POWER SERIES.

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1. Power series

Let $s_n(x) = \sum_{k=0}^{n} a_k x^k$ be a sequence of functions. Let D be the subset of $\mathbb{R}$ consisting of those $x \in \mathbb{R}$ for which the sequence $(s_n(x))_{n\geq 0}$ converges. Then the limits define a function $f\colon D \to \mathbb{R}$, that is we set $f(x) = \lim_{n\to\infty} s_n(x)$. The set D is known as the *domain of convergence* of (s_n). The purpose of this section is to investigate the kind of functions f that can arise in this way. We first show that f is continuous, but in fact we will soon see that it is much nicer than that. We start with a technical lemma.

Lemma 1.1. *Suppose that we have a subset A of D on which we have*

$$|a_k x^k| \leq C\alpha^k, \quad k \geq 0,$$

where $\alpha, C \in \mathbb{R}$, and α is such that $0 \leq \alpha < 1$. Then the sequence of functions (s_n) converges uniformly on A.

Proof. We use the Weierstrass test, *i.e.* we show the sequence (s_n) is uniformly Cauchy on A. Let $\varepsilon > 0$ be given. Consider

$$|\sum_{k=n}^{m} a_k x^k| \leq \sum_{k=n}^{m} |a_k x^k| \leq \sum_{k=n}^{m} M\alpha^k.$$

Now the last expression is a geometric series, and so:

$$\begin{aligned}\sum_{k=n}^{m} M\alpha^k &= M\alpha^n \sum_{k=0}^{m-n} \alpha^k \\ &= M\alpha^n(1-\alpha^{m-n+1})/(1-\alpha) \\ &< M\alpha^n/(1-\alpha).\end{aligned}$$

Now since (α^k) converges to 0, we see that there is an $N > 0$ such that for $n \geq N$ $M\alpha^n/(1-\alpha) < \varepsilon$, and hence

$$|\sum_{k=n}^{m} a_k x^k| \leq M\alpha^n/(1-\alpha) < \varepsilon$$

for all $m \geq n \geq N$. ∎

It follows that on any such subset A the function f is continuous, because it is the uniform limit of a sequence of polynomials, which are continuous. The following proposition shows that it is easy to find intervals where the hypothesis of the lemma holds:

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Proposition 1.2. *Suppose that $x_0 \in \mathbb{R}$ is such that the sequence $(s_n(x_0))$ converges. Then the sequence (s_n) converges uniformly on any interval $[-r, r]$ where $r < |x_0|$.*

Proof. Since the series $\sum_{k=0}^{n} a_k x_0^k$ converges, its terms must tend to zero, and hence they are certainly bounded, so we may find an $M > 0$ such that $|a_k x_0^k| < M$ for all k. Then we have for any $x \in [-r, r]$

$$|a_k x^k| = |a_k x_0^k||(\frac{x}{x_0})^k| < M(\frac{r}{|x_0|})^k.$$

Setting $\alpha = \frac{r}{|x_0|}$, we see immediately that the hypotheses of Lemma 1.1 are satisfied, and the result follows. □

Now let us see what this implies about the set D. First notice that if $a, b \in D$ have $a < b$, then the whole interval $[a, b]$ lies in D. This is because, if say $|a| < |b|$ then for any x with $a < x < b$ there is an $r > 0$ such that $|x| < r < |b|$ and hence the previous lemma shows the series converges (uniformly, and hence pointwise) on $[-r, r]$, and so in particular at x. Similarly the case $|a| > |b|$ follows in the same way by picking an r with $|x| < r < |b|$. Thus D is an interval. Moreover if $x_0 \in D$ then since all the intervals $[-r, r]$ lie in D as r ranges over the set $[0, |x_0|)$, again by the previous lemma, we see that the D contains the interval $(-|x_0|, |x_0|)$ and so D is symmetric about 0, except perhaps at it's end points. That is, D is either the whole real line, or D is an interval of the form $(-R, R)$, $[-R, R)$, $(-R, R]$ or $[-R, R]$. The number R is called the *radius of convergence* of the series. Moreover since for any $r < R$ the (s_n) converge uniformly on $[-r, r]$ the limiting function f is continuous there. But we can choose r arbitrarily close to R so in fact f is continuous on the interval $(-R, R)$. (It requires a little more work to show, but is nevertheless true, that f is continuous at the endpoints of the domain of convergence where it is defined).

