AN ANALYTIC APPROACH TO THE THEOREMS OF RIEMANN-ROCH AND ABEL

MICHAEL WONG

Abstract. Developing foundational notions in the theory of Riemann surfaces, we prove the Riemann-Roch Theorem and Abel's Theorem. These notions include sheaf cohomology, with particular focus on the zeroth and first cohomology groups, exact cohomology sequences induced by short exact sequences of sheaves, divisors, the Jacobian variety, and the Abel-Jacobi map. The general method of proof involves basic algebra and complex analysis.

Contents

1. Introduction

The theorems of Riemann-Roch and Abel are two fundamental results in algebraic geometry. As part of the "amazing synthesis" which David Mumford describes in the appendix to *The Red Book of Varieties and Schemes*, these theorems can be proved analytically when we restrict to algebraic curves over $\mathbb{C}$. In the following exposition, we pursue the analytic viewpoint from the theory of Riemann surfaces, incorporating sheaf cohomology, Jacobian varieties, and the Abel-Jacobi map. Our hope is that the method of proof, while drawing only from basic algebra and complex analysis, gives a small hint of some "synthesis."

Definition 1.1. A Riemann surface is a pair (X, Σ) where X is a connected manifold of real dimension 2 and Σ is a complex structure.

We assume basic knowledge of Riemann surfaces, including complex structure, holomorphic and meromorphic functions, differential forms, and integration of differential forms. Also, elementary facts about meromorphic functions on compact

Date: August 26, 2011.

Riemann surfaces will be used freely, e.g., the number of poles minus the number of zeros of a meromorphic function, counted with multiplicity, is 0.

Notation 1.2. For each open $U \subset X$, let

(1) $\mathcal{E}(U)$, $\mathcal{O}(U)$, $\mathcal{M}(U)$ be the sets of smooth functions (in the $\mathbb{R}^2$ sense), holomorphic functions, and meromorphic functions on U, respectively,
(2) $\mathcal{E}^1(U)$, $\mathcal{E}^2(U)$ $\mathcal{E}^{(1,0)}(U)$, and $\mathcal{E}^{(0,1)}(U)$ be the sets of smooth 1-forms, smooth 2-forms, smooth 1-forms of type $(1,0)$, and smooth 1-forms of type $(0,1)$ on U, respectively,
(3) and $\Omega(U)$ be the set of holomorphic 1-forms on U.

Each set is an abelian group under pointwise addition of functions.

Additionally, we assume basic knowledge about holomorphic maps between Riemann surfaces. We state briefly the main results we shall need (See [4]).

Theorem 1.3 (Local Normal Form)**.** *Let $f : X \to Y$ be a nonconstant holomorphic map between two Riemann surfaces. For each $x \in X$, let (V, ψ) be a chart centered at $f(x)$. Then there exists a chart (U, ϕ) centered at x and integer $m \geq 1$ such that*

$$F(z) := \psi \circ f \circ \phi^{-1}(z) = z^m.$$

The integer m is independent of charts and is called the multiplicity *of f at x, denoted $mult_x f$.*

A point $x \in X$ such that $mult_x f > 1$ is called a *ramification point* of f, and a point $y \in Y$ that is the image of a ramification point under f is called a *branch point* of f.

A nonconstant holomorphic map $f : X \to Y$ between Riemann surfaces is an open map. If in addition X is compact, then f is surjective and finite-to-one. Furthermore, f has only finitely many branch points, $y_1, y_2, \ldots, y_k$. It follows from Theorem 1.3 that f is an n-sheeted covering map over $Y - \{y_1, \ldots, y_k\}$ for some $n \geq 1$.

Finally, given an open covering of a compact Riemman surface, we shall assume the existence of a partition of unity subordinate to the open cover.

2. Sheaves and Sheaf Cohomology

In this section, we introduce the notions of sheaves and sheaf cohomology. With an eye toward the Riemann-Roch Theorem, we focus on the construction of only the zeroth and first cohomology groups, but the construction can be easily generalized to the higher-order groups.

Definition 2.1. Let X be a topological space. Let

$$\mathcal{F} = \{\mathcal{F}(U) \,|\, U \subset X \text{ open}\}$$

be a collection of abelian groups and

$$\rho = \{\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V) \,|\, V \subset U \subset X \text{ open}\}$$

be a collection of group homomorphisms with the conditions

$$\rho^U_U = id_{\mathcal{F}(U)}$$

and

$$\rho^V_W \circ \rho^U_V = \rho^U_W$$

for all open $W \subset V \subset U \subset X$. For $f \in U$, we shall denote $\rho_V^U(f)$ as $f|_V$. In the above notation, a *sheaf* on X is a pair $(\mathcal{F}, \rho)$ satisfying the following *sheaf axioms.* For any open $U \subset X$ and any open cover $\{U_i\}_{i\in I}$ of U,

(1) if $f, g \in \mathcal{F}(U)$ and $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$;
(2) if $f_i \in \mathcal{F}(U_i)$ and $f_i = f_j$ in $\mathcal{F}(U_i \cap U_j)$ for each $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

For each function listed in 1.2, the collection of abelian groups ranging over all open $U \subset X$ with the natural restriction maps makes a sheaf on X. The elements of ρ are called *restriction homomorphisms* in light of such examples.

Given a sheaf $\mathcal{F}$ on a topological space X, we would like to examine the sheaf structure locally, especially when the group elements are functions and the homomorphisms are the natural restriction maps. For $x \in X$, the set $\mathcal{I} := \{U \subset X \text{ open} \,|\, x \in U\}$ is a directed set under inclusion, and the sets $\{\mathcal{F}(U) \,|\, U \in \mathcal{I}\}$ and $\{\rho_V^U \,|\, V \subset U \text{ and } U, V \in \mathcal{I}\}$ together form a directed system. Then the *stalk* of $\mathcal{F}$ at x is the direct limit

$$\mathcal{F}_x := \varinjlim \mathcal{F}(U).$$

Definition 2.2. Let $\mathcal{F}$ be a sheaf on a topological space X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. A *q-cochain* is a tuple $(f_{i_0,i_1,\ldots,i_q})_{(i_0,i_1,\ldots,i_q)\in I^q}$ where

$$f_{i_0,i_1,\ldots,i_q} \in \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}).$$

The set of all such q-cochains under addition is an abelian group called the q^{th}-*cochain group with respect to* $\mathcal{U}$, denoted by $C^q(\mathcal{U}, \mathcal{F})$.

