STOCHASTIC CALCULUS

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Abstract. We go through some basics of stochastic calculus. Assuming knowledge of some probability and measure theory, we construct the Itô integral, develop some of its properties, and then show the existence and uniqueness of the solution of a type of stochastic differential equation. We then discuss Itô diffusions, and conclude by solving the stochastic Dirichlet problem.

Contents

1. Introduction

The motivation behind stochastic calculus is to define a calculus for situations where it would normally be useful, but in which normal methods of calculus do not apply due to randomness within the system. More specifically, one might encounter equations like:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“}noise\text{”}$$

where X_t is a stochastic process, b and σ are functions, and the noise is some sort of randomness in the problem. If σ is continuous and the noise term is of bounded variation (does not jump around "too much"), then one can take the Riemann-Stieltjes integral (see [4]). Unfortunately, the noise often has unbounded variation, and a new approach has to be taken. For this paper, we will address situations in which the randomness is derived from a Brownian motion. We follow Øksendal's approach for the following except where otherwise specified.

2. Probability and Brownian Motion

Before going through the construction and then application of the Itô integral to different topics, we must first review some probability concepts and list the basic properties of Brownian motion. Knowledge of basic probability theory and measure theory is assumed. We will provide a short reminder of some basic definitions.

Definition 2.1. Let Ω be a nonempty subset. Let $\mathcal{F}$ be a collection of subsets of Ω such that

(1) $\emptyset \in \mathcal{F}$
(2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and
(3) If $\{A_i\}_{i=1}^{\infty}$ is a collection of elements of $\mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Then $\mathcal{F}$ is a σ-algebra and the pair $(\Omega, \mathcal{F})$ is a measurable space. A measure $\mu : \mathcal{F} \to [0, \infty]$ is a function such that $\mu(\emptyset) = 0$, and if $\{A_i\}_{i=1}^{\infty}$ is a collection of pairwise disjoint elements of $\mathcal{F}$, then $\sum_{i=1}^{\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$. A probability measure is a measure such that $\mu(\Omega) = 1$.

I will use the convention that a.a. means "almost all", a.e. means "almost everywhere", and a.s. means "almost surely', where each means everywhere except a set of measure zero.

We follow [3] for the next definitions.

Definition 2.2. A (n-dimensional) stochastic process X is a function $X : T \times \Omega \to \mathbb{R}^n$, where T is an interval in $\{x \in \mathbb{R} | x \geq 0\}$, and $X(t, \cdot)$ is $\mathcal{F}$-measurable for each $t \in T$.

In other words, a stochastic process is a collection of random variables on the same probability space indexed by time. For convenience, we will often denote a function taking time t as an argument by the subscript t, and will usually omit ω as an argument. We will also use $x \vee y$ to denote $\max(x, y)$ and $x \wedge y$ to denote $\min(x, y)$.

Definition 2.3. If $\{X_t\}$ and $\{Y_t\}$ are n-dimensional stochastic processes, then we say that $\{Y_t\}$ is a version of $\{X_t\}$ if $P\{X_t = Y_t\} = 1$ for all t.

Definition 2.4. A filtration on $(\Omega, \mathcal{F})$ is a collection $\{\mathcal{M}_t : t \geq 0\}$ of σ-algebras such that $\mathcal{M}_t \subseteq \mathcal{F}$ and $\mathcal{M}_s \subseteq \mathcal{M}_t$ for $s < t$.

In probability, σ-algebras are thought of as containing information, and a filtration will usually represent the information we have about a stochastic process. This makes the following relevant:

Definition 2.5. If $\{\mathcal{M}_t\}$ is a filtration, a function $f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{M}_t$-adapted if for each $t \geq 0$ the function $\omega \mapsto f(t, \omega)$ is $\mathcal{M}_t$-measurable.

Definition 2.6. A stochastic proces $\{M_t\}_{t \geq 0}$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t \geq 0}$ and a probability measure P defined on $\mathcal{F}$ if, for all t,

(1) M_t is $\mathcal{M}_t$-measurable,
(2) $E[|M_t|] < \infty$, and
(3) $E[M_t \mid \mathcal{M}_s] = M_s$ for all $t \geq s$.

The most important property of a martingale is (3). We can think of a martingale as memoryless in some sense - if we start a martingale at a certain position and time, all that matters is that starting position and time, and not what happened earlier. One example of a martingale is a "fair game". If one were ever lucky enough to be playing a fair game in a casino (meaning your expected gain is zero, as opposed to negative in real life), one's expected winnings at a future time would be one's current winnings, regardless of any "hot streaks".

In our analysis, we will focus on Brownian motion, as it is relatively simple and has many nice properties that make it amenable to study. There are different ways to define Brownian motion, but one of the more intuitive is the following:

Definition 2.7. A Brownian motion starting at x is a stochastic process B_t, for $t \geq 0$, such that

1. $B_0 = x$ a.s.,
2. $B_t - B_s$ is normally distributed with mean zero and variance $t - s$ for $0 \leq s < t < \infty$,
3. $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent random variables for $0 \leq t_1 < \cdots < t_n < \infty$.

We will take the existence of Brownian motion as a given. The reader can see [1, p.36-46] for a construction of Brownian motion. Additionally, this definition extends in a straightforward way to higher dimensions:

Definition 2.8. An n-dimensional stochastic process $B_t = (B_1(t), \dots, B_n(t))$ starting at $x = (x_1, \dots, x_n)$, is an n-dimensional Brownian motion if

1. B_k is a Brownian motion for $1 \leq k \leq n$,
2. the σ-algebras generated by the B_k are independent.

Proposition 2.9. *We have the following properties of Brownian motion:*

1. *A Brownian motion B_t is a martingale with respect to the filtration generated by itself (meaning $\mathcal{F}_t$ is the σ-algebra generated by B_t for all $t > 0$).*
2. *Brownian motion is a.s. continuous.*
3. *Brownian motion is a.s. not differentiable.*

Proof. To prove (1), it is immediate from the definition that B_t is $\mathcal{F}_t$-measurable. Since it has a normal distribution, $E[|B_t|] < \infty$, and property (3) of Brownian motion implies that $E[B_t \mid B_s] = B_s$ for $t \geq s$. To show (2) and (3), one can consult [1, p.47-51]. □

We will let $\mathcal{F}_t$ denote the σ-algebra generated by the Brownian motion $\{B_s\}_{0 \leq s \leq t}$. Similarly, $\mathcal{F}_\infty$ represents the σ-algebra generated by $\{B_s\}_{0 \leq s}$. Depending on the context, this could be either one or n-dimensional Brownian motion. In the one-dimensional case, $\mathcal{F}_t$ is the σ-algebra generated by the sets $\{\omega \mid B_s(\omega) \in F,\ F$ a Borel set$\}$ for fixed s, $0 \leq s \leq t$.

3. Construction of the Itô Integral

The goal of developing the Itô integral is to "integrate against randomness". Specifically, we will focus on Brownian motion as the source of the randomness or noise, which is the realm in which the Itô integral applies. We thus focus on:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)\frac{dB_t}{dt}$$

where $\frac{dB_t}{dt}$ is not well defined. This could also be interpreted as:

$$dX = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{3.1}$$

where dB_t denotes the infinitesimal change in Brownian motion, not yet defined. This equation will be interpreted in its integral form:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

where the integral $\int_0^t \sigma(s, X_s)dB_s$ has yet to be defined.

Following this line of thought, our first task should be to define

$$\int_0^t \sigma(s, X_s)dB_s \tag{3.2}$$

for a suitably nice class of functions. To do this, we will first define (3.2) for simple functions that satisfy certain measurability criteria. We will then extend this definition by approximations to a broader class of functions.

We'd like to define the integral (3.2) first for functions $\phi(t, \omega)$ of the form:

$$\phi(t, \omega) = \sum_{j=0}^{k-1} e_j(\omega)\chi_{[t_j, t_{j+1})}(t) \tag{3.3}$$

Note that for any fixed ω, ϕ is just an ordinary simple function. However, we would like to put some conditions on ϕ so that its random component behaves nicely. For example, we don't want ϕ to act as if it had future knowledge of what changes B_t was about to undergo. This leads to the next definition:

Definition 3.4. Let $\{\mathcal{M}_t\}_{t\geq 0}$ be an increasing collection of σ-algebras of Ω. A stochastic process $f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{M}_t$-adapted if for any $t \geq 0$ the function $\omega \mapsto f(t, \omega)$ is $\mathcal{M}_t$-measurable.

Intuitively, a function will generally be $\mathcal{M}_t$-adapted if it does not foresee the future of $\mathcal{M}_t$. For example, we can consider $\{\mathcal{F}_t\}$, the filtration generated by a Brownian motion, and let $g(t, \omega) = B_{t+\delta}(\omega)$. In this case, g is not $\mathcal{F}_t$-adapted since it relies on future knowledge of the Brownian motion. However, if we were to let $h(t, \omega) = 2B_t(\omega)$, then h would be $\mathcal{F}_t$-adapted.

