

BILLIARDS HANDOUT, SECTION III

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3. POLARIZED FLAT STRUCTURES

Surfaces were defined in the previous section. A *curve* on a surface S can be thought of as a map $\gamma : I \rightarrow S$, but it is customary to also refer to the image of such a map on the surface as a curve. A *closed* curve has the property that $\gamma(0) = \gamma(1)$, or equivalently can be written as $\gamma : S^1 \rightarrow S$. A *simple* closed curve is non-self-intersecting.

For any two curves γ and γ' on S , they are *homotopic* if there exists a continuous map $F : I \times I \rightarrow S$ “sliding” γ to γ' ; that is, satisfying $F(0, t) = \gamma(t)$ and $F(1, t) = \gamma'(t)$. A homotopy *rel endpoints* is one that fixes the endpoints of the two curves throughout the process: $F(s, 0) = \gamma(0)$, $F(s, 1) = \gamma(1)$. One example of a homotopy, always available in the plane, is the *straight-line homotopy* $F(s, t) = (1-s)\gamma(t) + s\gamma'(t)$. Homotopy provides an equivalence relation on curves.

Definition 1. We will use \mathcal{S} to denote the homotopy classes of simple closed curves on a surface. (Or $\mathcal{S}(S)$ if the surface is in doubt.)

We have seen that a (rational) billiard gives rise to a polygonal system, and we have claimed (and seen for examples) that a surface is obtained from such a polygonal system by edge identifications. In this section, we see that billiards are only a small subset of the much larger classes of polygonal systems and translation surfaces. We will formalize the definitions of those two kinds of objects and show that they are equivalent.

Definition 2. Let

$$\Delta^i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, \ y \geq 0\}$$

be the upper half of a unit disk with its lower boundary divided into two named edges:

$$A^i = \{(x, 0) : -1 < x \leq 0\} \text{ and } B^i = \{(x, 0) : 0 \leq x < 1\}.$$

Similarly, let $\Delta_i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, \ y \leq 0\}$, with boundary edges $A_i = \{(x, 0) : -1 < x \leq 0\}$ and $B_i = \{(x, 0) : 0 \leq x < 1\}$. A *k-sheet cone neighborhood*, which we'll denote \mathcal{C}_k , is defined to be the surface obtained by gluing the $2k$ half-disks in an alternating pattern:

$$\mathcal{C}_k := [\Delta_1 \sqcup \Delta^1 \sqcup \cdots \sqcup \Delta_k \sqcup \Delta^k] / \sim,$$

$$\text{where } \begin{cases} B_i \sim B^i, & 1 \leq i \leq k \\ A_i \sim A^{i+1}, & 1 \leq i \leq k-1 \\ A_k \sim A^1 \end{cases}$$

This is a formal definition for a straightforward idea: an arc going around the center point will turn through an angle of $2\pi k$ before closing up.

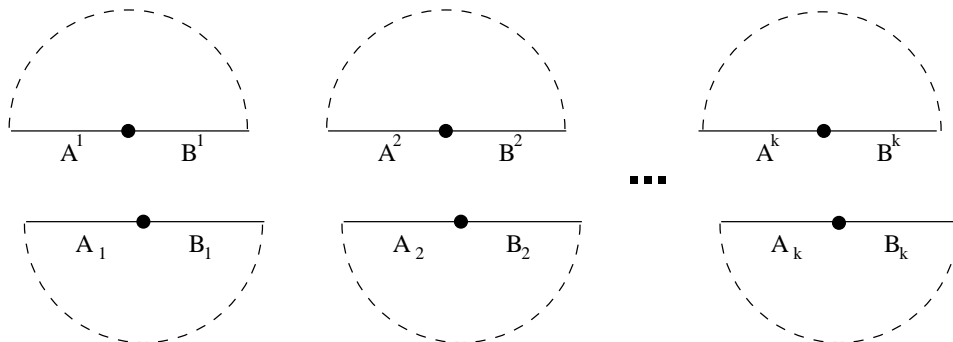


FIGURE 1. Making a k -sheet cone neighborhood out of half-disks.

Lemma. \mathcal{C}_k is homeomorphic to the open unit disk D in \mathbb{R}^2 .

Proof. Every point in \mathcal{C}_k can be uniquely assigned generalized polar coordinates (r, θ) with $0 \leq r < 1$ and $0 \leq \theta < 2\pi k$ once the $\theta = 0$ line is fixed. Then the map $f : \mathcal{C}_k \rightarrow D$ given by $f(r, \theta) = (r^{1/k}, \frac{1}{k}\theta)$ is a bijection, and it is clearly continuous in both directions. \square

Remark. In fact, this map $\mathcal{C}_k \rightarrow D$ is a diffeomorphism. Note that it can be written in rectangular coordinates as $f(r \cos \theta, r \sin \theta) = (r^{1/k} \cos \frac{1}{k}\theta, r^{1/k} \sin \frac{1}{k}\theta)$. In the language of complex analysis, \mathcal{C}_k is modeled on the branched cover of \mathbb{C} for $f(z) = z^{1/k}$.

