{3.18}$$

For every n,

$$\int_E f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq c \int_{E_n} s \, d\mu \tag{3.19}$$

Letting $n \to \infty$, we obtain

$$\alpha \geq c \int_E s \, d\mu \tag{3.20}$$

Letting $c \to 1$, we see that

$$\alpha \geq \int_E s \, d\mu \tag{3.21}$$

which by (3.13) implies

$$\alpha \geq c \int_E f \, d\mu \tag{3.22}$$

The theorem then follows from (3.15), (3.16), and (3.22). □

Theorem 3.23. *Suppose $E \in \mathfrak{M}$. If $\{f_n\}$ is a sequence of nonnegative measurable functions and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \ \ \forall x \in E \tag{3.24}$$

then

$$\int_E f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} f_n \, \mathrm{d}\mu \tag{3.25}$$

Proof. Apply Lebesgue's monotone convergence theorem to the partial sums of (3.15). □

Theorem 3.26. *(Fatou's theorem) Suppose $E \in \mathfrak{M}$. If f_n is a sequence of non-negative measurable functions and*

$$f(x) = \liminf_{n\to\infty} f_n(x) \;\; (x \in E) \tag{3.27}$$

then

$$\int_E f d\mu \leq \liminf_{n\to\infty} \int_E f_n d\mu \tag{3.28}$$

Proof. Let g_n be such that

$$g_n(x) = inf f_i(x) \;\; (i \geq n, n = 1, 2, 3, ...) \tag{3.29}$$

Then g_n is measurable on E and

$$0 \leq g_1(x) \leq g_2(x) \leq ... \tag{3.30}$$

$$g_n(x) \leq f_n(x) \tag{3.31}$$

$$g_n(x) \to f(x) \;\; (n \to \infty) \tag{3.32}$$

By Lebesgue's monotone convergence theorem, it follows that

$$\int_E g_n \, d\mu \to \int_E f \, d\mu \tag{3.33}$$

which, with (3.31), implies the desired inequality. □

Theorem 3.34. *If $f, g \in \mathcal{L}^2(\mu)$, then $f + g \in \mathcal{L}^2(\mu)$, and*

$$\|f + g\| \leq \|f\| + \|g\| \tag{3.35}$$

Theorem 3.36. *(Lebesgue's dominated convergence theorem) Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that*

$$f_n(x) \to f(x) \; (x \in E) \tag{3.37}$$

as $n \to \infty$. If there exists a function g in $\mathcal{L}\mu$ on E such that

$$|f_n(x)| \leq g(x) \; (x \in E, n = 1, 2, 3, ...) \tag{3.38}$$

then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu. \tag{3.39}$$

Proof. We have that $f_n \in \mathcal{L}(\mu)$ and $f \in \mathcal{L}(\mu)$ on E. Since $f_n + g \geq 0$, Fatou's theorem implies that

$$\int_E (f + g) \, d\mu \leq \liminf_{n\to\infty} \int_E (f_n + g) d\mu \tag{3.40}$$

or

$$\int_E f \, d\mu \leq \liminf_{n\to\infty} \int_E f_n d\mu \tag{3.41}$$

Since $g - f_n \geq 0$, we also have that

$$\int_E (g - f)\, d\mu \leq \liminf_{n\to\infty} \int_E (g - f_n) d\mu \tag{3.42}$$

Therefore

$$-\int_E f\, d\mu \leq \liminf_{n\to\infty} \left[-\int_E f_n\, d\mu \right] \tag{3.43}$$

which is the same as saying

$$\int_E f\, d\mu \geq \limsup_{n\to\infty} \int_E f_n\, d\mu \tag{3.44}$$

The conclusion of the theorem then follows from (3.40) and (3.44). □

Theorem 3.45. *(The set of continuous functions is dense in $\mathcal{L}^2$ on $[a, b]$.) For any $f \in \mathcal{L}^2$ on $[a, b]$ and any $\epsilon > 0$, there exists continuous function g on $[a, b]$ such that*

$$\|f - g\| = \left\{ \int_a^b (f - g)^2 \, \mathrm{d}x \right\}^{1/2} < \epsilon \tag{3.46}$$

Proof. Let A be a closed subset of $[a, b]$ and K_A be its characteristic function. Let

$$t(x) = inf|x - y| \ (y \in A) and \tag{3.47}$$

$$g_n(x) = 1/1 + nt(x) \ (n = 1, 2, 3, ...) \tag{3.48}$$

Then g_n is continuous on $[a, b]$, $g_n(x) = 1$, and $g_n(x) \to 0$ on B, defined by $B = [a, b] - A$. By Lebesgue's dominated convergence theorem, it follows that

$$\|g_n - K_A\| = \left(\int_X g_n(x)^2 \, \mathrm{d}x \right)^{1/2} \to 0 \tag{3.49}$$

Therefore characteristic functions of closed sets can be approximated by continuous functions in $\mathcal{L}^2$, which implies the same for simple measurable functions.