We will now describe the collection of functions on which we'll define the Itô integral:

Definition 3.5. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the collection of functions $f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$ such that

(1) $f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $[0, \infty)$,
(2) f is $\mathcal{F}_t$-adapted, and
(3) $E\left[\int_S^T f(t, \omega)^2 dt\right] < \infty$.

Out of these conditions, (1) is a normal condition needed to define an integral, (2) deals with knowledge of the future which was discussed above, and (3) has to do with the convergence of the integral, which will be more clear soon.

We can now begin to define the Itô integral. First, take a simple function ϕ (simple meaning satisfying (3.3)) in $\mathcal{V}(0, T)$. For such a function, we define the integral as follows:

$$\int_S^T \phi(t, \omega)dB_t(\omega) = \sum_{j=0}^{k-1} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega). \tag{3.6}$$

This seems straightforward, but there is the question of how this extends in a well-defined way to other functions. How do we know that two sequences of simple functions converging to the same function will give us the same integral? The answer lies with the following:

Lemma 3.7 (The Itô Isometry). *If $\phi(t,\omega) \in \mathcal{V}$ is simple, then*

$$E\left[\left(\int_S^T \phi(t,\omega)dB_t\right)^2\right] = E\left[\int_S^T \phi(t,\omega)^2 dt\right] \tag{3.8}$$

Proof. Set $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$E[e_i e_j \Delta B_i \Delta B_j] = \begin{cases} 0 & \text{if } i \neq j \\ E[e_i^2] \cdot (t_{i+1} - t_i) & \text{if } i = j, \end{cases}$$

since $e_i e_j \Delta B_i$ and ΔB_j are independent for $i < j$, and ΔB_i is normally distributed with mean zero and variance $t_{i+1} - t_i$. Similarly, $E[\Delta B_i \Delta B_j] = (t_{i+1} - t_i)\delta_{ij}$. It then follows that

$$\begin{aligned} E\left[\left(\int_S^T \phi(t,\omega)dB_t\right)^2\right] &= E\left[\sum_{0 \le i,j \le k-1} e_i e_j \Delta B_i \Delta B_j\right] \\ &= E\left[\sum_{i=0}^{k-1} e_i^2 (t_{i+1} - t_i)\right] = E\left[\int_S^T \phi(t,\omega)^2 dt\right]. \end{aligned}$$

□

We can now go about extending the integral to the class $\mathcal{V}$. Take a function f in $\mathcal{V}$. Let $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$. By condition (3) of Definition 3.5, f, f^+, and f^- are a.s. in $L^2(S,T)$. For a.a. ω, we can thus approximate f^+ by an increasing sequence of simple functions $\{\psi_{1,n}\}$ and f^- by a decreasing sequence of simple functions $\{\psi_{2,n}\}$. Letting $\phi_n = \psi_{1,n} + \psi_{2,n}$, ϕ_n converges to f in $L^2(S,T)$ for all ω. Thus $\int_S^T (f - \phi_n)^2 dt$ goes to 0 as n goes to ∞ a.s. By the dominated convergence theorem, since $E\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$ by (3) of Definition 3.5, we get

$$\lim_{n\to\infty} E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] = 0. \tag{3.9}$$

This completes the approximation.

This allows us to define the Itô integral:

Definition 3.10 (The Itô Integral). Let $f \in \mathcal{V}(S,T)$. Then the Itô integral of f from S to T is

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega) \tag{3.11}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$\lim_{n\to\infty} E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] = 0. \tag{3.12}$$

Note that the limit in (3.11) is in $L^2(P)$ because $\{\int_S^T \phi_n(t,\omega)dB_t(\omega)\}$ is a Cauchy sequence in $L^2(P)$ by Lemma 3.7. The construction of the $\{\phi_n\}$ above shows that there is such a sequence. Additionally, Itô's isometry also implies that the definition is well-defined: if $\{\phi_n\}$ and $\{\psi_n\}$ are two sequences of simple functions converging to f in $L^2(P)$, then

$$\lim_{n\to\infty} ||\phi_n - \psi_n||_{L^2(P)} \le \lim_{n\to\infty} ||\phi_n - f||_{L^2(P)} + \lim_{n\to\infty} ||f - \psi_n||_{L^2(P)} = 0.$$

In the above, we used Itô's isometry in conjunction with (3.12) the end equality.

The immediate consequences of (3.11) and Lemma 3.7 are the following:

Corollary 3.13 (The Itô Isometry)**.** *For any $f \in \mathcal{V}(S,T)$,*

$$E\left[\left(\int_S^T f(t,\omega)dB_t\right)^2\right] = E\left[\int_S^T f^2(t,\omega)dt\right]. \tag{3.14}$$

Corollary 3.15. *If $f, f_n \in \mathcal{V}(S,T)$ for $n = 1, 2, \ldots,$ and $E[\int_S^T (f - f_n)^2 dt] \to 0$ as $n \to \infty$, then*

$$\int_S^T f_n dB_t \to \int_S^T f dB_t$$

as $n \to \infty$ in $L^2(P)$.

4. Basic Properties of the Itô Integral

The Itô integral has the following properties:

Theorem 4.1. *For any $f, g \in \mathcal{V}(S,T)$, constant a, and constant U, $S < U < T$, we have:*

(1) $\int_S^T f\ dB_t = \int_S^U f\ dB_t + \int_U^T f\ dB_t$
(2) $\int_S^T (af + g)\ dB_t = a\int_S^T f\ dB_t + \int_S^T g\ dB_t$
(3) $E[\int_S^T f\ dB_t] = 0$
(4) $\int_S^T f\ dB_t$ *is $\mathcal{F}_T$-measurable*
(5) $E[(\int_S^T f\ dB_t)(\int_S^T g\ dB_t)] = E[\int_S^T fg\ dt]$.

Proof. Properties (1)-(4) all clearly hold for simple functions, and by taking limits, hold for f and g.

To show property (5), let $a = \int_S^T f\ dB_t$ and $b = \int_S^T g\ dB_t$. Since $2ab = (a+b)^2 - a^2 - b^2$, we have:

$$\begin{aligned} E[ab] &= \frac{1}{2}(E[(a+b)^2] - E[a^2] - E[b^2]) \\ &= \frac{1}{2}\left(E\left[\int_S^T (f+g)^2 dt\right] - E\left[\int_S^T f^2 dt\right] - E\left[\int_S^T g^2 dt\right]\right) \\ &= E\left[\int_S^T fg dt\right] \end{aligned}$$

by Itô's isometry, (3.14). □

Additionally, an Itô integral acts as a martingale:

Theorem 4.2. *Let $f(t,\omega) \in \mathcal{V}(0,T)$ for all T. Then*

$$M_t(\omega) = \int_0^t f(s,\omega)dB_t(\omega)$$

is a martingale with respect to $\mathcal{F}_t$.

Proof. First, M_t is $\mathcal{F}_t$-measurable by property (4) of 4.1. To show $E[|M_t|] < \infty$, letting $X = \{\omega | |M_t| < 1\}$,

$$\begin{aligned} E[|M_t|] &= \int_\Omega |M_t| dP(\omega) \\ &\leq \int_X |M_t| dP(\omega) + \int_\Omega M_t^2 dP(\omega) \\ &\leq 1 + E[M_t^2] \\ &< \infty. \end{aligned}$$

Last, we need to show that $E[M_T|\mathcal{F}_S] = M_S$ for all $T \geq S$. By the independence of intervals of Brownian motion and approximation by simple functions, $E[\int_S^T f dB_t|\mathcal{F}_S] = E[\int_S^T f dB_t]$. We then get

$$E[M_T|\mathcal{F}_S] = E[\int_S^T f dB_t|\mathcal{F}_S] + M_S = E[\int_S^T f dB_t] + M_S = M_S.$$

□

For future developments, we'll need the following:

Lemma 4.3 (Doob's martingale inequality)**.** *If M_t is a martingale such that $t \mapsto M_t(\omega)$ is continuous a.s., then for all $p \geq 1$, $T \geq 0$, and all $\lambda > 0$ we have*

$$P\left[\sum_{0 \leq t \leq T} |M_t| \geq \lambda\right] \leq \frac{1}{\lambda^p} E[|M_T|^p].$$

The proof is somewhat long and we'll only use the inequality twice, so we omit it. For the proof, see [1, p.32,118-120].