Definition 3. A *translation surface* is a closed topological surface X with a finite set of points $CP = \{x_1, \dots, x_k\}$ and an atlas $\{(U_i, \phi_i)\}$ such that

- If U_i and U_j are overlapping patches and they are disjoint from CP , then the transition map between the charts is of the form $\phi_i \phi_j^{-1}(\mathbf{v}) = \mathbf{v} + \mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^2$ depending on i and j .
- There is one patch U_{x_j} for each element of CP , and if U_i is a patch overlapping with U_{x_j} but disjoint from CP , then the transition map is of the form

$$\phi_i \phi_{x_j}^{-1}(r \cos \theta, r \sin \theta) = (r^n \cos(n\theta), r^n \sin(n\theta)) + \mathbf{c}$$

for some n depending on j and some $\mathbf{c} \in \mathbb{R}^2$ depending on i and j .

Two translation surfaces are equivalent if there is a homeomorphism (a bicontinuous bijection) $f : S_1 \rightarrow S_2$ of the underlying surfaces which takes singularities to singularities, and such that on nonsingular patches, the composition $\phi' \circ f \circ \phi$ of charts with f is just translation.

Definition 4. A *polygonal system* is a finite collection of finite-sided Euclidean polygons in \mathbb{R}^2 along with gluing data so that every edge of every polygon is glued to some parallel edge of equal length (subject to the rule that the interiors must be on opposite sides of glued edges). Two polygonal systems are equivalent if we can obtain one from the other by cutting, translating, and pasting according to the gluing data.

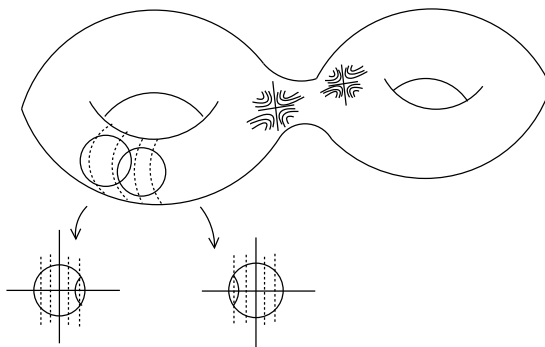


FIGURE 2. A schematic for a translation surface with genus two and two cone points of angle 4π .

Remark. Note that it follows from this definition that the total angle around any vertex, after gluing, is an integer multiple of 2π , and that therefore every cone point has a neighborhood that looks like \mathcal{C}_k for some k .

Theorem 5. *These definitions are equivalent: every translation surface can be presented as a (unique) polygonal system and vice versa.*

Remark. A translation surface or a polygonal system has a Euclidean (flat) metric away from its singularities— that this metric can be assigned should be clear from the respective definitions. An Euler characteristic argument shows that it has negative total curvature, concentrated in the singularities. From now on, we will sometimes refer to these objects as *polarized flat structures*— polarized because they have a notion of “vertical” and “horizontal.”

Proof. First we’ll show how to go from an X to a Q or vice versa; then we’ll argue that this correspondence is 1-1.

- (1) (translation surface \mapsto polygonal system)

Begin with a translation surface X . Without loss of generality, X has genus $g \geq 2$ (see Exercise 2), it has a nonempty set of singularities CP , and therefore it has saddle connections. We would like to cut along saddle connections until the remaining components are all simply connected (having every loop be homotopic to zero— equivalently, in this setting, they are all homeomorphic to disks). Here’s the inductive argument that cutting along saddle connections will eventually achieve this result: let Ω be a disjoint (except possibly at endpoints) collection of saddle connections on X . Let X_0, \dots, X_r be the connected components of $X \setminus \Omega$; each one of these is itself a translation surface with boundary. If some X_i were not homeomorphic to a disk, then it would have some closed curve λ not homotopic to the boundary ∂X_i . Take λ to be a geodesic. Then we have seen that it is either nonsingular, and so part of a cylinder, or it is a union of saddle connections. But indeed a core curve of a cylinder is homotopic to the boundary of the cylinder, which is a union of saddle connections. Thus we can break down X_i by adding saddle connections which make up λ to Ω , and continue inductively.

Being geodesics on the surface, the saddle connections are straight lines in the charts. A topological disk bounded by straight lines is a polygon, and note that we can in fact break the polygons down to triangles if we wish by adding saddle connections between the vertices. (This eliminates the problem of whether the polygons embed in the plane.)