The cochain groups lend themselves to a chain complex, constructed as follows. Fix a covering $\mathcal{U}$ of X. Define $\delta^0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ by

$$\delta^0(f_i)_{i\in I} = (f_{ij})_{i,j\in I}$$

where $f_{ij} = f_i - f_j$ for all $i, j \in I$; here, the operation is understood to be on the restrictions of f_i and f_j to $U_i \cap U_j$. In like manner, define $\delta^1 : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ by

$$\delta^1(f_{ij})_{i,j\in I} = (f_{ijk})_{i,j,k\in I}$$

where $f_{ijk} = f_{jk} - f_{ik} + f_{ij}$ for all $i, j, k \in I$. Again, the operations are understood to be on the restrictions to $U_i \cap U_j \cap U_k$. Clearly, δ^0 and δ^1 are homomorphisms, and $\delta^1 \circ \delta^0 = 0$. Hence, the sequence

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^1} C^2(\mathcal{U}, \mathcal{F})$$

is a chain complex, and we define the *first cohomology group* with respect to $\mathcal{U}$ to be

$$H^1(\mathcal{U}, \mathcal{F}) := Ker\delta^1/Im\delta^0.$$

Similarly, we define the *zeroth cohomology group* with respect to $\mathcal{U}$ to be

$$H^0(\mathcal{U}, \mathcal{F}) = Ker\delta^0/Im\big(0 \to C^0(\mathcal{U}, \mathcal{F})\big) = Ker\delta^0.$$

For $q = 0, 1$, elements of $Z^q(\mathcal{U}, \mathcal{F}) := Ker\delta^q$ are called *q-cocycles.* Observe that 0-cocycles satisfy

$$f_i = f_j, \text{ for all } i, j \in I \tag{2.3}$$

and 1-cocycles satisfy

$$f_{jk} = f_{ik} + f_{ij} \text{ for all } i, j, k \in I. \tag{2.4}$$

Elements of $Im\delta^q$ are called *q-coboundaries*. If $(f_{ij})_{i,j\in I} \in Z^1(\mathcal{U},\mathcal{F})$ is also a 0-coboundary, then there exists $(g_i) \in C^0(\mathcal{U},\mathcal{F})$, such that

$$f_{ij} = g_i - g_j \text{ for all } i,j \in I$$

and (f_{ij}) is said to *split*.

As defined, the zeroth cohomology group is practically independent of the cover. The relation (2.3) and sheaf axiom I imply that for every $(f_i) \in Ker\delta^0$, there exists $f \in \mathcal{F}(X)$ such that $f|_{U_i} = f_i$ for all $i \in I$. So $H^0(\mathcal{U},\mathcal{F}) \cong \mathcal{F}(X)$. We therefore define the *zeroth cohomology group of X with coefficients in $\mathcal{F}$* to be simply

$$H^0(X,\mathcal{F}) := \mathcal{F}(X).$$

However, the first cohomology group generally depends on the cover. To eliminate this dependence, we make the collection of open coverings of X into a directed set and take a direct limit. Let $\mathcal{U} = \{U_i\}$ be an open cover of X. An open cover $\mathcal{V} = \{V_k\}_{k\in K}$ of X is a *refinement* of $\mathcal{U}$, denoted $\mathcal{V} \leq \mathcal{U}$, if for every $k \in K$, there exists an $i \in I$ such that $V_k \subset U_i$. Given a refinement $\mathcal{V}$, let $t : K \to I$ be a map sending k to an i such that $V_k \subset U_i$. Then define $\tau^{\mathcal{U}}_{\mathcal{V}} : C^1(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{V},\mathcal{F})$ by $\tau^{\mathcal{U}}_{\mathcal{V}}(f_{ij}) = (g_{kl})$ where

$$g_{kl} = f_{t(k),t(l)}|_{V_k\cap V_l} \text{ for all } k,l \in K.$$

As is easily verified, $\tau^{\mathcal{U}}_{\mathcal{V}}$ preserves 1-cocycles and 0-coboundaries. Hence, $\tau^{\mathcal{U}}_{\mathcal{V}}$ induces a homomorphism

$$\tau^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F}).$$

One can demonstrate, in fact, that this map on cohomology with respect to open covers is independent of the function t, is injective, and satisfies two conditions:

$$\tau^{\mathcal{U}}_{\mathcal{U}} = id_{H^1(\mathcal{U},\mathcal{F})}$$

and

$$\tau^{\mathcal{V}}_{\mathcal{W}} \circ \tau^{\mathcal{U}}_{\mathcal{V}} = \tau^{\mathcal{U}}_{\mathcal{W}}, \text{ for all } \mathcal{W} \leq \mathcal{V} \leq \mathcal{U}.$$

Then we define the *first cohomology group of X with coefficients in the sheaf $\mathcal{F}$* to be

$$H^1(X,\mathcal{F}) := \varinjlim H^1(\mathcal{U},\mathcal{F}).$$

Proposition 2.5. *Let X be a compact Riemann surface. Then $H^1(X,\mathcal{E}) = 0$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. There exists a partition of unity $\{\psi_i\}_{i\in I}$ subordinate to $\mathcal{U}$. Take $(f_{ij})_{i,j\in I}$ to be a representative of an element of $H^1(\mathcal{U},\mathcal{E})$. Then for each $j \in I$, $\psi_j f_{ij}$ can be extended smoothly to U_i by giving it the value 0 outside its support. Let

$$g_i := \sum_{j\in I} \psi_j f_{ij}.$$

For each $x \in X$, $g_i(x)$ has only a finite number of summands, so g_i is indeed an element of $\mathcal{E}(U_i)$. Observe that, on $U_i \cap U_j$,

$$g_i - g_j = \sum_{k\in I} \psi_k f_{ik} - \sum_{k\in I} \psi_k f_{jk} = \sum_{k\in I} \psi_k (f_{ik} - f_{jk}) = \sum_{k\in I} \psi_k f_{ij} = f_{ij}.$$

So $(f_{ij}) \in Im\delta^0$, implying $H^1(\mathcal{U},\mathcal{E}) = 0$. $\square$

As the next theorem shows, cohomology is the same as cohomology relative to a special open cover, called a *Leray cover*.

Theorem 2.6 (Leray)**.** *Let $\mathcal{F}$ be a sheaf on the Riemann surface X. Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X such that*

$$H^1(U_i, \mathcal{F}) = 0 \text{ for all } i \in I.$$

Then $H^1(X, \mathcal{F}) \cong H^1(\mathcal{U}, \mathcal{F})$.

Proof. We show that for any refinement $\mathcal{V} = \{V_k\}_{k\in K}$ of $\mathcal{U}$, the homomorphism

$$\tau^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F})$$

is an isomorphism. This map is injective as noted above, so it remains only to show surjectivity.

Let $(f_{kl})_{k,l\in K} \in Z^1(\mathcal{V}, \mathcal{F})$. For each $i \in I$, $\mathcal{W} := \{U_i \cap V_k\}_{k\in K}$ is an open cover of U_i. Since $H^1(U_i, \mathcal{F}) = 0$ and the canonical homomorphism

$$H^1(\mathcal{W}, \mathcal{F}) \to H^1(U_i, \mathcal{F})$$

is injective, we have $H^1(\mathcal{W}, \mathcal{F}) = 0$. So there exists $(g_{ik})_{k\in K} \in C^0(\mathcal{W}, \mathcal{F})$ such that

$$f_{kl} = g_{ik} - g_{il} \text{ on } U_i \cap V_k \cap V_l. \tag{2.7}$$

Then for $j \in I$,

$$g_{ik} - g_{jk} = g_{il} - g_{jl} \text{ on } U_i \cap U_j \cap V_k \cap V_l.$$

So by sheaf axiom II, for each $k \in K$, there exists $h_{ij} \in \mathcal{F}(U_i \cap U_j)$ such that

$$h_{ij} = g_{ik} - g_{jk} \text{ on } U_i \cap U_j \cap V_k. \tag{2.8}$$

We see that (h_{ij}) satisfies Equation (2.4) and hence is in $Z^1(\mathcal{U}, \mathcal{F})$.