Example 1.3. Let us consider two simple examples.

(1) Consider the series $s_n(x) = \sum_{k=0}^{n} x^k$. Using our identity for geometric series we can write, for $x \neq 1$

$$\sum_{k=0}^{n} x^k = (1 - x^k)/(1 - x).$$

Thus we see that the series converges if and only if $(x^n)_{n \geq 0}$ converges to zero. This happens precisely when $|x| < 1$, as we have seen before in class. Thus the domain of convergence is $(-1, 1)$ and the radius of convergence is 1 (since the series manifestly does not converge for $x = 1$.)

(2) Now consider the series given by $t_n = \sum_{k=1}^{n} kx^{k-1}$. Once again we can give a "closed form" expression for this sum: If $x \neq 1$ we have:

$$\sum_{k=1}^{n} kx^{k-1} = (1 - (n + 1)x^n + nx^{n+1})/(1 - x)^2.$$

This is completely straightforward to prove by induction, (though of course I got the formula by differentiating the identity that sums a geometric series). Thus the series (t_n) will converge exactly when the sequence (nx^{n-1}) converges – because if the series converges then it terms tend to 0 and so certainly converge, whereas if the sequence (nx^{n-1}) converges it is clear that the sequences $((n+1)x^n)$ and (nx^{n+1}) both converge, and so the right-hand side of the above identity will converge. So to find the domain of convergence we just need to establish when the sequence (nx^{n-1}) converges. I claim this is the case exactly when $|x| < 1$. To see this note that if $|x| \geq 1$ then the sequence is unbounded, so certainly we must have $|x| < 1$. On the other hand, if $|x| < 1$ we can pick an r with $|x| < r < 1$. Now recall how we showed that (r^n) converges to zero: let $1/r = 1 + s$ for some $s > 0$ so $(1+s)^n \geq 1 + ns$ by Bernoulli's inequality and hence $r^n \leq 1/(1+ns)$. Using this we see that

$$|nx^{n-1}| = |nr^{n-1}||(x/r)|^{n-1} \leq |n/(1+(n-1)s)||x/r|^{n-1}.$$

But since the sequence $(n/(1+(n-1)s))_{n\geq 1}$ converges to $1/s$ and the sequence $(|\frac{x}{r}|^{n-1})$ converges to zero, their product tends to zero, and hence by the squeezing principle we see that (nx^{n-1}) tends to 0 as claimed. (This argument follows the idea used in the proof of Proposition 1.2, a different proof is given in the textbook as Exercise 8 in section 2.1). Thus we see that the domain of convergence of (t_n) is exactly $(-1, 1)$.

We now wish to investigate the differentiability of the function f. Notice that the functions s_n are all differentiable since they are polynomials, indeed we also know what their derivatives s_n' are:

$$s_n'(x) = \sum_{k=1}^{n} ka_k x^{k-1}$$

Now we have seen (or rather, to be more honest, I have told you to believe) that the pointwise limit of a sequence of continuously differentiable functions is continuously differentiable if the derivatives converge uniformly. Hence we need to determine where the sequence of functions s_n' converges uniformly.

Lemma 1.4. *The series (s_n') has the same radius of convergence as the series (s_n).*

Proof. Let R be the radius of convergence of the series (s_n), and let x be in $(-R, R)$. We must show that the sequence $(s_n'(x))$ converges. Pick x_0 such that $|x| < |x_0| < R$, so that the series $(s_n(x_0))$ converges, and fix r so that $|x| < r < |x_0|$. Since the series $(s_n(x_0))$ converges we may as before pick an M such that $|a_k x_o^k| < M$ for all k. Then setting $\alpha = r/|x_0|$ we see that

$$|\sum_{k=1}^{n} ka_k x^{k-1}| \leq \sum_{k=1}^{n} |ka_k x_0^k||x_0|^{-1}|(\frac{x}{x_0})|^{k-1} < M|x_0|^{-1}\sum_{k=1}^{n} k\alpha^{k-1}$$