Additionally, we'll want the following:

Lemma 4.4 (Borel-Cantelli Lemma)**.** *Let A_1, A_2,... be in the σ-algebra $\mathcal{F}$ such that*

$$\sum_{k=1}^{\infty} P(A_k) < \infty.$$

Then

$$P\left[\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right] = 0.$$

Proof. Since $\cup_{k=m}^{\infty} A_k$ are nested decreasing sets with respect to m,

$$P\left[\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right] = \lim_{m\to\infty} P\left[\bigcup_{k=m}^{\infty} A_k\right] \leq \lim_{m\to\infty} \sum_{k=m}^{\infty} P[A_k] = 0$$

since $\sum_{k=1}^{\infty} P[A_k] < \infty$. □

Note that the conclusion to the Borel-Cantelli lemma is equivalent to saying that the probability that infinitely many of the A_k occur is zero.

With these new tools, we can now show the following:

Theorem 4.5. *Let $f \in \mathcal{V}(0, T)$. Then there is a t-continuous version of*

$$\int_0^t f(s, \omega) dB_s(\omega).$$

Proof. Approximate f in $L^2([0,T] \times P)$ by a sequence of simple functions $\phi_n(t) = \sum_j e_j^{(n)} \chi_{[t_j^{(n)}, t_{j+1}^{(n)})}(t) \in \mathcal{V}(0,T)$. Put

$$I_n(t,\omega) = \int_0^t \phi_n(s,\omega) dB_s.$$

Note that each I_n is t-continuous. We will show that the sequence is Cauchy in $L^\infty(0,T)$, which would mean it converges to some continuous function, which would then be equal to $\int_0^t f(s,\omega)ds$ a.e.

First, we'd like to show that I_n is a martingale with respect to $\mathcal{F}_t$. For $s > t$, using basic properties of expectations, the law of total expectation, and the fact that B_t is a martingale with respect to $\mathcal{F}_t$,

$$\begin{aligned} E[I_n(s,\omega) \mid \mathcal{F}_t] &= \int_0^t \phi_n dB + E\left[\sum_{t \le t_i^{(n)} \le t_{i+1}^{(n)} \le s} e_i^{(n)} \Delta B_i | \mathcal{F}_t\right] \\ &= \int_0^t \phi_n dB + \sum_{t \le t_i^{(n)} \le t_{i+1}^{(n)} \le s} E[e_i^{(n)} E[\Delta B_i | \mathcal{F}_{t_i^{(n)}}] | \mathcal{F}_t] \\ &= \int_0^t \phi_n dB = I_n(t,\omega). \end{aligned}$$

Thus I_n is a martingale for any n.

This implies that $I_n - I_m$ is a martingale. Using Doob's martingale inequality (Lemma 4.3) and Itô's isometry (3.14), we get

$$\begin{aligned} P\left[\sup_{0 \le t \le T} |I_n(t) - I_m(t)| > \epsilon\right] &\le \frac{1}{\epsilon^2} E[|I_n(t) - I_m(t)|^2] \\ &= \frac{1}{\epsilon^2} E\left[\int_0^t (\phi_n - \phi_m)^2 ds\right] \end{aligned}$$

which goes to zero as $n, m \to \infty$.

Take a subsequence I_{n_k} such that

$$P\left[\sup_{0 \le t \le T} |I_{n_{k+1}}(t) - I_{n_k}(t)| > \frac{1}{2^k}\right] < \frac{1}{2^k}. \tag{4.6}$$

The Borel-Cantelli lemma implies that this subsequence will converge in $L^\infty(0,T)$ a.s. This limit will be a t-continuous function, equal to $\int_0^t f(s,\omega)dB_s$ in $L^2(P)$. Thus $\int_0^t f(s,\omega)dB_s$ is t-continous a.s. □

Additionally, we can define the multi-dimensional Itô integral in a straightforward way:

Definition 4.7. Let $B = (B_1, \ldots, B_n)$ be n-dimensional Brownian motion. Let $\mathcal{V}^{m \times n}(S,T)$ be the collection of $m \times n$ matrices $v(t,\omega)$ where all entries v_{ij} are in $\mathcal{V}(S,T)$. We then define the multidimensional Itô integral to be the m-dimensional vector

$$\int_S^T v dB = \left(\sum_{j=1}^n \int_S^T v_{1j} dB_j, \ldots, \sum_{j=1}^n \int_S^T v_{mj} dB_j\right).$$

5. Itô's Formula

Everything with the Itô integral has been relatively familiar so far. However, when we begin to evaluate actual integrals, we get formulas that seem strange.

Example 5.1. Let B_t be a Brownian motion starting at 0. Then

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t. \tag{5.2}$$

Proof. Let $\phi_n(s,\omega) = \sum_{k=0}^{n-1} B_{t_k}(\omega)\chi_{[t_k,t_{k+1})}(s)$, where $t_{k+1} - t_k = \frac{t}{n}$. Then

$$\begin{aligned} E\left[\int_0^t (B_s - \phi_n)^2 ds\right] &= E\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (B_s - B_{t_k})^2 ds\right] \\ &= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (s - t_k) ds \\ &= \sum_{k=0}^{n-1} (s-t_k)^2 = \frac{t^2}{n} \end{aligned}$$

which goes to 0 as n goes to ∞.

This implies that

$$\int_0^t B_s dB_s = \lim_{n\to\infty}\sum_{k=0}^{n-1} B_{t_k}\Delta B_{t_k}.$$

Looking at small changes in B_s,

$$\begin{aligned} \Delta(B_{t_k}^2) = B_{t_{k+1}}^2 - B_{t_k}^2 &= (B_{t_{k+1}} - B_{t_k})^2 + 2B_{t_k}(B_{t_{k+1}} - B_{t_k}) \\ &= (\Delta B_{t_k})^2 + 2B_{t_k}\Delta B_{t_k}. \end{aligned}$$

We then get

$$B_t^2 = \sum_{k=0}^{n-1}\Delta(B_{t_k}^2) = \sum_{k=0}^{n-1}((\Delta B_{t_k})^2 + 2B_{t_k}\Delta B_{t_k}),$$

which can be rewritten

$$\sum_{k=0}^{n-1} B_{t_k}\Delta B_{t_k} = \frac{1}{2}B_t^2 + \frac{1}{2}\sum_{k=0}^{n-1}(\Delta B_{t_k})^2.$$

We then get

$$\int_0^t B_s dB_s = \lim_{n\to\infty}\sum_{k=0}^{n-1} B_{t_k}\Delta B_{t_k} = \lim_{n\to\infty}\frac{1}{2}B_t^2 + \frac{1}{2}\sum_{k=0}^{n-1}(\Delta B_{t_k})^2 = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

since the limit is in $L^2(P)$, and $E[\Delta B_{t_k}^2] = t_{k+1} - t_k$. □

This example shows that the Itô integral fails to behave like a normal integral, since in this case it yields an extra term which would not be present in a Riemann-Stieltjes integral. We could rewrite (5.2) as

$$\frac{1}{2}B_t^2 = \int_0^t \frac{1}{2}ds + \int_0^t B_s dB_s. \tag{5.3}$$

This form is reminiscent of (3.1) - the term on the left is equal to a normal "ds" integral and a "dB_s" Itô integral.

This discussion leads us to the following definition:

Definition 5.4. A (1-dimensional) Itô process (or stochastic integral) is a stochastic process X_t that can be written

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s, \tag{5.5}$$

where $v \in \mathcal{V}(0,t)$ for all $t > 0$, u is $\mathcal{F}_t$-adapted, and

$$P[\int_0^t |u(s,\omega)|ds < \infty \text{ for all } t \geq 0] = 1.$$

If X_t is an Itô process, 5.5 can be written in the shorter differential form

$$dX_t = udt + vdB_t. \tag{5.6}$$

In normal calculus, computing differentials directly is a tiresome task. What is instead done is to compute the derivatives of some elementary functions, like ax, x^n, e^x, etc. The derivatives of some more complicated formulas constructed from elementary functions can then be easily found with the chain rule. Alhough the normal chain rule doesn't apply for stochastic integrals, we have Itô's formula as a replacement.

Theorem 5.7 (Itô's Formula)**.** *Let X_t be an Itô process given by*

$$dX_t = udt + vdB_t$$

Let $f(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Y_t = f(t, X_t)$$

is an Itô process since

$$dY_t = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)dX_t^2 \tag{5.8}$$

where $dt^2 = dt \cdot dB_t = 0$, and $(dB_t)^2 = dt$.

Itô's formula can be thought of as a stochastic version of the chain rule in normal calculus. For the normal chain rule, one expands the function as a Taylor series, and takes the rate of change as the first order terms. Itô's formula involves the same process, except now part of the second order term has a significant effect.

This results from the surprising rule that $dB_t^2 = dt$. The reason why both dB_t and dB_t^2 can both have an effect (as opposed to dt versus dt^2, where one is infinitely larger than the other) is that dB_t has larger variations, but they're going to be both positive and negative where they cancel to some extent.

The proof of Itô's formula is simply formalizing the above argument.

