1. In class on October sixth, the empty set was discussed. The students in the classroom all seemed to agree, eventually, that the empty set is a subset of every set (this is the content of Exercise 1.11 of Script #1). In symbols:

\[ \forall A \text{ s.t. } A \text{ is a set, } \emptyset \subset A. \]

However, it was noted in the discussion of Exercise 1.18 that, in the particular case where \( A = \{1, 2, 3\} \), the empty set was also a member of \( \mathcal{P}(A) \). In symbols:

\[ A = \{1, 2, 3\} \Rightarrow \emptyset \in \mathcal{P}(A). \]

Let \( A \) be a set. Consider the three following statements:

\[ \emptyset \in A \Rightarrow \exists B \text{ s.t. } A = \mathcal{P}(B); \]  
\[ \emptyset \in A \iff \exists B \text{ s.t. } A = \mathcal{P}(B); \]

\[ \emptyset \in A \iff \exists B \text{ s.t. } A = \mathcal{P}(B). \]

Determine, with proof, which of statements (1), (2), and (3) is true.

2. Compute \( \mathcal{P}(\emptyset) \), \( \mathcal{P}(\mathcal{P}(\emptyset)) \), and \( \mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) \). Do you see any pattern emerging?

3. Let \( A \subset \mathbb{Z} \) and let \( f : A \rightarrow \mathbb{Z} \) be a function. We say that

- \( f \) is monotonically increasing if \( m < n \Rightarrow f(m) \leq f(n) \);
- \( f \) is monotonically decreasing if \( m < n \Rightarrow f(m) \geq f(n) \);
- \( f \) is strictly increasing if \( m < n \Rightarrow f(m) < f(n) \);
- \( f \) is strictly decreasing if \( m < n \Rightarrow f(m) > f(n) \); and
- \( f \) is monotonic if \( f \) is monotonically increasing or \( f \) is monotonically decreasing.

Let \( A \subset \mathbb{Z} \) and let \( f : A \rightarrow \mathbb{Z} \) be a function.

(a) Prove or disprove: if \( f \) is monotonic, then \( f \) is injective.
(b) Prove or disprove: if \( f \) is strictly increasing or strictly decreasing, then \( f \) is injective.
(c) Prove or disprove: if \( f \) is injective, then \( f \) is monotonic.
(d) Prove or disprove: if \( f \) is injective, then \( f \) is strictly increasing or strictly decreasing.

We will revisit these ideas later in the year.