Now, for the refinement $\mathcal{V}$ of $\mathcal{U}$, fix a map $t : K \to I$ as above, and let $m_k = g_{tk,k} \in \mathcal{F}(V_k)$. Then by Equations (2.7) and (2.8) , on $V_k \cap V_l$,

$$h_{tk,tl} - f_{kl} = g_{tk,k} - g_{tl,k} - (g_{tl,k} - g_{tl,l}) = g_{tk,k} - g_{tl,l} = m_k - m_l.$$

Thus, $(h_{tk,tl})$ and (f_{kl}) are cohomologous, implying $\tau^{\mathcal{U}}_{\mathcal{V}}$ is surjective. □

3. The Exact Sequence of Sheaves

In this section, we explain how maps between sheaves on a topological space X induce maps between cohomologies with coefficients in these sheaves. This concept will be key in proving the Riemann-Roch Theorem.

Let $(\mathcal{F}, \rho)$ and $(\mathcal{G}, \varrho)$ be two sheaves on a topological space X. A *sheaf homomorphism* $\alpha : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms

$$\{\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U) \,|\, U \subset X \text{ open}\}$$

such that for all open $V \subset U \subset X$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\rho^U_V"'] & \mathcal{G}(U) \arrow[d, "\varrho^U_V"] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes. One can easily check that a sheaf homomorphism induces a homomorphism on stalks, $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$. Then a sequence of sheaves

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is said to be *exact* if, for all $x \in X$, the induced sequence

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$$

is exact. In particular, a sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is said to be short exact if the induced sequence on stalks is short exact.

The following proposition follows quickly from the definitions.

Proposition 3.1. *Suppose* $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ *is an exact sequence of sheaves. Then for every open subset* $U \subset X$*, the sequence*

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U) \xrightarrow{\beta} \mathcal{H}(U)$$

is exact.

As expected, a sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ induces homomorphisms on cohomology,

$$\alpha^q : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G}),\ q = 0, 1.$$

The map α^0 is just $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$, while the map α^1 is obtained as follows. The homomorphism α^* defined by $\alpha^*(f_{ij}) = (\alpha_{U_i \cap U_j} f_{ij})$ preserves cocycles and coboundaries. Thus, it induces a homomorphism

$$\alpha^* : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{U}, \mathcal{G}).$$

Since α^* commutes with τ_V^U as defined above, it induces a homomorphism

$$\alpha^1 : H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}).$$

To see how a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \tag{3.2}$$

induces an exact cohomology sequence, we must define a map

$$\delta^* : H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F})$$

called the *connecting homomorphism.* Let $h \in H^0(X, \mathcal{H}) = \mathcal{H}(X)$. Since $\beta_x : \mathcal{G}_x \to \mathcal{H}_x$ is surjective for each $x \in X$, there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ and a 0-cochain $(g_i) \in C^0(\mathcal{U}, \mathcal{G})$ such that

$$\beta(g_i) = h|_{U_i} \text{ for every } i \in I.$$

Thus, $\beta(g_i - g_j) = 0$ on $U_i \cap U_j$ for all $i, j \in I$. By Proposition 3.1, there exists $(f_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$ such that

$$\alpha(f_{ij}) = g_i - g_j \text{ for all } i, j \in I.$$

So $\alpha(f_{jk} - f_{ik} + f_{ij}) = 0$. Again by Proposition 3.1, this implies $f_{jk} - f_{ik} + f_{ij} = 0$, so $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$. Let $\delta^*(h)$ be the cohomology class of (f_{ij}) in $H^1(X, \mathcal{F})$. One can verify that $\delta^*(h)$ is independent of the choices made, so δ^* is well-defined.

Theorem 3.3. *The short exact sequence of sheaves in* (3.2) *induces an exact cohomology sequence,*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X,\mathcal{F}) & \xrightarrow{\alpha^0} & H^0(X,\mathcal{G}) & \xrightarrow{\beta^0} & H^0(X,\mathcal{H}) \\ & & & & & & \downarrow{\scriptstyle \delta^*} \\ & & H^1(X,\mathcal{H}) & \xleftarrow[\beta^1]{} & H^1(X,\mathcal{G}) & \xleftarrow[\alpha^1]{} & H^1(X,\mathcal{F}) \end{array}$$

The proof is tedious but elementary. See [1] for detials.

4. The Genus of a Compact Riemann Surface

Appearing in the Riemann-Roch formula, the genus g of a Riemann surface X is defined to be the dimension of $H^1(X,\mathcal{O})$ as a $\mathbb{C}$-vector space. The task of this section is to show that g is finite if X is compact. We first solve the inhomogeneous Cauchy-Riemann equation, from which we determine a Leray cover for the sheaf $\mathcal{O}$.

Lemma 4.1. *Suppose $g \in \mathcal{E}(\mathbb{C})$ has compact support. Then there exists $f \in \mathcal{E}(\mathbb{C})$ such that*

$$\frac{\partial f}{\partial \bar{z}} = g$$

Proof. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(\zeta + z)}{z} dz \wedge d\bar{z}.$$

Letting $z = re^{i\theta}$, we have

$$f(\zeta) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{g(\zeta + re^{i\theta})}{e^{i\theta}} dr\, d\theta.$$

Since g has compact support, this integral is finite. Furthermore, we have all the conditions for differentiating under the integral,

$$\frac{\partial f}{\partial \bar{\zeta}} = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial g(\zeta + re^{i\theta})}{\partial \bar{\zeta}} e^{-i\theta} dr\, d\theta.$$

Now, let $B_\epsilon = \{z \in \mathbb{C} \,|\, \epsilon \leq |z| \leq R\}$ where R is sufficiently large so that B_ϵ properly contains the support of g. In the original coordinates,

$$\begin{aligned} \frac{\partial f}{\partial \bar{\zeta}} &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \iint_{B_\epsilon} \frac{\partial g(\zeta + z)}{\partial \bar{\zeta}} \frac{1}{z} dz \wedge d\bar{z} \\ &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \iint_{B_\epsilon} \frac{\partial}{\partial \bar{z}} \Big(\frac{g(\zeta + z)}{z} \Big) dz \wedge d\bar{z} \\ &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \frac{g(\zeta + z)}{z} dz \end{aligned}$$

where we have used Stoke's Theorem to obtain the last integral. This integral is just the average of g over the circle of radius ϵ centered at ζ, so the limit is $g(\zeta)$. □

A standard compact exhaustion argument generalizes the result in Lemma 4.1 to g without compact support.

Theorem 4.2. *Let $0 < R \leq \infty$, and let $X = \{z \in \mathbb{C} \,|\, |z| < R\}$. Suppose $g \in \mathcal{E}(X)$. Then there exists $f \in \mathcal{E}(X)$ such that*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

Corollary 4.3. *With X as in the theorem, $H^1(X, \mathcal{O}) = 0$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Take $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F}) \subset Z^1(\mathcal{U}, \mathcal{E})$. By Proposition 2.5, $H^1(\mathcal{U}, \mathcal{E}) = 0$, so there exists $(g_i) \in C^0(\mathcal{U}, \mathcal{E})$ such that

$$f_{ij} = g_i - g_j \text{ on } U_i \cap U_j.$$

Then $\bar{\partial} g_i = \bar{\partial} g_j$ for all $i, j \in I$, so by sheaf axiom II, there exists $h \in \mathcal{E}(X)$ such that $h|_{U_i} = \bar{\partial} g_i$. By the theorem, there exists $g \in \mathcal{E}(X)$ such that $\bar{\partial} g = h$. Now, observe that $\bar{\partial}(g_i - g|_{U_i}) = 0$, so $g_i - g|_{U_i}$ is holomorphic for all i and $(g_i - g|_{U_i}) \in C^0(\mathcal{U}, \mathcal{O})$. Moreover,

$$g_i - g - (g_j - g) = g_i - g_j = f_{ij} \text{ on } U_i \cap U_j.$$

So $(f_{ij}) \in Im\big(C^0(\mathcal{U}, \mathcal{O}) \xrightarrow{\delta^1} C^1(\mathcal{U}, \mathcal{O}) \big)$, implying $H^1(\mathcal{U}, \mathcal{O}) = 0$. □

Corollary 4.4. $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Proof. Let $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$. Recall that (U_1, ϕ_1) and (U_2, ϕ_2) where $\phi_1 = id_{\mathbb{C}}$ and $\phi_2(z) = 1/z$ form a complex structure on $\mathbb{P}^1$. As is easy to check, $H^1(U_i, \mathcal{O}) \cong H^1(\phi_i(U_i), \mathcal{O})$ and $\phi_i(U_i) = \mathbb{C}$. Therefore, by Corollary 4.3, $\mathcal{U} := \{U_1, U_2\}$ is a Leray covering for $\mathcal{O}$.

