Axioms (Peano’s Postulates). The natural numbers are defined as a set \( \mathbb{N} \) together with a unary “successor” function \( S : \mathbb{N} \to \mathbb{N} \) and a special element \( 1 \in \mathbb{N} \) satisfying the following postulates:

I. \( 1 \in \mathbb{N} \).

II. If \( n \in \mathbb{N} \), then \( S(n) \in \mathbb{N} \).

III. There is no \( n \in \mathbb{N} \) such that \( S(n) = 1 \).

IV. If \( n, m \in \mathbb{N} \) and \( S(n) = S(m) \), then \( n = m \).

V. If \( A \subset \mathbb{N} \) is a subset satisfying the two properties:

   • \( 1 \in A \)
   
   • if \( n \in A \), then \( S(n) \in A \),

then \( A = \mathbb{N} \).

Theorem (Mathematical Induction). For each \( n \in \mathbb{N} \), let \( P(n) \) be a proposition. Suppose the following two results:

(A) \( P(1) \) is true.

(B) If \( P(n) \) is true, then \( P(S(n)) \) is true.

Then \( P(n) \) is true for all \( n \in \mathbb{N} \).

Statement (A) is called the base case and statement (B) is called the inductive step. The assumption that \( P \) is true of \( n \) is called the inductive hypothesis.

1. Prove the theorem.

2. Assuming now that we have defined addition, multiplication etc. on \( \mathbb{N} \) so that all the usual properties hold, prove that for all positive integers \( n \),

\[
1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}.
\]

3. Prove that if \( n \geq 4 \), then \( n^2 \leq 2^n \).

4. Prove Bernoulli’s Inequality:

   If \( 1 + x > 0 \), then \( (1 + x)^n \geq 1 + nx \) for any \( n \in \mathbb{N} \).

Remark. From now on we assume that all the usual properties of \( \mathbb{N} \) and \( \mathbb{Z} \) hold. A list of properties will be posted.