Proof of Itô's formula. Note that by integrating, (5.8) can be rewritten:

$$f(t,X_t) = f(0,X_0) + \int_0^t \left(\frac{\partial f}{\partial t}(s,X_s) + \frac{\partial f}{\partial x}(s,X_s)u(s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,X_s)v^2(s) \right) ds$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s,X_s)v(s)dB_s.$$

We can assume that f, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, and $\frac{\partial^2 f}{\partial x^2}$ are bounded, since otherwise we could approximate f by such functions. Additionally, we can approximate u and v by elementary functions.

Applying Taylor's theorem to $f(t, X_t)$, we get

$$\begin{aligned} f(t,X_t) =& f(0,X_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial t}\Delta t_i + \sum_{i=1}^{n} \frac{\partial f}{\partial x}\Delta X_i + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial t^2}(\Delta t_i)^2 + \\ & \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial t \partial x}(\Delta t_i)(\Delta X_i) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}(\Delta X_i)^2 + \sum_{i=1}^{n} R_i \end{aligned}$$

where $\Delta t_i = t_{i+1} - t_i$, $\Delta X_i = X_{i+1} - X_i$, and $R_i = o((\Delta t_i)^2 + (\Delta X_i)^2)$. If we take the limit at $n \to \infty$ (equivalently, $\Delta t_j \to 0$), then the first two sums converge to $\int_0^t \frac{\partial f}{\partial t}(s, X_s)ds$ and $\int_0^t \frac{\partial f}{\partial x}(s, X_s)u(s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)v(s)dB_s$, respectively. As long as we show the other sums are finite, then the final sum must go to zero by the definition of the remainder.

Additionally, the terms $\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial t^2}(\Delta t_i)^2$ and $\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial t \partial x}(\Delta t_i)(\Delta X_i)$ go to zero as $n \to \infty$ since we have a finite integral (integrating with respect to dt or dB) times a term that approaches the infinitesimal dt, which goes to zero.

Next we evaluate $\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 g}{\partial x^2}(\Delta X_i)^2$. Since u and v can be approximated by simple functions,

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}(\Delta X_i)^2 \approx \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}u_i^2(\Delta t_i)^2 + 2\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}u_i v_i(\Delta t_i)(\Delta B_i) + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}v_i^2(\Delta B_i)^2$$

where $u_i = u(t_i)$ and $v_i = v(t_i)$. The approximation becomes equality in the limit. In the same way as above, the first two terms go to zero as $n \to \infty$. We'd like to prove that for the last term we get

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}v_i^2(\Delta B_i)^2 \longrightarrow \int_0^t \frac{\partial^2 f}{\partial x^2}v^2 ds.$$

To show this, put $a(t) = \frac{\partial^2 f}{\partial x^2}(t, X_t)v^2(t)$ and $a_i = a(t_i)$ to simplify notation. If $i > j$, then $a_i a_j((\Delta B_j)^2 - \Delta t_j)$ and $((\Delta B_i)^2 - \Delta t_i)$ are independent, so

$$\begin{aligned} E[a_i a_j((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)] &= E[[a_i a_j((\Delta B_j)^2 - \Delta t_j)]E[(\Delta B_i)^2 - \Delta t_i] \\ &= 0 \end{aligned}$$

if $i > j$ and similarly if $j > i$. We then get

$$E\left[\left(\sum_{i=1}^{n} a_i(\Delta B_i)^2 - \sum_{i=1}^{n} a_i \Delta t_i\right)^2\right] = \sum_{i=1}^{n} E[a_i^2((\Delta B_i)^2 - \Delta t_i)^2] \tag{5.9}$$

$$= \sum_{i=1}^{n} E[a_i^2]E[(\Delta B_i)^4 - 2(\Delta B_i)^2\Delta t_i + (\Delta t_i)^2] \tag{5.10}$$

$$= \sum_{i=1}^{n} (E[a_i^2]E[(\Delta B_i)^4] - 2(\Delta t_i)^2 + (\Delta t_i)^2) \tag{5.11}$$

Using the normal distribution of ΔB_i, we can get

$$E[(\Delta B_i)^4] = \frac{1}{\sqrt{2\pi\Delta t_i}}\int_{-\infty}^{\infty} (\Delta B_i)^4 e^{-(\Delta B_i)^2/2\Delta t_i} d(\Delta B_i) = 3(\Delta t_i)^2.$$

Substituting into (5.11), we get

$$\sum_{i=1}^{n} E[a_i^2]E[(\Delta B_i)^4] - 2(\Delta t_i)^2 + (\Delta t_i)^2 = \sum_{i=1}^{n} E[a_i^2]2(\Delta t_i)^2$$

which goes to zero as $n \to \infty$ since we have a finite integral times a time increment that goes to zero. As a result,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2} v_i^2 (\Delta B_i)^2 = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \Delta t_i = \int_0^t \frac{\partial^2 f}{\partial x^2} v^2 ds \tag{5.12}$$

in $L^2(P)$. □

Theorem 5.13 (Integration by parts)**.** *Suppose $f(s,\omega)$ is continuous and of bounded variation with respect to $s \in [0,t]$ for a.a. ω. Set $B_0 = 0$. Then*

$$\int_0^t f(s) dB_s = f(t)B_t - \int_0^t B_s df_s \tag{5.14}$$

where the integral on the right is the Riemann-Stieltjes integral.

Proof. Let $f(t,x) = f(t)x$ and $Y_t = f(t,B_t) = f(t)B_t$. Then by Itô's formula,

$$dY_t = \frac{df}{dt}(t)B_t dt + f(t)dB_t.$$

This is equivalent to

$$f(t)B_t = \int_0^t \frac{df}{dt}(t)B_t dt + \int_0^t f(t)dB_t.$$

By the definition of the Riemann-Stieltjes integral, we get 5.14. □

Furthermore, we can generalize the above in a straightforward way:

Definition 5.15. An n-dimensional Itô process X_t is an n-dimensional stochastic process given by

$$dX_t(\omega) = u(t,\omega)dt + v(t,\omega)dB_t \tag{5.16}$$

where u is an n-dimensional vector, B_t is n-dimensional Brownian motion, and v is an an n-by-n matrix. Additionally, each u_i and each v_{ij} satisfies the conditions in definition 5.4.

Theorem 5.17. *Let*

$$dX_t = udt + vdB_t$$

be an n-dimensional Itô process, and let $f(t,x) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^p$ be C^2. Then

$$Y_t = f(t, X_t)$$

is a p-dimensional Itô process, whose k^{th} component is given by

$$dY_k = \frac{\partial f_k}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} dX_i + \frac{1}{2} \sum_{1\le i,j\le n} \frac{\partial^2 f_k}{\partial x_i \partial x_j} dX_i dX_j$$

where $dtdB_i = dt^2 = 0$ and $dB_i dB_j = \delta_{ij} dt$.

The proof proceeds the same way as in the 1-dimensional case, this time repeating the argument componentwise.

6. Stochastic Differential Equations

We will now return to the broad type of equation of the form (3.1). With some basic tools to apply to evaluate stochastic integrals, we can now try to attack these differential equations:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

If one has a possible solution, which is generally of the form $X_t = f(t, B_t)$ for some function f, checking it is generally easy by using Itô's formula.

Example 6.1. Let us examine an equation of the form

$$\frac{dN_t}{dt} = (r + \alpha\frac{dB_t}{dt})N_t \tag{6.2}$$

where r and α are constants and N_0 is given, and $\frac{dB_t}{dt}$ is not a derivative in the normal sense. This can be thought of as modelling population growth. The rate of growth, $\frac{dN}{dt}$, varies proportionally to the size of the population N_t. There is inevitably some randomness in this growth, so we build that into the problem by adding in a factor of Brownian motion. Note that it makes sense for this noise to be Brownian motion. Absent large scale cataclysms, we have N_t different individuals with their own randomness, and the central limit theorem says that their sum will be normally distributed with standard deviation proportional to N_t, just like $\alpha B_t N_t$.

