LIMITS AND CONTINUITY.

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1. Limit points

For a subset D of $\mathbb{R}$ we consider the set of limit points of D – that is the points $a \in \mathbb{R}$ such that there is a sequence of real numbers (x_n) with $x_n \in D \backslash \{a\}$ and $\lim_{n\to\infty} x_n = a$. The *closure* of D is the union of D together with all its limit points, we will denote it by $\bar{D}$. (This matches the conventions of the book, which the older version of this note did not).

Example 1.1. If $D = (0, 1)$ then the limit points are $[0, 1]$ and $\bar{D}$ is $[0, 1]$. If $D = \mathbb{Q}$, then since the rational numbers are dense in the real numbers. every real number is a limit point of $\mathbb{Q}$, and so $\bar{D} = \mathbb{R}$. Finally consider the example $D = \{\frac{1}{n} : n \in \mathbb{N}\}$. There is exactly one limit point of this set – the point 0, and $\bar{D} = \{0\} \cup D$. Note that in this last example the points of D are *not* limit points of D under the above definition.

An alternative definition for a limit point of D is given in the following lemma.

Lemma 1.2. *Let D be subset of $\mathbb{R}$. Then*

(1) *x_0 is a limit point of D if and only if for every $\varepsilon > 0$ there is a point $y \in D \backslash \{x_0\}$ such that $|x_0 - y| < \varepsilon$.*
(2) *We have $\overline{(\bar{D})} = \bar{D}$.*

Proof. For the first part note that if $(x_n)_{n\geq 1}$ is a sequence in $D\backslash\{x_0\}$ tending to x_0, then by the definition of convergence there is an $N > 0$ such that $|x_0 - x_n| < \varepsilon$ for all $n \geq N$. Setting $y = x_N$ we see one implication. To get the other, assume that for every $\varepsilon > 0$ there is a point $y \in D\backslash\{x_0\}$ such that $|x_0 - y| < \varepsilon$. Then for each $n > 0$ we can find an $x_n \in D\backslash\{x_0\}$ such that $|x_0 - x_n| < 1/n$. It follows immediately that $(x_n)_{n\geq 1}$ converges to x_0.

For the second part, we use our new criterion. Let $\varepsilon > 0$ and suppose that $x_0 \in \overline{(\bar{D})}$. Then we may find a $y \in \bar{D}$ such that $|x_0 - y| < \varepsilon$. Similarly we can find a $z \in D$ such that $|y - z| < \varepsilon$, and hence by the triangle inequality we have

$$|x_0 - z| \leq |x_0 - y| + |y - z| < \varepsilon.$$

□

We say that a set $D \subset \mathbb{R}$ is *closed* if it contains all its limit points. Thus the second part of the previous lemma says that the closure of a set is closed.

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2. LIMITS OF FUNCTIONS

Given $f\colon D \to \mathbb{R}$ a continuous function and x_0 a limit point of D, we say that $\lim_{x\to x_0} f(x)$ exists, if for every sequence $(x_n)_{n\geq 1}$ such that $x_n \in D$ and (x_n) converges to x_0, we have $(f(x_n))$ tends to $\lim_{x\to x_0} f(x)$. We can also give an "ε-δ" version of this definition as follows: We say that $l = \lim_{x\to x_0} f(x)$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x \in D$ with $|x - x_0| < \delta$ we have $|l - f(x)| < \varepsilon$. You should check that these two definitions are equivalent (mimic the proof of the equivalence of the two definitions of continuity).

Now suppose we have a continuous function $f\colon D \to \mathbb{R}$, such that for every limit point of D we have $\lim_{x\to x_0} f(x)$ exists. Then we can define a new function $\bar{f}\colon \bar{D} \to \mathbb{R}$ by setting:

$$\bar{f}(x_0) = \begin{cases} f(x_0), & x_0 \in D, \\ \lim_{x\to x_0} f(x), & x_0 \in \bar{D}\backslash D. \end{cases}$$

Since we are assuming that f is continuous on D we could equally have defined

$$\bar{f}(x_0) = \lim_{x\to x_0} f(x), \qquad x_0 \in \bar{D},$$

and not split the definition into two cases. It does however emphasize that $\bar{f}$ agrees with f wherever f was defined. We have the following result, which is almost, but not quite, tautological.

Proposition 2.1. *With the definitions as above, $\bar{f}\colon \bar{D} \to \mathbb{R}$ is continuous.*

Proof. We have to show that for any sequence $(x_n)_{n\geq 1}$ converging to $z \in \bar{D}$, the sequence $(\bar{f}(x_n))$ converges to $\bar{f}(z)$. As always, let $\varepsilon > 0$ be given. If all the terms in the sequence (x_n) lay in D, not just in $\bar{D}$ then we would be done immediately from the definitions. Since it is possible that some x_n lie in $\bar{D}\backslash D$ we first approximate them with elements of D. Thus we define a new sequence (y_n) by letting $y_n = x_n$ if x_n lies in D, while if $x_n \notin D$ we pick $y_n \in D$ such that $|x_n - y_n| < 1/n$ and $|\bar{f}(x_n) - \bar{f}(y_n)| < \varepsilon/2$. (To see that we can do this, use the ε-δ definition of a limit with $\varepsilon/2$ rather than ε to find a δ such that for $y \in D$ with $|y - x_n| < \delta$ we have $|\bar{f}(x_n) - f(y)| < \varepsilon/2$, and then note that since x_n is a limit point of D there must be a y_n with $|x_n - y_n| < \min\{\delta, 1/n\}$.) Then since $x_n - 1/n < y_n < x_n + 1/n$ we see that (y_n) converges to z. Now since the sequence (y_n) lies in D, we know from the definition of $\bar{f}(z)$ that $(\bar{f}(y_n))$ converges to $\bar{f}(z)$. Thus there is an $N > 0$ such that for $n \geq N$ we have $|\bar{f}(z) - \bar{f}(y_n)| < \varepsilon/2$. But then by the triangle inequality we get

$$|\bar{f}(z) - \bar{f}(x_n)| \leq |\bar{f}(z) - \bar{f}(y_n)| + |\bar{f}(y_n) - \bar{f}(x_n)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $n \geq N$. Thus $\bar{f}$ is continuous at z as required. □

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