We want to show that $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O})$ splits. Since f_{12} is holomorphic on $\mathbb{C}^*$, it has a Laurent expansion at 0

$$f_{12}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

which converges for all $z \in \mathbb{C}^*$. Then let

$$g_1(z) = \sum_{n=0}^{\infty} c_n z^n \text{ on } U_1, \quad g_2(z) = -\sum_{n=-\infty}^{-1} c_n z^n \text{ on } U_2.$$

Clearly, $g_i \in \mathcal{O}(U_i)$ for $i = 1, 2$ and $f_{12} = g_1 - g_2$. □

We need one more lemma, which we shall take for granted. Given a topological space X and a subset V, we say that a subset U is a *relatively compact* subset of V, denoted $U \Subset V$, if $\bar{U}$ is compact and $\bar{U} \subset V$.

Lemma 4.5. *Suppose X is a Riemann surface. Let $\mathcal{U}^* = \{U_i^*\}_{i \in I}$ be a finite collection of coordinate neighborhoods such that $z_i(U_i^*) \subset \mathbb{C}$ is the unit disk. Let $\mathcal{W} = \{W_i\}_{i \in I}$ and $\mathcal{U} = \{U_i\}_{i \in I}$ be collections of open sets such that $W_i \Subset U_i \Subset U_i^*$ for each $i \in I$. Then the natural restriction map*

$$H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{W}, \mathcal{O})$$

has finite dimensional image

Remark 4.6. Note that we do not assume the collections of open sets are covers of X. The lemma refers to the cohomology of

$$X_{\mathcal{U}} = \bigcup_{i \in I} U_i, \quad X_{\mathcal{W}} = \bigcup_{i \in I} W_i.$$

The shortest proof of this result of which we are aware involves sophisticated methods in functional analysis. For fear of digressing from the main point, we only refer the reader to [1].

Now, let X be a compact Riemann surface. Let $\mathcal{U}^* = \{U_i\}_{i\in I}$ be as in the lemma, but suppose further that $\mathcal{U}^*$ *covers* X. It is easy to show that there exist collections of open sets $\mathcal{W} = \{W_i\}_{i\in I}$ and $\mathcal{U} = \{U_i\}_{i\in I}$ as in the lemma but with the additional properties

(1) $\mathcal{W}$ and $\mathcal{U}$ *cover* X, and
(2) $z_i(U_i)$ is an open disk for all i.

Then the natural restriction mapping

$$\tau^{\mathcal{U}}_{\mathcal{W}} : H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{W}, \mathcal{O})$$

has finite dimensional image. As noted in Section 2, it is also injective. We have $H^1(U_i, \mathcal{O}) \cong H^1(z(U_i), \mathcal{O})$, so by Corollary 4.3, $\mathcal{U}$ is a Leray covering. Therefore, $H^1(\mathcal{U}, \mathcal{O}) \cong H^1(X, \mathcal{O})$. This proves that $g = dim H^1(X, \mathcal{O}) < \infty$.

Remark 4.7. It is a consequence of Serre duality that the genus g of a compact Riemann surface equals $dim\Omega(X)$. We will use this fact in the proof of Abel's Theorem, though we will not develop the theory. See [1] and [4].

5. The Riemann-Roch Theorem

The Riemann-Roch Theorem provides a way for computing the dimension of the space of meromorphic functions on a compact Riemann surface X with restrictions on poles and zeros. The restrictions are introduced via the notion of a divisor.

Definition 5.1. A *divisor* D on a Riemann surface X is a function $D : X \to \mathbb{Z}$ such that for any compact $K \subset X$, $D(x) \neq 0$ for only finitely many $x \in K$.

Remark 5.2. If X is compact, then D can alternatively be defined as a finite formal sum of points in X with coefficients in $\mathbb{Z}$,

$$D = \sum_{i=1}^{m} n_i\, p_i$$

where $m \in \mathbb{Z}^+$, $n_i \in \mathbb{Z}$, and $p_i \in X$ for all i.

We denote by $Div(X)$ the group of divisors on X under addition. Moreover, with X compact and D as in the remark, we define a group homomorphism $deg : Div(X) \to \mathbb{Z}$ by

$$deg D = \sum_{i=1}^{m} n_i,$$

called the *degree* map. The kernel of the degree map is denoted $Div_0(X)$.

For the rest of this section, X will be a compact Riemann surface. Each nonzero meromorphic function $f \in \mathcal{M}(X)$ determines a divisor. Recall that the order function of f is defined by

$$ord_x(f) := \begin{cases} 0 & \text{if } f(x) \neq 0 \\ k & \text{if } x \text{ is a zero of order } k \\ -k & \text{if } x \text{ is a pole of order } k. \end{cases}$$

Since $f \neq 0$, the zeros (and poles) of f are isolated. Then because X is compact, there are only finitely many of them. Hence, the order function of f is a divisor, written as (f). A divisor D which is the order function of a nonzero meromorphic function f is called a *principal divisor*, and f is said to be a *meromorphic solution* of D. The subgroup of all such divisors is denoted $Div_P(X)$.

Let $D \in Div(X)$. We let D be a pointwise lower bound for the order function and thereby place a restriction on the zeros and poles of a nonzero meromorphic function. For every open set $U \subset X$, let

$$\mathcal{O}_D(U) := \{f \in \mathcal{M}(U) - \{0\} : ord_x(f) \geq -D(x) \text{ for all } x \in U\}.$$

With the natural restriction maps, $\mathcal{O}_D$ is a sheaf on X. Note that $\mathcal{O}_0$ is just the sheaf of holomorphic functions.

Theorem 5.3 (Riemann-Roch)**.** *Suppose X is a compact Riemann surface of genus g, and let D be a divisor on X. Then the cohomology groups $H^q(X, \mathcal{O}_D)$ for $q = 0, 1$ are finite dimensional $\mathbb{C}$-vector spaces, and*

$$dim H^0(X, \mathcal{O}_D) - dim H^1(X, \mathcal{O}_D) = 1 - g + deg D.$$

Our task is made easier if we first construct auxiliary sheaves. The idea of the proof is to obtain the formula from an exact cohomology sequence induced from a short exact sequence of sheaves.

First, let $p \in X$. Let P be the divisor which is 1 at p and 0 elsewhere, and let $D' = D + P$. Then we have the inclusion morphism $\mathcal{O}_D \hookrightarrow \mathcal{O}_{D'}$, i.e., the collection of natural inclusion maps $\mathcal{O}_D(U) \hookrightarrow \mathcal{O}'_D(U)$ for all open $U \subset X$.