If f is nonnegative and $f \in \mathcal{L}^2$, by Theorem 2.5 we may let $\{s_n(x)\}$ be a monotonically increasing sequence of simple nonnegative measurable functions such that $s_n(x) \to f(x)$. Since $|f - s_n|^2 \leq f^2$, Lebesgue's dominated convergence theorem shows that $\|f - s_n\| \to 0$. The general case follows from (2.10). □

Definition 3.50. Let f, $f_n \in \mathcal{L}^2(\mu)(n = 1, 2, 3, \ldots)$. We say that $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$ if $\|f_n - f\| \to 0$. We say that $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$ if for every $\epsilon > 0$ there exists an integer N such that $n \geq N, m \geq N$ implies $\|f_n - f_m\| < \epsilon$

We will now give a proof that every Cauchy sequence in $\mathcal{L}^2(\mu)$ converges - that is, that $\mathcal{L}^2(\mu)$ is complete.

Theorem 3.51. *($\mathcal{L}^2(\mu)$ is complete.) If $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$, then there exists a function $f \in \mathcal{L}^2(\mu)$ such that $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$.*

Proof. By definition, there exists a sequence $\{n_k\}(k \in \mathbb{N})$ such that

$$\|f_(n_k) - f_(n_{k+1})\| < \frac{1}{2^k} \ \ (k \in \mathbb{N}) \tag{3.52}$$

Choose a function $g \in \mathcal{L}^2(\mu)$. By the Schwarz inequality,

$$\int_X |g(f_(n_k) - f_(n_{k+1})|\, d\mu \leq \|f_(n_k) - f_(n_{k+1}\| \|g\| \leq \frac{g}{2^k} \tag{3.53}$$

Adding the inequalities

$$\int_X |g(f_(n_k) - f_(n_{k+1})|\, d\mu \leq \frac{g}{2^k} \tag{3.54}$$

for each k, we obtain

$$\sum_{k=1}^{\infty} \int_X |g(f_(n_k) - f_(n_{k+1})|\, d\mu \leq \|g\| \tag{3.55}$$

By Theorem 3.15 we may interchange the summation and integration above. It follows that

$$|g(x)| \sum_{k=1}^{\infty} |f_(n_k)(x) - f_(n_{k+1}(x)| < +\infty \tag{3.56}$$

almost everywhere on X (since its integral is also finite). Then

$$\sum_{k=1}^{\infty} |f_(n_k)(x) - f_(n_{k+1}(x)| < +\infty \tag{3.57}$$

almost everywhere on X. If the series above diverged on a set E of positive measure, then we could take $g(x)$ to be nonzero on a subset of E of positive measure, contradicting (3.56).

Since the kth partial sum of the telescoping series

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < +\infty \tag{3.58}$$

which converges almost everywhere on X, is

$$f_{n_{k+1}}(x) - f_{n_1}(x) \tag{3.59}$$

we see that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \tag{3.60}$$

defines $f(x)$ for almost all $x \in X$.

Now let $\epsilon > 0$ be given and choose an integer N such that $n \geq N, m \geq N$ implies $\|f_n - f_m\| \leq \epsilon$. By Fatou's theorem, if $n_k > N$ then

$$\|f - f_{n_k}\| \leq \liminf_{k\to\infty} f_{n_i} - f_{n_k}(x)\| \leq \epsilon \tag{3.61}$$

Thus $f - f_{n_k} \in \mathcal{L}^{\in}(\mu)$. Since $f = (f - f_{n_k}) + f_{n_k}$, we see that $f \in \mathcal{L}^{\in}(\mu)$ also. Since ϵ was arbitrary,

$$\lim_{k\to\infty} \|f - f_{n_k}\| = 0. \tag{3.62}$$

Finally, the inequality

$$\|f - f_n\| \leq \|f - f_{n_k}\| + \|f_{n_k} - f_n\| \tag{3.63}$$

shows that f_n converges to f in $\mathcal{L}^{\in}(\mu)$, since by taking n and n_k large enough each of the terms on the right can be made arbitrarily small.

□

In addition, by defining the **inner product** for $\mathcal{L}^2$ of two functions f and g on a measure space X with

$$\langle f, g\rangle = \int_X f\overline{g}d\mu \tag{3.64}$$

$\mathcal{L}^2(\mu)$ becomes a **Hilbert space**. A Hilbert space is a complete vector space with an inner product, and is also a complete metric space. In particular, $\mathcal{L}^2$ is an infinite-dimensional vector space. The space $\mathcal{L}^2$ is unique among $\mathcal{L}^p$ spaces as a Hilbert space.

Hilbert spaces have many useful properties. In particular, its similarity to Euclidean space makes possible the use of geometric notions such as distance and orthogonality. The Pythagorean identity also holds true in $\mathcal{L}^2$. In addition, Hilbert spaces, and in particular the $\mathcal{L}^2$ Hilbert space, are crucial to and arise naturally in areas as diverse as quantum mechanics, Fourier series, and stochastic calculus.

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References

[1] Rudin, Walter. Principles of Mathematical Analysis. McHraw-Hilll, Inc. 1974.
[2] Taylor, Michael E. Measure Theory and Integration. American Mathematical Society. 2006.
[3] Tung, Diana. Lebesgue Integration and Measure:Its Properties and Implications. `http://www.cims.nyu.edu/vigrenew/ug_research/DianaTung05.pdf`
[4] Bieren, Herman J. Introduction to Hilbert Spaces. `http://econ.la.psu.edu/~hbierens/HILBERT.PDF`