Hyperbolic Space Isometries

Theorem 1 $\text{Isom}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$

Consider the upper half-space model of the hyperbolic plane. The metric is $ds = 1/(Imz)\ dz$.

1. Translations of the form $z \mapsto z + b$, dilations of the form $z \mapsto az$, and inversions of the form $z \mapsto -1/z$ are all isometries.

2. The group of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ is the subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by translations, dilations, and inversions.

3. $PSL_2(\mathbb{R})$ is isomorphic to the group of the above Möbius transformations.

4. The group of Möbius transformations is transitive on the set of triples of points in $\mathbb{H}^2$.

5. An isometry is determined by its action on any three non-collinear points.

Theorem 2 *The set of geodesics of* $\mathbb{H}^2$ *is the set of lines/circles perpendicular to the real-axis.*

1. Vertical lines are geodesics.

2. The group of Möbius transformations acts transitively on the set of lines/circles perpendicular to the real axis.