Next, for $p \in X$, let

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

With the obvious restriction maps, $\mathbb{C}_p$ is a sheaf on X, called the *skyscraper sheaf* at p. We see immediately that

$$H^0(X, \mathbb{C}_P) = \mathbb{C}_p(X) = \mathbb{C}.$$

To compute the first cohomology group, let $\mathcal{U}$ be any open covering of X. Then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that precisely one element of $\mathcal{V}$ contains p. Therefore,

$$H^1(\mathcal{U}, \mathbb{C}_p) = H^1(\mathcal{V}, \mathbb{C}_p) = 0$$

implying $H^1(X, \mathbb{C}_p) = 0$.

Define a sheaf morphism $\alpha : \mathcal{O}_{D'} \to \mathbb{C}_p$ as follows. Fix a chart (V, z) centered at p. If $p \notin U$, let α_U be the zero morphism. If $p \in U$, then each $f \in \mathcal{O}_{D'}(U)$ has a Laurent expansion around p

$$\sum_{n=-D(p)-1}^{\infty} c_n z^n.$$

So set $\alpha_U(f) = c_{-D(p)-1}$. As one can easily check, for each $x \in X$, the sequence of stalks at x

$$0 \longrightarrow \mathcal{O}_{D,x} \xrightarrow{i_x} \mathcal{O}_{D',x} \xrightarrow{\alpha_x} \mathbb{C}_{p,x} \longrightarrow 0$$

is short exact. By Theorem 3.2, the corresponding short exact sequence of sheaves induces an exact cohomology sequence,

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D'}) \to \mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0. \tag{5.4}$$

Proof of Theorem 5.3. Generally, $D = P_1 + P_2 + \cdots + P_m - P_{m+1} - P_{m+2} \cdots - P_{m+n}$, where the P_i's are not necessarily distinct. We proceed by induction on m and n. First, suppose $D = 0$. Then $\mathcal{O}_D = \mathcal{O}$ is just the sheaf of holomorphic functions. By definition, $dim H^1(X, \mathcal{O}) = g$. As regards the zeroth cohomology group, a function $f \in \mathcal{O}(X)$ is a holomorphic mapping into $\mathbb{P}^1$ which is not surjective. Hence, f is constant, implying $dim H^0(X, \mathcal{O}) = 1$.

Next, assume the result holds for either D or D'. Let $A = Im(H^0(X, \mathcal{O}_{D'}) \to \mathbb{C})$ and $B = \mathbb{C}/A$. Note that the exact sequence in Equation (5.4) can be divided into two short exact sequences,

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D'}) \to A \to 0$$

and

$$0 \to B \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0$$

from which we derive

$$dim H^0(X, \mathcal{O}_D) + dim A = dim H^0(X, \mathcal{O}_{D'}) \tag{5.5}$$

and

$$dim B + dim H^1(X, \mathcal{O}_{D'}) = dim H^1(X, \mathcal{O}_D). \tag{5.6}$$

Thus, the cohomology groups are finite dimensional. Note furthermore that

$$dim A + dim B = deg D' - deg D = 1. \tag{5.7}$$

Adding Equations (5.5) and (5.6) and substituting from Equation (5.7), we obtain

$$\begin{aligned} & dim H^0(X, \mathcal{O}_D) - dim H^1(X, \mathcal{O}_D) - deg D \\ = \quad & dim H^0(X, \mathcal{O}_{D'}) - dim H^1(X, \mathcal{O}_{D'}) - deg D'. \end{aligned}$$

Therefore, if the result holds for D, then the result holds for D' and vice versa. $\square$

Notice the variety of techniques we employed to prove Riemann-Roch. While the finiteness of the genus of a compact Riemann surface was essentially an analytic result, the Riemann-Roch formula was an algebraic deduction from an exact cohomology sequence.

6. A Word on $\mathcal{E}^{(0,1)}(X)$

We now shift gears to develop the requisite theory for Abel's Theorem. Given a Riemann surface X, let $d'' : \mathcal{E}(U) \to \mathcal{E}^{(0,1)}(U)$ be the homomorphism sending a smooth function f on U to the 1-form on U given in local coordinates by $\bar{\partial} f \, d\bar{z}$. The following lemma will be very useful in the proof of Abel.

Lemma 6.1. *Let X be a compact Riemann surface, and suppose $\sigma \in \mathcal{E}^{(0,1)}(X)$. Then there exists $f \in \mathcal{E}(X)$ such that $d'' f = \sigma$ if and only if*

$$\iint_X \sigma \wedge \omega = 0 \; for \; all \, \omega \in \Omega(X). \tag{6.2}$$

As we will demonstrate, this result is a simple consequence of the algebraic properties of $\mathcal{E}^{(0,1)}(X)$ and Serre duality.

Proposition 6.3. *Let X be a compact Riemann surface. Then $H^1(X, \mathcal{O}) \cong \mathcal{E}^{(0,1)}(X)/d''\mathcal{E}(X)$*

Proof. It is easy to verify that the sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{i} \mathcal{E} \xrightarrow{d''} \mathcal{E}^{(0,1)} \longrightarrow 0\ ,$$

where i is the collection of natural inclusion maps and d'' is the collection of maps defined above, is an exact sequence of sheaves. In the resulting exact cohomology sequence, $H^1(X, \mathcal{E}) = 0$ by Proposition 2.5. □

Recall that $\omega \in \mathcal{E}^1(X)$ can be decomposed uniquely as $\omega = \omega_1 + \omega_2$ where $\omega_1 \in \mathcal{E}^{(1,0)}(X)$ and $\omega_2 \in \mathcal{E}^{(0,1)}(X)$ are given in local coordinates by $f\,dz$ and $g\,d\bar{z}$, respectively. We define the *complex conjugate* of ω as the smooth 1-form $\bar{\omega}$ given in local coordinates by $\bar{\omega} = \bar{f}\,d\bar{z} + \bar{g}\,dz$. In particular, if $\omega \in \Omega(X)$, then $\bar{\omega}$ is called an *antiholomorphic 1-form* on X, and the space of all antiholomorphic 1-forms on X is labeled $\bar{\Omega}(X)$. We then define a map $* : \mathcal{E}^1(X) \to \mathcal{E}^1(X)$ by

$$*\omega = i(\bar{\omega}_1 - \bar{\omega_2})$$

Clearly, this map is a group isomorphism from $\mathcal{E}^{(1,0)}(X)$ to $\mathcal{E}^{(0,1)}(X)$ and, in particular, from $\Omega(X)$ to $\bar{\Omega}(X)$.

We can endow $\mathcal{E}^1(X)$, where X is compact, with a scalar product. For all smooth 1-forms ω_1, ω_2, define

$$\langle \omega_1, \omega_2 \rangle = \iint_X \omega_1 \wedge *\omega_2.$$

As is quickly verified, this map satisfies all the axioms for a scalar product. Thus, we have a notion of orthogonality in $\mathcal{E}^1(X)$.