We can rewrite this equation to get

$$dN_t = rN_t dt + \alpha N_t dB_t \tag{6.3}$$

which yields

$$\frac{dN_t}{N_t} = rdt + \alpha dB_t. \tag{6.4}$$

Here N_t is an Itô process, and thus cannot be integrated normally to get $\log N_t$. However, we can try to use Itô's formula here. Let $f(t, x) = \log x$ for $x > 0$. Applying Itô's formula,

$$\begin{aligned} d(\log N_t) &= \frac{1}{N_t}dN_t - \frac{1}{2N_t^2}(dN_t)^2 \\ &= \frac{1}{N_t}dN_t - \frac{1}{2N_t^2}\alpha^2 N_t^2 dt \\ &= \frac{1}{N_t}dN_t - \frac{1}{2}\alpha^2 dt. \end{aligned}$$

Plugging this into (6.4),

$$d(\log N_t) + \frac{1}{2}\alpha^2 dt = rdt + \alpha dB_t.$$

Integrating,

$$\log\frac{N_t}{N_0} + \frac{1}{2}\alpha^2 t = rt + \alpha B_t.$$

Isolating the log and taking exponents on both sides,

$$\frac{N_t}{N_0} = e^{(r-\frac{1}{2}\alpha^2)t + \alpha B_t},$$

giving us

$$N_t = N_0 e^{(r-\frac{1}{2}\alpha^2)t+\alpha B_t}. \tag{6.5}$$

Here we see the second derivative term in Itô's formula gives us an unexpected $-\frac{1}{2}\alpha^2 t$ in the exponent. A surprising consequence of this is that if $r < \frac{1}{2}\alpha^2$, then N_t will decrease to 0 with probability one as $t \to \infty$. This can be seen by

$$\begin{aligned} E[\log(N_0 e^{(r-\frac{1}{2}\alpha^2)t+\alpha B_t})] &= E[\log N_0] + E[\log e^{(r-\frac{1}{2}\alpha^2)t}] + E[\log e^{\alpha B_t}] \\ &= \log N_0 + (r - \frac{1}{2}\alpha^2)t + E[\alpha B_t] = \log N_0 + (r - \frac{1}{2}\alpha^2)t \end{aligned}$$

since if $\lim_{t\to\infty} E[\log N_t] = -\infty$, then $\lim_{t\to\infty} E[N_t] = -\infty$.

However, if we take expectations and use the moment generating function of the normal distribution, we get the paradoxical

$$E[N_t] = N_0 e^{(r-\frac{1}{2}\alpha^2)t} E[e^{\alpha B_t}] = N_0 e^{(r-\frac{1}{2}\alpha^2)t} e^{\frac{1}{2}\alpha^2 t} = N_0 e^{rt}.$$

The average growth is the same with randomness, but in general, random population levels will be lower than the level without randomness. This is compensated by random peaks of especially large population sizes.

A natural question is when we have a solution to (3.1), and if we do, when such a solution is unique. This is answered by the following theorem:

Theorem 6.6. *Let $T > 0$, and let $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable with respect to $\mathcal{F}$ satisfying:*

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|) \text{ for } x \in \mathbb{R}^n,\ t \in [0,T],\ C \text{ a constant} \tag{6.7}$$

and,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le D|x-y|, \text{ for } x \in \mathbb{R}^n,\ t \in [0,T],\ D \text{ a constant} \tag{6.8}$$

where the absolute values refer to the $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{n\times m})$ norms. Let Z be a random variable independent of $\mathcal{F}_\infty$ such that $E[|Z|^2] < \infty$. Then the equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{6.9}$$

has a unique t-continuous solution $X_t(\omega)$ such that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $\{B_s\}_{s\le t}$, and

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty.$$

Lemma 6.10 (The Gronwall Inequality)**.** *Let $f(t)$ be a nonnegative function such that*

$$f(t) \le C + A\int_0^t f(s)ds$$

for $0 \le t \le T$ and for some constants C, A. Then

$$f(t) \le Ce^{At} \tag{6.11}$$

for $0 \le t \le T$.

Proof. If $A = 0$, (6.11) is clear, so assume $A \neq 0$. Let $g(t) = \int_0^t f(s)ds$. Then $g'(t) = f(t) \leq C + A\int_0^t f(s)ds = C + Ag(t)$. Now we'd like to show that

$$C + Ag(t) \leq Ce^{At}$$

from which (6.11) follows. This is equivalent to

$$g(t) \leq \frac{C}{A}(e^{At} - 1). \tag{6.12}$$

To show (6.12), let $h(t) = g(t)e^{-At}$. If we can show

$$h(t) \leq \frac{C}{A} - \frac{C}{A}e^{-At} \tag{6.13}$$

then we'll know (6.12). First note that (6.13) clearly holds for $t = 0$. If we look at the rate of change of both sides of the equation,

$$h'(t) = e^{-At}(f(t) - A\int_0^t f(s)ds) \leq Ce^{-At} = \frac{d}{dt}\left(\frac{C}{A} - \frac{C}{A}e^{-At}\right).$$

As a result, (6.13) always holds. □

Proof of the uniqueness of the solution. Suppose X_t and Y_t are two such solutions. First, we'd like to show that $E[|X_t - Y_t|^2] = 0$. Let $a(t) = b(t, X_t) - b(t, Y_t)$ and $\gamma(t) = \sigma(t, X_t) - \sigma(t, Y_t)$. Note the identity $(\int_0^t f(s)ds)^2 \leq t\int_0^t f^2(s)ds$, which follows from the identity $2xy \leq x^2 + y^2$ and approximation by simple functions. Using this, Itô's formula, and (6.8),

$$E[|X_t - Y_t|^2] = E\left[\left|\int_0^t a(s)ds + \int_0^t \gamma(s)dB_s\right|^2\right] \tag{6.14}$$

$$= E\left[\left|\int_0^t a(s)ds\right|^2\right] + 2E\left[\left|\int_0^t a(s)ds\int_0^t \gamma(s)dB_s\right|\right] \tag{6.15}$$

$$+ E\left[\left|\int_0^t \gamma(s)dB_s\right|^2\right] \tag{6.16}$$

$$\leq 3E\left[\left|\int_0^t a(s)ds\right|^2\right] + 3E\left[\left|\int_0^t \gamma(s)dB_s\right|^2\right] \tag{6.17}$$

$$= 3tE\left[\int_0^t a^2(s)ds\right] + 3E\left[\int_0^t \gamma^2(s)ds\right] \tag{6.18}$$

$$\leq 3(1+t)E\left[\int_0^t a^2(s) + \gamma^2(s)ds\right] \tag{6.19}$$

$$\leq 3(1+t)DE\left[\int_0^t |X_s - Y_s|^2 ds\right] \tag{6.20}$$

$$= 3(1+t)D\int_0^t E[|X_s - Y_s|^2]ds \tag{6.21}$$

We thus have

$$E[|X_t - Y_t|^2] \leq 3(1+t)D\int_0^t E[|X_s - Y_s|^2]ds.$$

By Gronwall's inquality, this implies that

$$E[|X_t - Y_t|^2] = 0.$$

This implies that $|X_t - Y_t|^2$ is nonzero only on a set of measure zero. As a result,

$$P[X(t,\omega) - Y(t,\omega) = 0 \text{ for all } t \in \mathbb{Q} \cap [0,T]] = 1$$

since this is the countable intersection of events of probability 1. If $X(t,\omega) - Y(t,\omega)$ is continuous and is zero on a dense subset, then it's zero everywhere, so we get

$$P[X(t,\omega) - Y(t,\omega) = 0 \text{ for all } t \in [0,T]] = 1 \tag{6.22}$$

which gives the uniqueness desired. □

Proof of the existence of the solution. To show the existence of a solution, we'll use Picard's approximation method that is used to show the existence of solutions for ordinary differential equations. To see how this method is applied for ordinary differential equations, see [5, p.720-741].

Let $X_t^{(0)} \equiv X_0$. For $k \geq 0$, define

$$X_t^{(k+1)} = X_0^{(k)} + \int_0^t X_s^{(k)} ds + \int_0^t X_s^{(k)} dB_s. \tag{6.23}$$

We'd like to show that $\{X_t^{(k)}\}$ is converges to our desired solution in $L^2(\mu \times P)$, where μ is Lebesgue measure on $(0,T)$. To do this, we'll first show that it's a Cauchy sequence. We start off with the difference of $X^{(1)}$ and $X^{(0)}$ to begin the estimate:

$$\begin{aligned}
||X_t^{(1)} - X_t^{(0)}||_{L^2(P)}^2 &= E[|\int_0^t b(s,X_0)ds + \int_0^t \sigma(s,X_0)dB_s|^2] \\
&\leq E[|\int_0^t C(1+|X_0|)ds|^2] \\
&= C^2t^2E[1+2|X_0|+|X_0|^2] \\
&\leq At
\end{aligned}$$

where the first inequality comes from (6.7) and A is a constant depending on T, C, and X_0.