The rest of the details are (relatively) straightforward to check.

- (2) (polygonal system \mapsto translation surface)

We need to show that the polygonal system, after identifications, gives a surface. Then we will consider the transition functions.

For any point on the interior of a polygon, choose an open neighborhood in the polygon and let the identity function be its chart to \mathbb{R}^2 . For a point on an edge, cut and paste so that the two copies of the edge are together, and do the same. If a vertex is not a cone point, then all polygons containing copies of it may be glued together in \mathbb{R}^2 , and the identity chart can be taken again.

So long as these open neighborhoods are chosen small enough that their intersections are connected, it follows from the construction that the transition functions are just translations in the plane.

The last case to consider is how to make the charts near cone points. But we have already remarked that every cone point has a neighborhood modeled on \mathcal{C}_k for some k . The map used in Lemma 2 then gives a chart to \mathbb{R}^2 with the properties we want.

- (3) Finally, we need to check that the correspondence is 1-1. This is just the observation that a cutting and pasting operation which gives the equivalence of polygonal systems is exactly a homeomorphism which witnesses the equivalence of translation surfaces.

□

Example 1. Skew-tori surface.

Consider two tori, one given by the standard unit square and the second given by a second copy of the square, rotated by an irrational multiple of π . Open a small vertical slit in each and glue them to each other as indicated in Figure 3. The resulting surface is a two-holed torus, and its two singularities occur at the endpoints of the slit.

This is a translation surface and can easily be realized as a polygonal system, but it does not come from a rational billiard.

On this surface, there is a vertical cylinder on the left-hand torus, but there is not a complete cylinder decomposition in the vertical direction because that direction is not closed on the right-hand torus. Indeed, no direction has a complete cylinder decomposition.

Definition 6. A *foliation* of a surface S is a partition of S into path-connected subsets F_α so that there is some atlas of charts sending the F_α to straight lines. The *vertical foliation* will be the collection of curves on the surface which are vertical lines in every chart. Likewise the *horizontal foliation*. (These are well-defined because of the translation structure, which maintains slope.) Each such curve is called a *leaf* of the foliation, and it must either be a closed leaf (and therefore part of a cylinder), a leaf which hits a cone point (called a *singular leaf*), or an infinite leaf. These are not mutually exclusive.

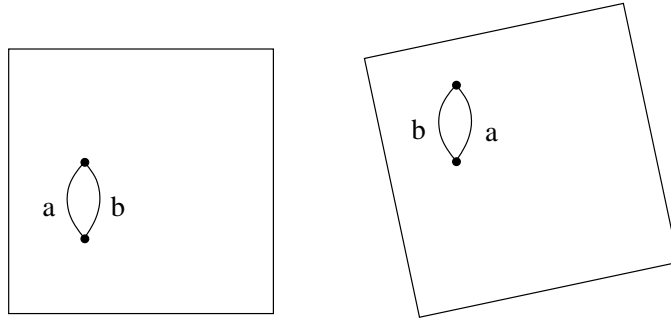


FIGURE 3. Two square tori, glued along parallel (vertical) slits.

Exercise 1. Show that rational billiards have measure zero in the space of polarized flat structures.

Exercise 2. Show that a translation surface can't be homeomorphic to the sphere. This follows from a general statement of Gauss-Bonnet, but you can do it without.

Exercise 3. Using the Kenyon-Smillie-Puchta-Ward theorem from Section 1, give a concrete example of a translation surface that does not “behave like a square.” Try to analyze your example to find out how it fails, say by finding a direction with an incomplete cylinder decomposition or an infinite orbit that fails to equidistribute.

Exercise 4. Choose a non-rational billiard. Try to describe its unfolding.

Exercise 5. If X has r singularities, show that the number of disjoint saddle connections (possibly sharing endpoints) is no more than $6g - 6 + 3r$. (If you don't get that far, show that it's finite.) That it's finite is needed for the inductive argument in the equivalence proof to terminate.

Exercise 6. Show that the link of every vertex is connected in a polygonal system. (The link is the boundary of an ϵ -neighborhood.)

Exercise 7. For X arising from a rational billiard, prove the formula relating the genus g of X to the angles $\frac{m_i\pi}{n_i}$ at the corners of the billiard polygon. Then give a formula relating g to the cone angles $2\pi k_i$ at the singularities of X . (This should turn out nice in terms of the orders of the singularities $j_i = k_i - 1$.)

Exercise 8. Consider a horizontal strip of four squares. Glue the left edge to the right edge by translation. How many gluing patterns of top to bottom yield polygonal systems? How many of these arise from billiards?

Harder: for such a translation surface X , try to find the affine maps $X \rightarrow X$. That is, look for linear transformations of the plane taking X to something that can be cut-and-pasted back to the original X .