Lemma 6.4. *Suppose X is a compact Riemann surface. Then $d''\mathcal{E}(X)$ and $\bar{\Omega}(X)$ are orthogonal subspaces of $\mathcal{E}^{(0,1)}(X)$.*

Proof. Given $\omega \in \Omega(X)$ and $f \in \mathcal{E}(X)$, observe that

$$\bar{\omega} \wedge *d''f = i\bar{\omega} \wedge d'\bar{f} = -id(\bar{f}\bar{\omega}).$$

So by Stokes' Theorem,

$$\langle \bar{\omega}, d''f \rangle = -i \iint_X d(\bar{f}\bar{w}) = 0.$$

□

Proposition 6.5. $\mathcal{E}^{(0,1)}(X) \cong d''\mathcal{E}(X) \oplus \bar{\Omega}(X)$.

Proof. By Proposition 6.3, $\mathcal{E}^{(0,1)}/d''\mathcal{E}(X)$ has dimension g as a $\mathbb{C}$-vector space. By Serre duality, $\bar{\Omega}(X)$ has dimension g as well. The result then follows from Lemma 6.4 □

The condition stated in Equation (6.2) is equivalent to the condition that σ is orthogonal to every element of $\bar{\Omega}(X)$, i.e. $\sigma \in d''\mathcal{E}(X)$. So the proposition implies Lemma 6.1.

7. The Jacobian Variety and the Abel-Jacobi Map

Our formulation of Abel's Theorem is that the principal divisors constitute the kernel of a particular homomorphism, called the Abel-Jacobi map. To define this map, we introduce the notion of homology with smooth maps and construct the Jacobian variety of a compact Riemann surface.

Let X be a Riemann surface, and let Δ_n be the standard n-simplex. A *smooth singular n-simplex* is a continuous map $\sigma : \Delta_n \to X$ that is smooth in local coordinates. Let $C_n^\infty(X)$ be the free abelian group generated by all smooth singular n-simplices. An element of this group is called a smooth 1-chain. Clearly, $C_n^\infty(X)$ is a subgroup of $C_n(X)$, the free abelian group generated by all n-simplices. Hence, with $\partial : C_n(X) \to C_{n-1}(X)$ the boundary homomorphism defined in the usual way,

$$\cdots \longrightarrow C_n^\infty(X) \xrightarrow{\partial} C_{n-1}^\infty(X) \xrightarrow{\partial} \cdots \longrightarrow C_1^\infty(X) \xrightarrow{\partial} C_0^\infty(X) \longrightarrow 0$$

is a chain complex. Then we define the n^{th} *smooth singular homology group* to be

$$H_n^\infty(X) := Ker\big(\, C_n^\infty(X) \xrightarrow{\partial} C_{n-1}^\infty(X)\,\big)/Im\big(\, C_{n+1}^\infty(X) \xrightarrow{\partial} C_n^\infty(X)\,\big).$$

The inclusion map $C_n^\infty(X) \hookrightarrow C_n(X)$ obviously commutes with ∂ and so induces a homomorphism $H_n^\infty(X) \to H_n(X)$. In fact, this map is an isomorphism, so we lose no homological information by considering only smooth singular simplices [2].

A smooth singular 1-simplex is just a smooth path, i.e., a continuous map $[0,1] \to X$ which is smooth in local coordinates, upon identifying Δ_1 with $[0,1]$. The integral of a smooth 1-form ω over a smooth path σ is defined, as usual, by

$$\int_\sigma \omega := \int_{\Delta_1} \sigma^*\omega.$$

Then we define the integral over a smooth 1-chain $c = \sum_{i\in I} n_i\sigma_i$ by

$$\int_c \omega = \sum_i n_i \int_{\sigma_i} \omega.$$

With these two definitions, it is straightforward to prove Stokes' Theorem for smooth 1-chains [2].

Theorem 7.1. *Let $c \in C_2^\infty(X)$ and $\omega \in \mathcal{E}^1(X)$. Then*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Let ω be a closed, smooth 1-form. Define a map

$$\int_{-} \omega : H_1^\infty(X) \to \mathbb{C}$$

by sending $[c] \in H_1^\infty(X)$ to the integral of ω over a representative of $[c]$. This map is well-defined by Theorem 7.1 and is clearly a homomorphism. In light of the above discussion, we alternatively may take the domain of this map to be $H_1(X)$.

Now, suppose X is a compact Riemann surface of genus g, and let $\mathfrak{B} = \{\omega_1, \omega_2, \ldots, \omega_g\}$ be a basis for $\Omega(X)$. The *period* subgroup of $\mathbb{C}^g$ with respect to $\mathfrak{B}$, denoted $Per(\mathfrak{B})$, is defined as the image of the homomorphism

$$\Big(\int_{-} \omega_1, \int_{-} \omega_2, \ldots, \int_{-} \omega_g\Big) : H_1^\infty(X) \to \mathbb{C}^g.$$

The *Jacobian variety* of X is then the quotient

$$Jac(X) := \mathbb{C}^g / Per(\mathfrak{B})$$

Although the $Jac(X)$ depends on the basis for $\Omega(X)$, different bases yield isomorphic Jacobian varieties.

Note that, for a compact Riemann surface X, a divisor is just a 0-chain, i.e. $Div(X) = C_0(X)$. Thus, the boundary map

$$\partial : C_1^\infty(X) \to C_0^\infty(X)$$

is a homomorphism from smooth 1-chains to divisors on X, sending a 1-simplex to the divisor that is the formal difference of its terminal points. In particular, if $c : [0,1] \to X$ is a smooth path, then ∂c is the divisor that is 1 at $c(1)$ and -1 at $c(0)$.

Proposition 7.2. *Let X be a compact Riemann surface. Then for any $D \in Div_0(X)$, there exists $c \in C_1^\infty(X)$ such that $\partial c = D$.*

Proof. Since $degD = 0$, D can be decomposed as

$$D = P_1 + P_2 + \cdots + P_m - P_{m+1} - \cdots - P_{2m}$$

where P_k is the divisor that is 1 at a point and 0 elsewhere (the P_k's are not necessarily distinct). Any Riemann surface is path-connected, so let $\gamma_k : [0,1] \to X$ be a path from P_{m+k} to P_k for $k = 1, 2, \ldots, m$.

We claim that there exists a smooth 1-chain c_k such that $\partial c_k = P_k - P_{m+k}$. Because $[0,1]$ is compact, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ and charts $\{(U_i, z_i)\}_{i=1,\ldots,n}$ such that, for all i,

(1) $z_i(U_i) \subset \mathbb{C}$ is the unit disk, and
(2) $\gamma_i([t_{i-1}, t_i]) \subset U_i$.

For each i, let $\lambda_i = z_i \circ \gamma|_{[t_{i-1},t_i]}$. Let $l_i : [0,1] \to \mathbb{C}$ be the line between the terminal points of λ_i, $l_i(s) = (1-s)\lambda_i(t_{i-1}) + s\lambda_i(t_i)$. We see that $\sigma_i := z_i^{-1} \circ l_i$ is a smooth singular 1-simplex. So let $c_k = \sigma_1 + \cdots + \sigma_n \in C_1^\infty(X)$ and observe that

$$\partial(c_1 + \cdots + c_n) = P_k - P_{m+k}.$$

Then $\sum_{k=1}^m c_k$ is the desired smooth 1-chain. $\square$

With X and $\mathfrak{B}$ as above, we define the Abel-Jacobi map $A : Div_0(X) \to Jac(X)$ as follows. For $D \in Div_0(X)$, take $c \in C_1^\infty(X)$ as in the proposition. Then let

$$A(D) = \Big(\int_c \omega_1, \int_c \omega_2, \ldots, \int_c \omega_g \Big) \; mod \; Per(\mathfrak{B}).$$

For any other smooth 1-chain c_0 with $\partial c_0 = D$, $c - c_0 \in Ker\partial$. Hence,

$$\Big(\int_{c-c_0} \omega_1, \int_{c-c_0} \omega_2, \ldots, \int_{c-c_0} \omega_g \Big) \in Per(\mathfrak{B})$$

implying A is well-defined. Indeed, A is a homomorphism.