For $k > 1$, suppose that $||X_t^{(k)} - X_t^{(k-1)}||_{L^2(P)}^2 \leq A\frac{\alpha^{k-1}t^k}{k!}$, where $\alpha = 3(1+T)D^2$ is a constant. By plugging in (6.23) X_t^{k+1} and X_t^k into 6.14, we get

$$\begin{aligned}
||X_t^{(k+1)} - X_t^{(k)}||_{L^2(P)}^2 &\leq 3(1+T)D^2 \int_0^t ||X_t^{(k)} - X_t^{(k-1)}||_{L^2(P)}^2 ds \\
&\leq 3(1+T)D^2 \int_0^t ||X_t^{(k)} - X_t^{(k-1)}||_{L^2(P)}^2 ds \\
&\leq 3(1+T)D^2 A\frac{\alpha^{k-1}}{k!} \int_0^t s^k ds \\
&\leq A\frac{\alpha^k}{k!}\frac{t^{k+1}}{k+1} \leq A\frac{\alpha^k t^{k+1}}{(k+1)!}.
\end{aligned}$$

By induction, and Fubini's theorem, we get

$$||X_t^{(k+1)} - X_t^{(k)}||_{L^2(\mu\times P)} \leq \left(\int_0^T A\frac{\alpha^k t^{k+1}}{(k+1)!}dt\right)^{1/2} = \left(A\frac{\alpha^k T^{k+2}}{(k+2)!}\right)^{1/2}.$$

Since the difference has a factorial term in the denominator, $\{X_t^{(k)}\}$ is a Cauchy sequence in $L^2(\mu\times P)$. Since this space is complete, the sequence converges to some $X_t \in L^2(\mu \times P)$. We now show that X_t is the desired solution to our equation. We know X_t is adapted to the filtration $\mathcal{F}_t^Z$ since each $X_t^{(k)}$ is and measurability is preserved by taking limits. Additionally, the construction implies that

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty$$

since $X_t \in L^2(\mu \times P)$.

Next, we want to show that $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$. First,

$$\begin{aligned}
&\lim_{k\to\infty} E\left[\left|\int_0^t b(s, X_s^{(k)})ds - \int_0^t b(s, X_s)ds\right|^2\right]^{1/2} \\
&\leq \lim_{k\to\infty} E\left[\left(\int_0^t |b(s, X_s^{(k)}) - b(s, X_s)|^2 ds\right) t^{1/2}\right]^{1/2} \\
&\leq \lim_{k\to\infty} T^{1/4} E\left[D^2 \int_0^t |X_s^{(k+1)} - X_s|^2 ds\right]^{1/2} = 0
\end{aligned}$$

where the first inequality comes from Hölder's inequality and the second inequality results from (6.8).

Similarly, by Itô's isometry and (6.8),

$$\begin{aligned}
&\lim_{k\to\infty} E\left[\left|\int_0^t \sigma(s, X_s^{(k)})dB_s - \int_0^t \sigma(s, X_s)dB_s\right|^2\right]^{1/2} \\
&\leq n \lim_{k\to\infty} E\left[\int_0^t |\sigma(s, X_s^{(k)}) - \sigma(s, X_s)|^2 ds\right]^{1/2} = 0
\end{aligned}$$

where we get an inequality and n, the dimension of B, when applying Itô's isometry since B is n-dimensional.

As a result of the $L^2(P)$ convergence of the integrals of b and σ we get

$$X_t = \lim_{k\to\infty} X_t^{(k+1)} = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

in $L^2(P)$. Thus X_t satisfies (6.9).

By Theorem 4.5, there is a t-continuous version of X_t, and this version satisfies the conditions above. $\square$

7. Diffusions

An Itô process X_t can be thought of as the position of a particle in a fluid at time t. Under this view, $b(t, X_t)$ is the rate and direction in which the particle is expected to move, and is thus called the drift coefficient. On the other hand,

$\sigma(t, X_t)$ describes the random motions of the particle, and σ, or sometimes $\frac{1}{2}\sigma\sigma^T$ (for reasons that will become apparent soon) is called the diffusion coefficient.

If b and σ only take X_t as an argument and satisfy the conditions of 6.6, then we call X_t an Itô diffusion. This can be reworded as follows:

Definition 7.1. An Itô diffusion is an n-dimensional stochastic proces $X_t(\omega)$ satisfying an equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \tag{7.2}$$

for $t \geq s$, $X_s = x$, B_t is m-dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y| \tag{7.3}$$

for x, $y \in \mathbb{R}^n$.

Because of the lack of explicit time dependence in an Itô diffusion, Itô diffusions have the Markov property, meaning that the only variable that affects its future behaviour is where it is at present. This means that the time at a certain point is irrelevant in determining where the diffusion ends up a certain amount of time later; all that matters is its current position.

If we let E^x denote the expectation given that the process starts with $X_0 = x$, this can be restated more precisely as follows:

Proposition 7.4. *Let f be a bounded Borel function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, for t, $h \geq 0$,*

$$E^x[f(X_{t+h})|\mathcal{F}_t](\omega) = E^{X_t(\omega)}[f(X_h)]. \tag{7.5}$$

The proposition follows immediately from the independence of X_{t+h} from $\mathcal{F}_t$.

Something that we will want for later are random times that depend on some condition.

Definition 7.6. Let $\{\mathcal{N}_t\}$ be a filtration. A function $\tau : \Omega \to [0, \infty]$ is a stopping time with respect to $\{\mathcal{N}_t\}$ if

$$\{\tau \leq t\} \in \mathcal{N}_t$$

for all $t \geq 0$.

This condition means that we should know whether or not τ has occurred at a specific time t based on our knowledge of $\mathcal{N}_t$.

Proposition 7.7. *Let $\{\mathcal{N}_t\}$ be a right-continuous family of σ-algebras (meaning that for each t, $\mathcal{N}_t = \bigcap_{s>t} \mathcal{N}_s$), containing all sets of measure zero.*

(1) Let τ_1, τ_2 be stopping times. Then $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ are stopping times.
(2) If $\{\tau_n\}$ is a decreasing family of stopping times then $\lim_{n\to\infty} \tau_n$ is a stopping time.

Proof. To show (1),

$$\{\omega|(\tau_1 \wedge \tau_2) \leq t\} = \{\omega|\tau_1 \leq t\} \cup \{\omega|\tau_2 \leq t\} \in \mathcal{N}_t$$

and, similarly,

$$\{\omega|(\tau_1 \vee \tau_2) \leq t\} = \{\omega|\tau_1 \leq t\} \cap \{\omega|\tau_2 \leq t\} \in \mathcal{N}_t.$$

To show (2), let $\tau = \lim_{n\to\infty} \tau_n$. Then

$$\{\omega | \tau \leq t\} = \bigcap_{n=1}^{\infty} \{\omega | \tau \leq t\} \in \mathcal{N}_t.$$

□

Example 7.8. Let X_t be an Itô diffusion and $U \subseteq \mathbb{R}^n$ be an open or closed set. The first exit time of X_t from U,

$$\tau_U := \inf\{t > 0 | X_t \notin U\}$$

is a stopping time with respect to $\{\mathcal{M}_t\}$, the σ-algebra generated by X_t.

This doesn't immediately result because it's possible that $\tau(\omega) = t$, but $X_t(\omega) \in U$ while $X_s(\omega) \notin U$ for $s > t$, which we wouldn't necessarily know from $\mathcal{M}_t$. To show that it is a stopping time, first assume that U is open. Then

$$\{\tau \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{\{t_i \in \mathbb{Q} | t_i < t\}} \{X_{t_i} \notin U_n\}$$

where $\{U_n\}$ is an increasing sequence of closed sets whose union is U. Each of these sets is measurable with respect to $\mathcal{M}_t$, and thus the countable union and intersection is. Thus if U is open, τ is a stopping time.

Next suppose that U is closed. Let $\{U_n\}$ be a decreasing sequence of closed sets, each containing U, such that their intersection is U. Then τ_{U_n} is a stopping time for all n, and thus $\tau_U = \lim_{n\to\infty} \tau_{U_n}$ is a stopping time by Proposition 7.7.

Equipped with this, we can make a more general statement about Itô diffusions than the Markov property. Specifically, they satisfy the strong Markov property:

Theorem 7.9. *Let f be a bounded Borel function on $\mathbb{R}^n$, τ a stopping time with respect to $\mathcal{F}_t$, $\tau < \infty$ a.s. Then*

$$E^x[f(X_{\tau+h}) \mid \mathcal{F}_\tau] = E^{X_\tau}[f(X_h)] \tag{7.10}$$

for all $h \geq 0$.

The difference between the Markov property and the strong Markov property is that the strong Markov property allows the starting time to be randomly determined. The proof of the strong Markov property involves some attention to detail, and is ommitted. Consult [2, p.118-119] for the proof.

A useful operator that comes from a diffusion is the generator of the diffusion.

Definition 7.11. Let X_t be an n-dimensional Itô diffusion. The generator A of X_t is

$$Af(x) = \lim_{t\downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t}$$

where $x \in \mathbb{R}^n$. We represent the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x by $\mathcal{D}_A(x)$, and the set of functions where the limit exists everywhere by $\mathcal{D}_A$.

