

Moduli of Finite Flat Group Schemes with G -Structure

Michael Broshi
University of Chicago

November 3, 2007

Brief Timeline

- ▶ 1631: Fermat conjectures that $a^n + b^n = c^n$ has no nontrivial solutions for $n > 2$.
- ▶ 1995: Wiles and Taylor-Wiles prove Fermat's conjecture by proving modularity of semistable elliptic curves over \mathbb{Q} .
- ▶ 2001: Breuil, Conrad, Diamond and Taylor prove modularity of all elliptic curves over \mathbb{Q} .

Modularity overview

- ▶ Modularity of an elliptic curve is equivalent to modularity of its Tate module.
- ▶ Wiles proved modularity of elliptic curves by proving a theorem of the form

$$\bar{\rho} \text{ is modular} \implies \rho \text{ is modular.}$$

- ▶ To prove this he establishes $R \xrightarrow{\sim} \mathbb{T}$. This isomorphism is proven by showing an isomorphism $R_\infty \xrightarrow{\sim} \mathbb{T}_\infty$ of “patched rings.”
- ▶ To deduce modularity of elliptic curves, Wiles considers representations on 3 and 5-torsion points.

Motivation

- ▶ To prove a more general $R = \mathbb{T}$ result, Kisin uses Taylor-Wiles patching argument, considering R_∞ as an algebra over a local deformation ring R_p .
- ▶ He shows that R_∞ is a power series over R_p , and studies connected components of R_p .
- ▶ He studies a “resolution” \mathcal{G} of R_p that has mild singularities, and is able to study its connected components.

Deformations

K is a finite extension of \mathbb{Q}_p ; $G_K = \text{Gal}(\bar{K}/K)$; $G = \text{Sp}_{2n}$. Fix $\bar{\rho} : G_K \rightarrow G(\mathbb{F})$.

- ▶ $D_{\bar{\rho}}$ is the functor of liftings $G_K \rightarrow G(A)$ of $G_K \rightarrow G(\mathbb{F})$. By Mazur, this is representable by a complete local $W(\mathbb{F})$ -algebra, R .
- ▶ $D_{\bar{\rho}}^{\text{fl}}$ is the subfunctor of flat representations. By Ramakrishna, this is representable by R^{fl} .

Remark

R^{fl} will have singularities since finite flat models are not unique.

Main Result

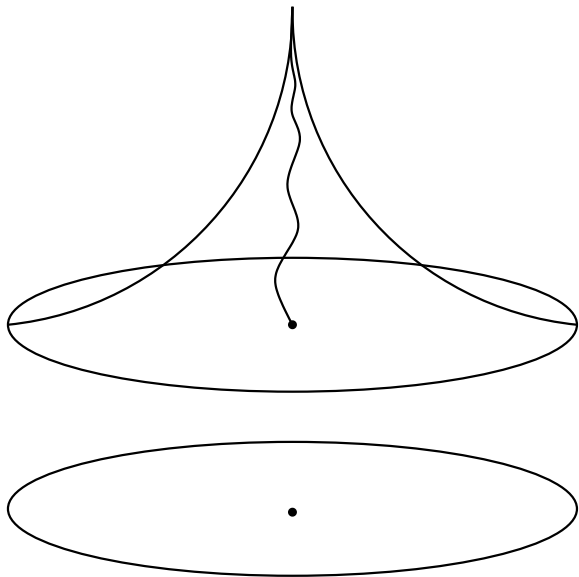
Theorem

There is a projective R^{fl} -scheme, \mathcal{G} with the following properties

- (i) $\mathcal{G} \rightarrow R^{\text{fl}}$ becomes an isomorphism after inverting p*
- (ii) \mathcal{G}^{can} is normal and Cohen-Macaulay*
- (iii) $\bar{\mathcal{G}}^{\text{can}}$ is reduced and normal with rational singularities*

\mathcal{G} is a “moduli space for finite flat group schemes with polarization.”

A Picture



Existence of \mathcal{G}

- ▶ To construct moduli space, we consider auxiliary objects that are “algebraic.”
- ▶ Breuil modules replace finite flat groups schemes with linear algebra data.
- ▶ Deformations of linear algebra data embeds in an affine grassmannian.
- ▶ As an upshot of Breuil modules, one gets tensor products, which allows for “ G -structure.”

Proof Ingredients

- (i) This is a geometric version of Tate's theorem on p -divisible groups, and relies on this, as well as Raynaud's Extension Theorem.

- (ii)/(iii) One compares \mathcal{G} with certain local models of Shimura varieties. In the symplectic case, Pappas and Rapoport show these have the above properties.

Directions

- ▶ Local models of Shimura varieties:
Pappas and Rapoport have a series of papers for certain classical groups. Expand results to all G .

- ▶ Modularity lifting results:
Using techniques of Kisin, and Langlands correspondence for GSp_4 , one hopes to prove an $R = \mathbb{T}$ theorem in new cases.