8. Abel's Theorem

We are now prepared to state the second main theorem of this paper.

Theorem 8.1 (Abel)**.** *Let X be a compact Riemann surface of genus g. The kernel of $A : Div_0(X) \to Jac(X)$ is precisely $Div_P(X)$.*

The theorem gives a necessary and sufficient condition for the existence of a meromorphic function of prescribed zeros and poles. Specifically, if we encode the desired zeros and poles in a divisor D, then $degD$ must be 0, and D will have a meromorphic solution if and only if $D \in KerA$. Most of the work in proving the theorem goes into constructing a meromorphic solution, for which we need the notion of a *weak solution* of a divisor.

Let X be a Riemann surface, and let D be a divisor on X. Let

$$X_D = \{x \in X \mid D(x) \geq 0\}.$$

A *weak solution* of D is a function $f \in \mathcal{E}(X_D)$ with the following property: for each $p \in X$, there exists a coordinate neighborhood (U, z) centered at x and $\psi \in \mathcal{E}(U)$ with $\psi(p) \neq 0$ such that

$$f = \psi z^k \text{ on } U \cap X_D$$

where $k = D(p)$. We list a few immediate properties of weak solutions.

(1) A weak solution f of D is a meromorphic solution if and only if f is holomorphic on X_D.
(2) Let f_i be a weak solution of D_i for $i = 1, 2$. Then $f_1 f_2$ is a weak solution of $D := D_1 + D_2$ if we extend $f_1 f_2$ smoothly to the points $p \in X$ where $D(p) \geq 0$ but $D_1(p)$ or $D_2(p)$ is less than 0.
(3) Let f be a weak solution of D. Then df/f is a smooth 1-form on $X'_D := \{x \in X_D \mid D(x) = 0\}$. For $p \notin X'_D$, write $f = \psi z^k$ around p as above. There exists a sufficiently small neighborhood U of p such that, on $U - \{p\}$,
$$\frac{df}{f} = \frac{d\psi}{\psi} + \frac{k \cdot dz}{z}$$
is well-defined and smooth. On the same domain,
$$\frac{d''f}{f} = \frac{d''\psi}{\psi}$$
which can be smoothly extended to p and thus to all of X.

Lemma 8.2. *Let X be a Riemann surface. Let $a_1, \ldots, a_n \in X$ be distinct points and $k_1, \ldots, k_n \in \mathbb{Z}$. Suppose $D \in Div(X)$ is the divisor with $D(a_i) = k_i$ for $i = 1, \ldots, n$ and $D(x) = 0$ otherwise. Let f be a weak solution of D. Then for any $g \in \mathcal{E}(X)$ with compact support,*

$$\frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge dg = \sum_{i=1}^{n} k_i g(a_i).$$

Proof. Let $(U_1, z_1), \ldots, (U_n, z_n)$ be disjoint coordinate neighborhoods centered at $a_1, \ldots, a_n$, respectively, as in property (3) above. Without loss of generality, we may assume that $z_i(U_i)$ is the unit disk. Then for $0 < r_1 < r_2 < 1$, there exist $\phi_i \in \mathcal{E}(U_i)$ satisfying

$$\phi_i(z_i) = 1 \text{ if } |z_i| \leq r_1, \quad \phi_i(z_i) = 0 \text{ if } |z_i| \geq r_2$$

for all $i = 1, \dots, n$.

Let $g_i = \phi_i g$, and let $g_0 = g - (g_1 + \cdots + g_n)$. Then g_0 has compact support in $X' := X - \{a_1, \dots, a_n\}$, and

$$\iint_X \frac{df}{f} \wedge dg_0 = -\iint_{X'} d\Big(g_0 \frac{df}{f}\Big).$$

By Stokes' theorem, this integral is 0. Hence,

$$\iint_X \frac{df}{f} \wedge dg = \sum_{i=1}^{n} \iint_{U_i} \frac{df}{f} \wedge dg_i.$$

On $U_i - \{a_i\}$, let

$$\frac{df}{f} = \frac{d\psi_i}{\psi_i} + \frac{k \cdot dz_i}{z_i}.$$

Again by Stokes' Theorem,

$$\iint_{U_i} \frac{d\psi_i}{\psi_i} \wedge dg_i = 0.$$

Therefore, we have

$$\begin{aligned} \iint_{U_i} \frac{df}{f} \wedge dg_i &= k_i \iint_{U_i} \frac{dz_i}{z_i} \wedge dg_i \\ &= -\lim_{\epsilon \to 0} \iint_{\epsilon \le |z_i| \le r_2} d\Big(g_i \frac{dz_i}{z_i}\Big) = \lim_{\epsilon \to 0} \int_{|z_i| = \epsilon} g_i \frac{dz_i}{z_i} = 2\pi i g(a_i). \end{aligned}$$

□

Lemma 8.3. *Let X be a Riemann surface and $c : [0,1] \to X$ be a smooth path. Then there exists a weak solution f of the divisor ∂c such that for every closed, smooth form $\omega \in \mathcal{E}^1(X)$,*

$$\int_c \omega = \frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge \omega. \tag{8.4}$$

Proof. First, assume there exists a coordinate neighborhood (U, z) such that $c([0,1]) \subset U$. Without loss of generality, assume $z(U)$ is the unit disk, and let r, r' be such that

(1) $0 < r < r' < 1$, and
(2) $c([0,1])$ is contained in the disk of radius r.

There exists $\phi \in \mathcal{E}(U)$ such that

$$\phi(z) = 1 \text{ if } |z| \le r, \quad \phi(z) = 0 \text{ if } |z| \ge r'.$$

Let $a = c(0)$ and $b = c(1)$. Then define a function $f_0 \in \mathcal{E}(U - \{a\})$ by

$$f_0(z) = \begin{cases} exp(\phi \cdot log(\frac{z-b}{z-a})) & \text{if } r \le |z| < 1 \\ \frac{z-b}{z-a} & \text{if } |z| \le r. \end{cases}$$

We see f_0 can be extended to $f \in \mathcal{E}(X - \{a\})$ by setting it equal to 1 on $X - U$. Clearly, f is a weak solution of ∂c.

Let ω be a closed, smooth 1-form on X. Since U is simply connected, ω has a primitive $g_0 \in \mathcal{E}(U)$. With r' as above and $0 < r' < r'' < 1$, there exists $\psi \in \mathcal{E}(U)$ such that

$$\psi(z) = 1 \text{ if } |z| \le r', \quad \psi(z) = 0 \text{ if } |z| \ge r''.$$

Then $g := \psi g_0$ is a primitive for ω on the disk of radius r' that has compact support in U. Thus, by the previous lemma,

$$\frac{1}{2\pi i}\iint_X \frac{df}{f}\wedge\omega = \frac{1}{2\pi i}\iint_U \frac{df}{f}\wedge dg = g(b)-g(a) = \int_c \omega.$$

In the general case, since $[0,1]$ is compact, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and coordinate neighborhoods (U_i, z_i) such that

(1) $z_i(U_i)$ is the unit disk for all i and
(2) $c([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \ldots, n$.