The definition of the generator bears resemblance to the definition of a derivative, and this intuition is justified by a formula for the generator that we will derive shortly. We will first establish the following lemma:

Lemma 7.12. *Let X_t be an n-dimensional Itô process of the form*

$$X_t(\omega) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s(\omega)$$

where B is m-dimensional. Let $f \in C_0^2(\mathbb{R}^n)$, and let τ be a stopping time with respect to $\{\mathcal{F}_t\}$ such that $E^x[\tau] < \infty$. Assume u and v are bounded on the set of (t,ω) such that $X_t(\omega)$ is in the support of f. Then

$$E^x[f(X_\tau)] = f(x) +$$
$$E^x\left[\int_0^\tau \left(\sum_{i=1}^n u_i(s,\omega)\frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2}\sum_{1\le i,j\le n}(vv^T)_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\right) ds\right].$$

Proof. Letting $Y_t = f(X_t)$ and applying Itô's formula (without writing the time dependence to simplify notation),

$$\begin{aligned} dY &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X)dX_i + \frac{1}{2}\sum_{1\le i,j\le n}\frac{\partial^2 f}{\partial x_i\partial x_j}(X)dX_idX_j \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X)(u_idx_i + \sum_j v_{i,j}dB_j) \\ &\quad + \frac{1}{2}\sum_{1\le i,j\le n}\frac{\partial^2 f}{\partial x_i\partial x_j}(X)(\sum_k v_{i,k}dB_k)(\sum_m v_{j,m}dB_m). \end{aligned}$$

Taking integrals and then expectations to find $E^x[Y_\tau]$, we find that the term involving $\sum_j v_{i,j}dB_j$ has expectation 0. As for $(\sum_k v_{i,k}dB_k)(\sum_m v_{j,m}dB_m)$,

$$\begin{aligned} (\sum_k v_{i,k}dB_k)(\sum_m v_{j,m}dB_m) &= \sum_k v_{i,k}v_{j,k}(dB_k)^2 + \sum_{k\ne m} v_{i,k}v_{j,m}dB_kdB_m \\ &= \sum_k v_{i,k}v_{j,k}(dB_k)^2 = (vv^T)_{i,j}dt. \end{aligned}$$

This gives us

$$\begin{aligned} E^x[f(X_\tau)] = f(x) &+ E^x\left[\int_0^\tau \left(\sum_{i=1}^n u_i\frac{\partial f}{\partial x_i}(X) + \frac{1}{2}\sum_j (vv^T)_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}(X)\right) ds\right] \\ &+ \sum_{i,j} E^x\left[\int_0^\tau \frac{\partial f}{\partial x_i}(X)v_{i,j}dB_j\right]. \end{aligned}$$

We'd like to show that for each pair of i,j, the last term on the right,

$$E^x\left[\int_0^\tau \frac{\partial f}{\partial x_i}(X)v_{i,j}dB_j\right], \tag{7.13}$$

is zero. The reason that we don't automatically know it's zero from the basic properties of the Itô integral is that the integral ends at τ instead of a fixed time t. We'll show that this property holds by approximation. Let g be a bounded borel function.

$$E^x\left[\int_0^{\tau\wedge k} g(X_s)dB_s\right] = E^x\left[\int_0^k g(X_s)\chi_{s\le\tau}dB_s\right] = 0.$$

We now approximate integrals of the form 7.13 by normal integrals:

$$E^x\left[\left(\int_0^\tau g(X_s)dB_s - \int_0^{\tau\wedge k} g(X_s)dB_s\right)^2\right] = E^x\left[\left(\int_{\tau\wedge k}^\tau g(X_s)dB_s\right)^2\right]$$
$$= E^x\left[\int_{\tau\wedge k}^\tau g^2(X_s)ds\right]$$

which goes to zero as k goes to ∞. This implies that 7.13 is zero, which proves the lemma. $\square$

This lemma immediately yields two results:

Theorem 7.14. *Let X_t be an n-dimensional Itô diffusion given by*

$$X_t = b(X_t)dt + \sigma(X_t)dB_t$$

and let $f \in C_0^2(\mathbb{R}^n)$. Then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{1\le i,j\le n}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{7.15}$$

Proof. This results by plugging X_t starting at x and f into 7.12 along with the definition of the generator. $\square$

Theorem 7.16 (Dynkin's Formula)**.** *Let X_t be an n-dimensional Itô diffusion, let $f \in C_0^2(\mathbb{R}^n)$, and let τ be a stopping time with $E^x[\tau] < \infty$. Then*

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right]. \tag{7.17}$$

Proof. This results by plugging 7.15 into Lemma 7.12. $\square$

Suppose $f \in C^2(\mathbb{R}^n)$ and $E^x[\tau] < \infty$ where τ is the exit time of a bounded set A. Then by modifying f so that it maintains its original values in A and is zero outside of a bounded set, we still get (7.17).

Dynkin's formula can be used to derive some properties of Brownian motion.

Example 7.18. Let B_t be one-dimensional Brownian motion starting at $x \in (0,\infty)$. Put

$$\tau = \inf\{t > 0 | B_t = 0\}.$$

Then

(1) $P[\tau < \infty] = 1$
(2) $E^x[\tau] = \infty$.

To show (1), Let α_k denote the first exit time from $A_k = (0, 2^k x)$. Only consider k such that $2^k > x$. Let $f_k \in C_0^2(\mathbb{R})$ such that $f_k = x$ for $x \in \overline{A_k}$. Since B_t is normally distributed with mean x and variance t, $P[\alpha_k < \infty] = 1$. Because $\frac{d^2 f}{dx^2}(x) = 0$ in A_k, Dynkin's formula implies that

$$E[f_k(B_{\alpha_k})] = f_k(x) = x.$$

Put $p_k = P[B_{\alpha_k} = 0]$. We then have

$$0 \cdot p_k + 2^k x(1 - p_k) = E[f_k(B_{\alpha_k})] = x$$

which implies

$$p_k = 1 - \frac{1}{2^k}.$$

Taking limits, $\lim_{k\to\infty} p_k = 1$. Thus $P[\tau < \infty] = 1$.

To show (2), let α_k, A_k be as before. Let $g_k(x) = x^2$ on $(0, 2^k)$ and in $C_0^2(\mathbb{R})$. Then by Dynkin's formula,

$$E[f(B_{\alpha_k})] = f(x) + E\left[\int_0^{\alpha_k} \frac{1}{2}\frac{d^2 f}{dx^x}(B_s)ds\right].$$

This gives

$$2^k x = \frac{x}{2^k}(2^k)^2 = x^2 + E[\alpha_k].$$

Thus

$$E[\alpha_k] = 2^k x - x^2.$$

Again taking limits, we get $E[\tau] = \lim_{k\to\infty} E[\alpha_k] = \infty$.

8. Harmonic Measure

Take an (nonconstant) Itô diffusion X_t starting at x and a measurable set $G \subseteq \mathbb{R}^n$ containing x with compact closure. Note that if the Brownian motion coefficient is nonzero, then $\tau_G < \infty$ a.s. since Brownian motion is normally distributed with variance t. If $\tau_G < \infty$ a.s., we define

Definition 8.1. The harmonic measure μ_G^x of X on ∂G is given by

$$\mu_G^x(F) = P[X_{\tau_G} \in F]$$

for $F \subseteq \partial G$, and $X_0 = x \in G$.

It is immediate that the harmonic measure is a probability measure by its definition. The reason that it is named the harmonic measure is the following:

Proposition 8.2. *Let f be a bounded measurable function. Then*

$$\phi(x) = E^x[f(X_{\tau_H})]$$

satisfies the mean value property:

$$\phi(x) = \int_{\partial G} \phi(y)d\mu_G^x(y) \tag{8.3}$$

for all $x \in G$ and for all Borel sets $G \subseteq H$ where G have compact closure.

Remark 8.4. For a normal function $f : \mathbb{R}^n \to \mathbb{R}$ to be harmonic in a set means that $f \in C^2$ and the Laplacian of f, $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$, is zero for all points in the set. For a function to be harmonic is equivalent to it being C^2 and satisfying the mean value property. This is where the term harmonic measure comes in.

Proof. This results immediately from the law of total expectation and the Markov property:

$$\begin{aligned} E^x[f(X_{\tau_H})] &= E^x[E^x[f(X_{\tau_H})]|X_{\tau_G} = y] \\ &= E^x[E^y[f(X_{\tau_H})]|X_{\tau_G} = y] = \int_{\partial G} \phi(y)d\mu_G^x(y). \end{aligned}$$

□

9. Boundary Value Problems

We would like to use what we have developed thus far to try to answer the Dirichlet problem. We begin with the classical Dirichlet problem as an example.

Example 9.1 (Classical Dirichlet Problem)**.** Let $D \subset \mathbb{R}^n$ be a bounded open set and let ϕ be a bounded function on ∂D. Suppose there is some $w \in C^2(D)$ such that

(1) $\Delta w = 0$ in D, where Δ is the Laplacian, and
(2) $\lim_{x \in D, x \to y} w(x) = \phi(y)$ for all $y \in \partial D$.

Then $w(x) = E^x[\phi(B_{\tau_D})]$.