But now note that since $\alpha < 1$ the series $\sum_{k=1}^{n} k\alpha^{k-1}$ converges as we saw in the above example, and hence by the comparison test that the series $s_n'(x)$ converges absolutely. Thus we see that the sequence (s_n') converges pointwise on the interval $(-R, R)$. To see that it cannot converge on an interval of greater length one has to show that the convergence of $(s_n'(x))$ implies the convergence of $(s_n(x))$, which is similar, but easier than what we just did. □

Lemma 1.5. *Let R be the radius of convergence of the series (s_n). Then for any $r < R$ the limit function f is continuously differentiable on $(-r, r)$ with derivative $f'(x) = lim_{n\to\infty} s_n'(x)$.*

Proof. To see this we apply the theorem (9.19 in the textbook) on sequences of continuously differentiable functions, hence we just need to verify that the hypotheses of the theorem hold in our case. By what we have done above, on an interval $(-r, r)$ as in the statement, we know that the two sequences (s_n) and (s'_n) converge uniformly, which is in fact stronger than the hypotheses in the theorem □

Corollary 1.6. *If $s_n(x) = \sum_{k=0}^{n} a_k x^k$ is a power series with radius of convergence R, then the limiting function f is infinitely differentiable on $(-R, R)$. Moreover if $f^{(n)}(x)$ is the n-th derivative of f, then the constants a_k are given by*

$$a_k = \frac{1}{k!} f^{(k)}(0).$$

Proof. Use the previous lemma and induction. □

We remark that there exist infinitely differentiable functions which cannot be expressed as power series (in the terminology of calculus, there are infinitely differentiable function which are not equal to their Taylor series expansions).

Finally it is worth pointing out that this section allows us to define various "standard" functions in calculus. We give just one example.

Example 1.7. Consider the series

$$e_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k.$$

It is easy to see that this sequence of functions converges uniformly on any interval $[-r, r]$, ($r \geq 0$), indeed using the ratio test, the ratio of consecutive terms is $\frac{1}{(n+1)!} x^{n+1}/(x^n/n!) = x/(n+1)$ which is smaller than 1 for all $n > 1/r$. The limit is called the exponential function, written $exp(x)$. To see why this function is indeed what you normally think of as the exponential function, consider the product $s_{2n}(x)s_{2n}(y)$. You can check using the binomial theorem that if you expand out this product, the terms of degree at most n (in x and y combined) give you exactly $s_n(x + y)$. For example,

$$\begin{aligned} s_4(x)s_4(y) &= (1 + x + 1/2x^2 + 1/6x^3 + 1/24x^4)(1 + y + 1/2y^2 + 1/6y^3 + 1/24y^4) \\ &= 1 + (x + y) + (1/2x^2 + xy + 1/2y^2) + \ldots \\ &= 1 + (x + y) + 1/2(x + y)^2 + \ldots \\ &= s_2(x + y) + \ldots \end{aligned}$$

With a little care you can then show that

$$exp(x)exp(y) = \lim_{n\to\infty}(s_{2n}(x)s_{2n}(y)) = \lim_{n\to\infty} s_n(x + y) = exp(x + y). \tag{1.1}$$

In fact one can prove a general theorem which shows that the product of two power series is again a power series, with the coefficients you expect by multiplying the infinite series termwise.

If we let $e = exp(1)$ then it is easy to see using 1.1 and induction, that $exp(n) = e^n$ and since $exp(0) = 1$ that $exp(-x) = exp(x)^{-1}$. This is then enough to show that for $x \in \mathbb{Q}$ we have $exp(x) = e^x$. The extension to $\mathbb{R}$ follows since $exp(x)$ is continuous and agrees with the continuous function e^x at a dense set ($\mathbb{Q} \subset \mathbb{R}$). (Of course you should be thinking here about what is meant by e^x for x irrational in the first place — one way to make sense of this is to show that if (q_n) is a sequence of rational numbers tending to x then e^{q_n} tends to a limit which depends only on x. This, however, is a little painful to do, and it's maybe just as easy to notice that $exp(x) = e^x$ for $x \in \mathbb{Q}$ and then simply to *define* e^x to be $exp(x)$. Since we know that exp is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$ this gives the unique continuous extension of e^x to the real numbers.

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