Let $c_i = c|_{[t_{i-1},t_i]}$. By the first part, for each i, there exists a weak solution f_i of ∂c_i such that $f_i|_{X-U_i} = 1$ and for each closed, smooth 1-form ω on X,

$$\int_{c_i}\omega = \frac{1}{2\pi i}\iint_X \frac{df_i}{f_i}\wedge\omega.$$

Then let $f = f_1 \cdot f_2 \cdots f_n$. By property (2) in the discussion preceding Lemma 8.2, f is a weak solution of $\partial c = \partial c_1 + \cdots + \partial c_n$ such that

$$\int_c \omega = \iint_X \frac{df}{f}\wedge\omega.$$

□

We can immediately generalize the result to $c = \sum_i n_i\gamma_i \in C_1^\infty(X)$. By the lemma, there exists a weak solution f_i for $\partial\gamma_i$ satisfying Equation 8.4. Then $f := \prod_i f_i^{n_i}$ is a weak solution of ∂c satisfying the same equation.

With X a compact Riemann surface, we have completed most of the work for constructing a meromorphic solution to $D \in KerA$. For the converse, given a nonzero meromorphic function f, we wish to show $(f) \in KerA$. It clearly suffices to show there exists $c \in C_1^\infty(X)$ with $\partial c = (f)$ such that

$$\int_c \omega = 0 \text{ for all } \omega \in \Omega(X).$$

If f is constant, this follows immediately. Otherwise, f is a holomorphic map onto $\mathbb{P}^1$. From Corollary 4.4, $\Omega(\mathbb{P}^1) = 0$. We capitalize on this fact by constructing a global holomorphic 1-form on $\mathbb{P}^1$ from the pullbacks of ω under local inverses of f.

The global 1-form we have in mind is called the *trace* of ω. Let $y_1, \ldots, y_k$ be the branch points of f, and let $Y := \mathbb{P}^1 - \{y_1, \ldots, y_k\}$. Recall that f is an n-sheeted covering space map over Y for some $n \geq 1$. Hence, for each $y \in Y$, there exists a neighborhood V and disjoint neighborhoods $U_1, \ldots, U_n \subset X$ such that $f|_{U_i}$ is a biholomorphic map onto V for each $i = 1, \ldots, n$. Let $\phi_i : V \to U_i$ be the inverse of $f|_{U_i}$. Define a holomorphic 1-form on V by

$$Tr(\omega)_V := \phi_1^*\omega + \phi_2^*\omega + \cdots + \phi_n^*\omega.$$

The collection of all such neighborhoods V for $y \in Y$ is an open cover of Y. Moreover, if V_1 and V_2 are in this cover, then $Tr(\omega)_{V_1} = Tr(\omega)_{V_2}$ on $V_1 \cap V_2$. By sheaf axiom II, there exists a holomorphic 1-form $Tr(\omega)$ on Y that equals $Tr(\omega)_V$ on V.

We claim that $Tr(\omega)$ can be extended holomorphically to $\mathbb{P}^1$. For each branch point y_i, let V_i be a neighborhood of y_i not containing any other branch point and let $f^{-1}(y_i) = \{x_{i,1}, x_{i,2}, \ldots, x_{i,m}\}$. Since ω is holomorphic on X, there exists a chart (U_k, z_k) centered at $x_{i,k}$ such that, if $\omega = g\,dz_k$ on U_k, g is bounded on U_k.

But f is an open map, so $W := \bigcup_k f(U_k) \cap V_i$ is a neighborhood of y_i. On $W - \{y_i\}$, $Tr(\omega)$ is bounded. Then $Tr(\omega)$ can be extended to y_i by the Riemann removable singularities theorem. Consequently, $Tr(\omega) \in \Omega(\mathbb{P}^1)$, implying $Tr(\omega) = 0$.

Proof of Theorem 8.1. Let $\mathfrak{B} = \{\omega_1, \ldots, \omega_g\}$ be a basis for $\Omega(X)$. First, we prove $D \in KerA$ implies $D \in Div_P(X)$. There exists $c \in C_1^\infty(X)$ such that $\partial c = D$ and

$$v := \Big(\int_c \omega_1, \ldots, \int_c \omega_g \Big) \in Per(\mathfrak{B}).$$

We may assume $v = 0$, for otherwise there exists $[c_0] \in H_1^\infty(X)$ such that $\partial(c - c_0) = D$ and

$$\Big(\int_{c-c_0} \omega_1, \ldots, \int_{c-c_0} \omega_g \Big) = 0.$$

As observed after Lemma 8.3, there exists a weak solution f of ∂c such that for any $\omega \in \Omega(X)$,

$$\int_c \omega = \iint_X \frac{df}{f} \wedge \omega = \iint_X \frac{d''f}{f} \wedge \omega.$$

This integral is 0 by assumption. Lemma 6.1 then gives us $g \in \mathcal{E}(X)$ such that $d''g = d''f/f$. Letting

$$h = e^{-g} f$$

we see h is a weak solution of ∂c as well, and

$$d''h = -e^{-g} f d''g + e^{-g} d''f = 0 \text{ on } X_D.$$

Hence, h is a meromorphic solution of $\partial c = D$, implying $D \in Div_P(X)$.

For the converse, assume f is nonconstant. We can use the procedure in Proposition 7.2 to construct a piecewise smooth path $\gamma : [0, 1] \to \mathbb{P}^1$ from ∞ to 0 which avoids branch points, except perhaps at the endpoints. Since f is an n-sheeted covering map, $f^{-1}(\gamma)$ comprises n piecewise smooth paths $c_1, \ldots, c_n$ from the poles to the zeros of f. If $p \in X$ is a pole or zero of f of order k, then the local normal form of f at p is $F(z) = z^k$. In particular, the number of paths among $c_1, \ldots, c_n$ *from* p if p is a pole or *to* p if p is a zero is equal to the order of f at p. Therefore, letting $c := c_1 + \cdots + c_n$, we have $\partial c = (f) = D$ and

$$\int_c \omega = \int_\gamma Tr(\omega) = 0.$$

So $D \in KerA$. □

9. Closing Remarks

The theorems of Riemann-Roch and Abel form a basis for deeper forays into the theory of Riemann surfaces and algebraic geometry. Torelli's Theorem, for example, states that a compact Riemann surface is completely determined by its Jacobian variety endowed with certain extra structure. Henrik Martens proves this result as a "combinatorial consequence of the Riemann-Roch theorem and Abel's theorem" [3]. In exploring Torelli's Theorem and beyond, one uncovers more of beautiful "synthesis" for which we have given only a slight hint.

Acknowledgments. I sincerely thank my mentors, Professors Ben McReynolds and David Constantine, for their patience, their knowledge, their guidance, and their corrections. I thank Wolfgang Schmaltz for constructive suggestions and for listening to my mathematical explanations. I am also deeply thankful to Professor Madhav Nori for taking the time to help me on diverse mathematical topics, far beyond this paper. Finally, Professor Peter May has once again done a wonderful job in providing feedback on my project and in organizing the REU.

References

[1] Otto Forster. Lectures on Riemann Surfaces. Springer. 1980.
[2] John M. Lee Introduction to Smooth Manifolds. Springer. 2000
[3] Henrik H. Martens. A New Proof of Torelli's Theorem. The Annals of Mathematics. Second Series, Vol. 78, No. 1 (Jul., 1963), pp. 107-111.
[4] Rick Miranda. Algebraic Curves and Riemann Surfaces. American Mathematical Society. 1991.