We can modify w outside of D so that $w \in C_0^2(\mathbb{R}^n)$. Take an increasing sequence $\{D_k\}$ such that $\overline{D_k} \subseteq D$ and each D_k is open. Then, by Dynkin's formula,

$$E^x[w(B_{\tau_{D_k}})] = w(x) + \int_0^{\tau_{D_k}} Aw(B_s)ds = w(x) + \int_0^{\tau_{D_k}} \frac{1}{2}\Delta w(B_s)ds = w(x).$$

Since B is continuous a.s. and $\lim_{x \in D, x \to y} w(x) = \phi(y)$, taking limits gives us

$$\lim_{k \to \infty} w(B_{\tau_{D_k}}) = \phi(B_{\tau_D})$$

a.s. By the dominated convergence theorem (since ϕ is bounded and w satisfies the mean value property), we then get

$$w(x) = \lim_{k \to \infty} E^x[w(B_{\tau_{D_k}})] = E^x[\lim_{k \to \infty} w(B_{\tau_{D_k}})] = E^x[\phi(B_{\tau_D})].$$

We can solve more general boundary value problems using the tools we have developed. Let D be a domain (connected open set) in $\mathbb{R}^n$, $\phi \in C(\partial D)$ be a given function, and

$$L = \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + \sum_{1 \le i,j \le n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} \tag{9.2}$$

where $b_i(x)$ and $a_{ij}(x) = a_{ji}(x)$ are continuous functions and all eigenvalues of the matrix $a = (a_{ij})$ are non-negative.

Find $u \in C^2(D)$ such that:

$$Lu = 0 \tag{9.3}$$

and

$$\lim_{x \in D, x \to y} u(x) = \phi(y) \tag{9.4}$$

for all $y \in \partial D$.

Note that given a semi-elliptic partial differential operator L on $C^2(\mathbb{R}^2)$ as above, we can imagine an Itô diffusion of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

such that $\frac{1}{2}(\sigma\sigma^T)_{ij} = a_{ij}$. If we can choose such an Itô diffusion, then its generator is equal to L, and we can use what we already know to approach the problem. In this section we will only address L that have such a corresponding σ.

Keeping in mind Dynkin's formula, we'd like to show that, in certain circumstances, there is a solution of the form $E^x[\phi(X_{\tau_D})]$. The motivation for this is that we'd expect something of this form to have the mean value property, which ought to imply something like 9.3 and 9.4. This leads us to the following:

Definition 9.5. Let f be a locally bounded, measurable function on D, and let X be an Itô diffusion. Then f is called X-harmonic on D if

$$f(x) = E^x[f(X_{\tau_U})]$$

for all $x \in D$ and open U with $\overline{U} \subseteq D$.

We then immediately get:

Lemma 9.6. *If f is X-harmonic in D, then $Af = 0$ in D. Suppose $f \in C^2(D)$. Then f is X-harmonic in D if and only if $Af = 0$ in D.*

Proof. If f is X-harmonic in D, $Af(x) = \lim_{t\downarrow 0} \frac{E^x[f(X_t)]-f(x)}{t} = 0$. If $Af = 0$ in D,

$$\begin{aligned} E^x[f(X_{\tau_U})] &= \lim_{k\to\infty} E^x[f(X_{\tau_U \wedge k})] \\ &= f(x) + \lim_{k\to\infty} E^x\left[\int_0^{\tau_U\wedge k} Af(X_s)ds\right] = f(x). \end{aligned}$$

□

The tie between an X-harmonic function and the harmonic measure of X is immediate:

Lemma 9.7. *Let ϕ be a bounded, measurable function on ∂D and put*

$$u(x) = E^x[\phi(X_{\tau_D})]$$

for $x \in D$. Then u is X-harmonic.

Proof. This follows from the mean value property 8.3. Taking any bounded open set U with $\overline{U} \subseteq D$,

$$u(x) = \int_{\partial U} u(y)d\mu_U^x(y) = \int_{\partial U} u(y)P[X_{\tau_U} \in dy] = E^x[u(X_{\tau_U})].$$

□

We will also need the following fact about martingale convergence:

Lemma 9.8. *Let $X \in L^1(P)$ be a stochastic process, $\{\mathcal{N}_k\}_{k=1}^\infty$ be an increasing family of σ-algebras, $\mathcal{N}_k \subseteq \mathcal{F}$ and let $\mathcal{N}_\infty$ be the σ-algebra generated by the $\mathcal{N}_k$. Then*

$$E[X|\mathcal{N}_k] \to E[X|\mathcal{N}_\infty]$$

as $k \to \infty$ in $L^1(P)$.

This lemma should make intuitive sense: the σ-algebra $\mathcal{N}_\infty$ should contain the same amount of information as all the $\mathcal{N}_k$. Since the sequence is "bounded above" by $\mathcal{F}$, the $\mathcal{N}_k$ should get very close to $\mathcal{N}_\infty$, at which point the difference will affect X by a very small amount. For the proof, we refer the reader to [2, p.315].

We now have the tools to tackle a stronger version of the Dirichlet problem, namely, the stochastic Dirichlet problem.

Theorem 9.9 (Solution of the stochastic Dirichlet problem)**.** *Let D be a domain, ϕ a bounded measurable function on ∂D. If we define*

$$u(x) = E^x[\phi(X_{\tau_D})] \tag{9.10}$$

then u satisfies:

(1) u is X-harmonic
(2) $\lim_{t\uparrow\tau_D} u(X_t) = \phi(X_{\tau_D})$ *a.s. for* $X_0 = x$*,* $x \in D$*.*

Additionally, if a bounded function u on D satisfies (1) *and* (2)*, then*

$$u(x) = E^x[\phi(X_{\tau_D})].$$

Proof. First we want to show the existence of the solution, i.e. $u(x) = E^x[\phi(X_{\tau_D})]$ satisfies (1) and (2). Lemma 9.6 implies (1). Fix $x \in D$. Let $\{D_k\}$ be an increasing sequence of open sets such that $\overline{D_k} \subseteq D$ and $D = \cup_{k=1}^{\infty} D_k$. The strong Markov property, described in theorem 7.9, implies

$$u(X_{\tau_{D_k}}) = E^{X_{\tau_{D_k}}}[\phi(X_{\tau_D})] = E^x[\phi(X_{\tau_D}) \mid \mathcal{F}_{\tau_{D_k}}].$$

Additionally, applying the convergence lemma 9.8,

$$\lim_{k\to\infty} u(X_{\tau_{D_k}}) = \lim_{k\to\infty} E^x[\phi(X_{\tau_D}) \mid \mathcal{F}_{\tau_{D_k}}] = \phi(X_{\tau_D}) \tag{9.11}$$

where the limit holds in $L^1(P)$.

Additionally,

$$u(X_{\tau_{D_k} \vee (t \wedge \tau_{D_{k+1}})}) - u(X_{\tau_{D_k}})$$

is a martingale with respect to $\mathcal{F}_{\tau_{D_k} \vee (t\wedge\tau_{D_{k+1}})}$. This is equal to $u(X_t) - u(X_{\tau_{D_k}})$ for $\tau_{D_k} < t < \tau_{D_{k+1}}$, zero for $t < \tau_{D_k}$, and $u(X_{\tau_{D_{k+1}}}) - u(X_{\tau_{D_k}})$ for $t > \tau_{D_{k+1}}$. The reason for analyzing this expression is to use Doob's martingale inequality to show that $u(X_t)$ approaches $\phi(X_{\tau_D})$. Specifically, Doob's martingale inequality yields:

$$P\left[\sup_{\tau_{D_k} \le t \le \tau_{D_{k+1}}} |u(X_t - u(X_{\tau_{D_k}})| > \epsilon\right] \le \frac{1}{\epsilon^2} E^x[|u(X_{\tau_{D_{k+1}}}) - u(X_{\tau_{D_k}})|^2] \tag{9.12}$$

which goes to zero as $k \to \infty$ for all $\epsilon > 0$. Combining (9.11) and (9.12), we get

$$\lim_{t\uparrow\tau_D} u(X_t) = \lim_{k\to\infty} u(X_{\tau_{D_k}}) = \phi(X_{\tau_D})$$

pointwise, which finishes the proof of existence of the solution.

Finally, to show uniqueness, f is X-harmonic, so

$$f(x) = E^x[f(X_{\tau_{D_k}})]$$

for all k. We know that $\lim_{k\to\infty} f(X_{\tau_k}) = \phi(X_{\tau_D})$ a.s. Since ϕ and g are bounded, the dominated convergence theorem implies that

$$f(x) = \lim_{k\to\infty} E^x[f(X_{\tau_{D_k}})] = E^x[\phi(X_{\tau_D})].$$

□

Here we have used the tools we have developed in stochastic calculus to solve a non-stochastic problem in partial differential equations. This result highlights some of the power of stochastic calculus.

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